# On Complete Convergence for Weighted Sums of Pairwise Negatively Quadrant Dependent Sequences 

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#### Abstract

In this paper we prove the complete convergence for weighted sums of pairwise negatively quadrant dependent random variables. Some results on identically distributed and negatively associated setting of Liang and Su (1999) are generalized and extended to the pairwise negative quadrant dependence case.


Keywords: Negative quadrant dependence, complete convergence, weighted sums, negative association.

## 1. Introduction

Lehmann (1966) introduced the concept of negative quadrant dependence as follows: Two random variables $X$ and $Y$ are said to be negatively quadrant dependent(NQD) if

$$
\begin{equation*}
P(X \leq x, Y \leq y) \leq P(X \leq x) P(Y \leq y), \tag{1.1}
\end{equation*}
$$

for all real numbers $x$ and $y$. A collection of random variables is said to be pairwise NQD if every pair of random variables in the collection satisfies (1.1). It is important to note that (1.1) implies

$$
\begin{equation*}
P(X>x, Y>y) \leq P(X>x) P(Y>y) \tag{1.2}
\end{equation*}
$$

for all real numbers $x$ and $y$. Moreover, it follows that (1.2) implies (1.1), and hence, (1.1) and (1.2) are equivalent.

Consider a special case of the F-G-M system where both marginals are exponentials. The joint distribution function is then of the form,

$$
F(x, y)=\left(1-e^{\lambda_{1} x}\right)\left(1-e^{\lambda_{2} y}\right)\left(1+\rho e^{\lambda_{1} x-\lambda_{2} y}\right), \quad-1 \leq \rho \leq 1, \lambda_{1}, \lambda_{2}>0
$$

(see Johnson and Kotz, 1972, pp. 262-263). It is obvious that $X$ and $Y$ are negatively quadrant dependent if $\rho \leq 0$.

The F-G-M bivariate distribution has been studied extensively. It has several applications in various contexts, for example, in competing risk problems (Tolley and Norman, 1979).

In 1983, Joag-Dev and Proschan introduced another concept of negative dependence: A finite family $\left\{X_{i}, 1 \leq i \leq n\right\}$ of random variables is said to be negatively associated(NA) if for every disjoint subsets $A, B \subset\{1,2, \ldots, n\}$ and for any increasing functions $f$ and $g \operatorname{Cov}\left(f\left(X_{i}, i \in A\right), g\left(X_{j}, j \in B\right)\right) \leq$ 0.

[^0]These concepts of dependent random variables have been very useful in reliability theory and applications (Barlow and Proschan, 1975).

Obviously, NA implies pairwise NQD from the definition of NA and pairwise NQD. But pairwise NQD does not imply NA, so pairwise NQD is much weaker than NA. Hence, extending the limit properties of independent or NA random variables to the case of pairwise NQD variables is highly desirable and considerably significant in theory and application.

Let $\left\{X_{n}, n \geq 1\right\}$ be a sequence of random variables and $\left\{a_{n i}, i \geq 1,1 \leq i \leq n\right\}$ be an array of real numbers. The weighted sums $\sum_{i=1}^{n} a_{n i} X_{i}$ can play an important role in various applied and theoretical problems, such as those of the least square estimators (Kafles and Bhaskara Rao, 1982) and M-estimates (Rao and Zhao, 1992) in linear models, the nonparametric regression estimators (Cheng, 1995). Therefore, a So the study of the limiting behavior of the weighted sums is very important and significant.

Hsu and Robbins (1947) introduced the concept of the complete convergence of $\left\{X_{n}, n \geq 1\right\}$ as follows: A sequence $\left\{X_{n}, n \geq 1\right\}$ of random variables is said to converge completely to a constant $c$ if

$$
\sum_{n=1}^{\infty} P\left\{\left|X_{n}-c\right| \geq \epsilon\right\}<\infty, \quad \text { for all } \epsilon>0
$$

There have been many investigations in the complete convergences: For examples, Bai and Su (1985) proved the complete convergence of partial sums of i.i.d. random variables, Gut $(1992,1993)$ investigated the complete convergence of arrays of i.i.d. random variables, Li et al. (1995) studied complete convergence and almost sure convergence of weighted sums of independent random variables, Liang and Su (1999) and Liang (2000) obtained complete convergence of weighted sums of negatively associated sequence, Kuczmaszewska (2009) showed the complete convergence for arrays of rowwise negatively associated random variables.

Liang and Su (1999) proved that for identically distributed NA random variables $\left\{X_{n}, n \geq 1\right\}$, under some restrictions on weights $\left\{a_{n i}, 1 \leq i \leq n, n \geq 1\right\}$ there exists the equivalence between the convergence of series

$$
\sum_{n=1}^{\infty} n^{r-2} P\left\{\max _{1 \leq k \leq n}\left|\sum_{i=1}^{k} a_{n i} X_{i}\right|>\epsilon n^{\frac{1}{2}}\right\}, \quad \forall \epsilon>0
$$

and the existence of moment $E\left(|X|^{2(r-1)} \log |X|\right)$, for correctly chosen $p$ and $r$ (Liang and $\mathrm{Su}, 1999$, Theorem 2.1 (I)).

Moreover, they considered sequences without identical distribution and proved the following result.

Theorem 1. (Liang and Su, 1999, Theorem 2.2) Let $\left\{X_{n}, n \geq 1\right\}$ be a sequence of zero mean $N A$ random variables, and let $\left\{a_{n i}, 1 \leq i \leq n, n \geq 1\right\}$ be an array of real numbers satisfying the conditions $\sum_{n=1}^{n} a_{n i}^{2}=O\left(n^{\delta}\right)$ as $n \rightarrow \infty$ and $\left|a_{n i}\right|=O(1), 1 \leq i \leq n, n \geq 1$ for some $0<\delta<2 / p, p \geq$ 2. If $\beta=\sum_{k \geq 1} E\left|X_{k}\right|^{p}<\infty$, then $\forall \epsilon>0$,

$$
\sum_{n=1}^{\infty} n^{-1} P\left\{\max _{1 \leq k \leq n}\left|\sum_{i=1}^{k} a_{n i} X_{i}\right|>\epsilon n^{\frac{1}{p}}\right\}<\infty .
$$

In this paper we investigate the complete convergence for weighted sums of sequences of pairwise NQD random variables and extend Theorem 2.1(I) in Liang and Su (1999) to the pairwise NQD case without extra conditions and show that Theorem 1 still holds under the pairwise NQD assumption with restriction $p=2$.

## 2. Some Lemmas

We introduce a few lemmas needed in the further part of this paper.
Lemma 1. (Lehmann, 1966) Let $\left\{X_{n}, n \geq 1\right\}$ be a sequence of pairwise NQD random variables and $\left\{f_{n}, n \geq 1\right\}$ be a sequence of Borel functions, all of which are monotone increasing(decreasing). Then $\left\{f_{n}\left(X_{n}\right), n \geq 1\right\}$ is still a sequence of pairwise NQD random variables.
Lemma 2. (Wu, 2006) Let $\left\{X_{n}, n \geq 1\right\}$ be a sequence of pairwise $N Q D$ random variables with $E X_{n}=0$ and $E X_{n}^{2}<\infty$ for all $n \geq 1$. Then

$$
\begin{gather*}
E\left(\sum_{i=1}^{n} X_{i}\right)^{2} \leq \sum_{i=1}^{n} E X_{i}^{2}  \tag{2.1}\\
E\left(\max _{1 \leq k \leq n} \sum_{i=1}^{k} X_{i}\right)^{2} \leq\left(\log _{2} n\right)^{2} \sum_{i=1}^{n} E X_{i}^{2} . \tag{2.2}
\end{gather*}
$$

From Lemma 1 and Lemma 2 we obtain the following lemma.
Lemma 3. Let $\left\{X_{n}, n \geq 1\right\}$ be a sequence of pairwise NQD random variables. Then for any $x \geq 0$ and for all $n \geq 1$,

$$
\begin{equation*}
\left(1-P\left(\max _{1 \leq k \leq n}\left|X_{k}\right|>x\right)\right)^{2} \sum_{k=1}^{n} P\left(\left|X_{k}\right|>x\right) \leq 2 P\left(\max _{1 \leq k \leq n}\left|X_{k}\right|>x\right) . \tag{2.3}
\end{equation*}
$$

Proof: Let $A_{k}=\left\{\left|X_{k}\right|>x\right\}$ and

$$
\begin{equation*}
\alpha_{n}=1-P\left(\bigcup_{k=1}^{n} A_{k}\right)=1-P\left(\max _{1 \leq k \leq n}\left|X_{k}\right|>x\right) . \tag{2.4}
\end{equation*}
$$

Without loss of generality, we assume that $\alpha_{n}>0$. Note that $\left\{I\left[X_{k}>x\right]-E I\left[X_{k}>x\right], k \geq 1\right\}$ and $\left\{I\left[X_{k}<-x\right]-E I\left[X_{k}<-x\right], k \geq 1\right\}$ are still sequences of pairwise NQD random variables by Lemma 1. Hence, we get

$$
\begin{align*}
E\left(\sum_{k=1}^{n}\left(I_{A_{k}}-E I_{A_{k}}\right)\right)^{2} & =E\left(\sum_{k=1}^{n}\left(I\left[X_{k}>x\right]-E I\left[X_{k}>x\right]\right)+\sum_{k=1}^{n}\left(I\left[X_{k}<-x\right]-E I\left[X_{k}<-x\right]\right)\right)^{2}  \tag{2.5}\\
& \leq 2 E\left(\sum_{k=1}^{n}\left(I\left[X_{k}>x\right]-E I\left[X_{k}>x\right]\right)\right)^{2}+2 E\left(\sum_{k=1}^{n}\left(I\left[X_{k}<-x\right]-E I\left[X_{k}<-x\right]\right)\right)^{2} \\
& \leq 2 \sum_{k=1}^{n} E\left(I\left[X_{k}>x\right]-E I\left[X_{k}>x\right]\right)^{2}+2 E \sum_{k=1}^{n}\left(I\left[X_{k}<-x\right]-E I\left[X_{k}<-x\right]\right)^{2} \quad \text { by (2.1) } \\
& \leq 2 \sum_{k=1}^{n} E\left(I\left[X_{k}>x\right]\right)^{2}+\sum_{k=1}^{n} E\left(I\left[X_{k}<-x\right]\right)^{2} \\
& =2 \sum_{k=1}^{n} P\left(X_{k}>x\right)+\sum_{k=1}^{n} P\left(X_{k}<-x\right) \\
& =2 \sum_{k=1}^{n} P\left(A_{k}\right) .
\end{align*}
$$

By Cauchy-Schwarz inequality, (2.4) and (2.5) we obtain

$$
\begin{aligned}
\sum_{k=1}^{n} P\left(A_{k}\right) & =\sum_{k=1}^{n} P\left(A_{k} \cap\left(\cup_{j=1}^{n} A_{j}\right)\right)=\sum_{k=1}^{n} E\left(I_{A_{k}} I_{\cup_{j=1}^{n} A_{j}}\right) \\
& =E \sum_{k=1}^{n}\left(I_{A_{k}}-E I_{A_{k}}\right) I_{\cup_{j=1}^{n} A_{j}}+\sum_{k=1}^{n} P\left(A_{k}\right) P\left(\cup_{j=1}^{n} A_{j}\right) \\
& \leq\left(E\left(\sum_{k=1}^{n}\left(I_{A_{k}}-E I_{A_{k}}\right)\right)^{2} E\left(I_{\cup_{j=1}^{n} A_{j}}\right)^{2}\right)^{\frac{1}{2}}+\left(1-\alpha_{n}\right) \sum_{k=1}^{n} P\left(A_{k}\right) \\
& \leq\left(2\left(1-\alpha_{n}\right) \sum_{k=1}^{n} P\left(A_{k}\right)\right)^{\frac{1}{2}}+\left(1-\alpha_{n}\right) \sum_{k=1}^{n} P\left(A_{k}\right) \\
& \leq \frac{1}{2}\left(\frac{2\left(1-\alpha_{n}\right)}{\alpha_{n}}+\alpha_{n} \sum_{k=1}^{n} P\left(A_{k}\right)\right)+\left(1-\alpha_{n}\right) \sum_{k=1}^{n} P\left(A_{k}\right)
\end{aligned}
$$

which yields

$$
\begin{equation*}
\alpha_{n}^{2} \sum_{k=1}^{n} P\left(A_{k}\right) \leq 2\left(1-\alpha_{n}\right) \tag{2.6}
\end{equation*}
$$

Hence, by (2.4) and (2.6) we get (2.3).

## 3. Main Results

In the following, let $a_{n} \ll b_{n}$ denote that there exists a constant $c>0$ such that $a_{n} \leq c b_{n}$ for sufficiently large $n, a_{n} \approx b_{n}$ mean $a_{n} \ll b_{n}$ and $a_{n} \gg b_{n}$ and $S_{n}=\sum_{j=1}^{n} X_{j}$.

Theorem 2. Let $\left\{X, X_{n}, n \geq 1\right\}$ be a sequence of identically distributed pairwise NQD random variables, $\left\{a_{n k}, 1 \leq k \leq n, n \geq 1\right\}$ be an array of real numbers and let $r>1$. If

$$
\begin{align*}
N(n, m+1) & =\sharp\left\{k,\left|a_{n k}\right| \geq(m+1)^{-\frac{1}{2}}\right\} \approx m^{r-1}, \quad n, m \geq 1,  \tag{3.1}\\
E X & =0, \tag{3.2}
\end{align*}
$$

and for $1 \leq 2(r-1)$,

$$
\begin{equation*}
\sum_{k=1}^{n}\left|a_{n k}\right|^{2(r-1)}=O(1), \quad \text { as } n \rightarrow \infty \tag{3.3}
\end{equation*}
$$

Then,

$$
\begin{equation*}
E\left(|X|^{2(r-1)} \log |X|\right)<\infty \tag{3.4}
\end{equation*}
$$

if and only if

$$
\begin{equation*}
\sum_{n=1}^{\infty} n^{r-2} P\left(\max _{1 \leq k \leq n}\left|\sum_{i=1}^{k} a_{n i} X_{i}\right|>\epsilon n^{\frac{1}{2}}\right)<\infty, \quad \forall \epsilon>0 \tag{3.5}
\end{equation*}
$$

Proof: (3.4) $\Rightarrow$ (3.5) Without loss of generality, we can assume that $a_{n i}>0$ for all $1 \leq i \leq n, n \geq 1$. For $0<\alpha<1 / 2$ small enough let

$$
\begin{aligned}
& X_{n i}^{(1)}=-n^{\alpha} I_{\left[a_{n i} X_{i}<-n^{\alpha}\right]}+a_{n i} X_{i} I_{\left[a_{n i} \mid X_{i} \leq n^{\alpha}\right]}+n^{\alpha} I_{\left[a_{n i} X_{i}>n^{\alpha}\right]}, \\
& X_{n i}^{(2)}=\left(a_{n i} X_{i}-n^{\alpha}\right) I_{\left[n^{\alpha}<a_{n i} X_{i}<\epsilon n^{\frac{1}{2}} / 8\right]}, \\
& X_{n i}^{(3)}=\left(a_{n i} X_{i}+n^{\alpha}\right) I_{\left[-\epsilon n^{\frac{1}{2}} / 8<a_{n i} X_{i}<-n^{\alpha}\right]}, \\
& X_{n i}^{(4)}=\left(a_{n i} X_{i}+n^{\alpha}\right) I_{\left[a_{n i} X_{i}<\epsilon n^{\frac{1}{2}} / 8\right]}+\left(a_{n i} X_{i}-n^{\alpha}\right) I_{\left[a_{n i} X_{i}>\epsilon n^{\frac{1}{2}} / 8\right]},
\end{aligned}
$$

and

$$
S_{n k}^{(j)}=\sum_{i=1}^{k} X_{n i}^{(j)}, \quad j=1,2,3,4 ; 1 \leq k \leq n, n \geq 1
$$

Obviously, $S_{n k}=\sum_{j=1}^{4} S_{n k}^{(j)}$. Note that

$$
\left(\max _{1 \leq k \leq n}\left|S_{n k}\right|>\epsilon n^{\frac{1}{2}}\right) \subseteq \cup_{j=1}^{4}\left(\max _{1 \leq k \leq n}\left|S_{n k}^{(j)}\right|>\frac{\epsilon}{4} n^{\frac{1}{2}}\right)
$$

So, to prove (3.5) it suffices to show that

$$
\begin{equation*}
I_{j}=\sum_{n=1}^{\infty} n^{r-2} P\left(\max _{1 \leq k \leq n}\left|S_{n k}^{(j)}\right|>\frac{\epsilon}{4} n^{\frac{1}{2}}\right)<\infty, \quad j=1,2,3,4 \tag{3.6}
\end{equation*}
$$

First we prove that

$$
\begin{equation*}
n^{-\frac{1}{2}} \max _{1 \leq k \leq n}\left|E S_{n k}^{(1)}\right| \rightarrow 0, \quad \text { as } n \rightarrow \infty . \tag{3.7}
\end{equation*}
$$

For $1 \leq 2(r-1)$ by (3.3) and (3.4) we obtain

$$
\begin{aligned}
n^{-\frac{1}{2}} \max _{1 \leq k \leq n}\left|E S_{n k}^{(1)}\right| & \leq n^{-\frac{1}{2}} \sum_{i=1}^{n}\left(E\left|a_{n i} X_{i}\right| I_{\left[\left|a_{n} X_{i}\right|>n^{\alpha}\right]}+n^{\alpha} P\left(\left|a_{n i} X_{i}\right|>n^{\alpha}\right)\right) \\
& \leq n^{-\frac{1}{2}} \sum_{i=1}^{n}\left(E\left|a_{n i} X_{i}\right|\left(\frac{\left|a_{n i} X_{i}\right|}{n^{\alpha}}\right)^{2(r-1)-1} I_{\left[\left|a_{n i} X_{i}\right|>n^{\alpha}\right]}+n^{\alpha-2 \alpha(r-1)} E\left|a_{n i} X_{i}\right|^{2(r-1)}\right) \\
& \leq n^{-\frac{1}{2}+\alpha-2 \alpha(r-1)} E\left|X_{i}\right|^{2(r-1)} \sum_{i=1}^{n}\left|a_{n i}\right|^{2(r-1)} \\
& \ll n^{-\frac{1}{2}+\alpha-2 \alpha(r-1)} \rightarrow 0, \quad \text { as } n \rightarrow \infty \text { since }-\frac{1}{2}+\alpha-2 \alpha(r-1)<0 .
\end{aligned}
$$

Hence (3.7) holds. Therefore, to prove $I_{1}<\infty$ it is enough to show that

$$
\begin{equation*}
\tilde{I}_{1}=\sum_{n=1}^{\infty} n^{r-2} P\left(\max _{1 \leq k \leq n}\left|S_{n k}^{(1)}-E S_{n k}^{(1)}\right|>\frac{\epsilon}{4} n^{\frac{1}{2}}\right)<\infty, \quad \forall \epsilon>0 \tag{3.8}
\end{equation*}
$$

Note that $\left\{X_{n i}^{(1)}, 1 \leq i \leq n, n \geq 1\right\}$ is still pairwise NQD by definition of $X_{n i}^{(1)}$ and Lemma 2. Using Chebyshev's inequality and Lemma 1, we get

$$
\begin{align*}
\tilde{I}_{1} & \leq \sum_{n=1}^{\infty} n^{r-3} E\left(\max _{1 \leq k \leq n}\left|S_{n k}^{(1)}-E S_{n k}^{(1)}\right|\right)^{2} \\
& \ll \sum_{n=1}^{\infty} n^{r-3}\left(\log _{2} n\right)^{2}\left(\sum_{i=1}^{n} E\left(X_{n i}^{(1)}\right)^{2}\right) \\
& \leq \sum_{n=1}^{\infty} n^{r-3}\left(\log _{2} n\right)^{2} \sum_{i=1}^{n}\left(E\left(a_{n i} X_{i}\right)^{2} I_{\left[\mid a_{n i} X_{i} \leq n^{\alpha}\right]}+n^{2 \alpha} P\left(\left|a_{n i} X_{i}\right|>n^{\alpha}\right)\right) \\
& \leq \sum_{n=1}^{\infty} n^{r-3}\left(\log _{2} n\right)^{2} \sum_{i=1}^{n}\left(E\left|a_{i} X_{i}\right|^{2(r-1)} n^{\alpha(2-2(r-1))}+n^{\alpha(2-2(r-1))} E\left|a_{n i} X_{i}\right|^{2(r-1)}\right)  \tag{3.9}\\
& \leq \sum_{n=1}^{\infty} n^{r-3+2 \alpha-2 \alpha(r-1)}\left(\log _{2} n\right)^{2} \sum_{i=1}^{n}\left|a_{n i}\right|^{2(r-1)} E|X|^{2(r-1)} \\
& \ll \sum_{n=1}^{\infty} n^{r-3+2 \alpha-2 \alpha(r-1)}\left(\log _{2} n\right)^{2} \\
& <\infty .
\end{align*}
$$

Since, by the definition of $X_{n i}^{(2)}, S_{n k}^{(2)}>0$ and

$$
\begin{align*}
\left(\max _{1 \leq k \leq n}\left|S_{n k}^{(2)}\right|>\frac{\epsilon}{4} n^{\frac{1}{2}}\right) & =\left(\sum_{i=1}^{n} X_{n i}^{(2)}>\frac{\epsilon}{4} n^{\frac{1}{2}}\right)  \tag{3.10}\\
& =\left(\sum_{i=1}^{n}\left(a_{n i} X_{i}-n^{\alpha}\right) I_{\left[n^{\alpha}<a_{n i} X_{i}<\epsilon n^{\frac{1}{2}} / 8\right]}>\frac{\epsilon}{4} n^{\frac{1}{2}}\right) \\
& \subseteq\left\{\text { there exists at least } 2 \text { indices } k \text { such that } a_{n k} X_{k}>n^{\alpha}\right\},
\end{align*}
$$

we have

$$
P\left(\sum_{i=1}^{n} X_{n i}^{(2)}>\frac{\epsilon}{4} n^{\frac{1}{2}}\right) \leq \sum_{1 \leq i_{1}<i_{2} \leq n} P\left(a_{n i_{1}} X_{i_{1}}>n^{\alpha}, a_{n i_{2}} X_{i_{2}}>n^{\alpha}\right) .
$$

By Lemma $1\left\{a_{n i} X_{i}, 1 \leq i \leq n, n \geq 1\right\}$ is still a sequence of pairwise NQD random variables. Hence, we conclude that

$$
\begin{align*}
P\left(\max _{1 \leq k \leq n}\left|S_{n k}^{(2)}\right|>\frac{\epsilon}{4} n^{\frac{1}{2}}\right) & =P\left(\sum_{i=1}^{n} X_{n i}^{(2)}>\frac{\epsilon}{4} n^{\frac{1}{2}}\right)  \tag{3.11}\\
& \leq \sum_{1 \leq i_{1}<i_{2} \leq n} \prod_{j=1}^{2} P\left(a_{n i_{j}} X_{i_{j}}>n^{\alpha}\right) \\
& \leq\left(\sum_{i=1}^{n} P\left(\left|a_{n i} X_{i}\right|>n^{\alpha}\right)\right)^{2}
\end{align*}
$$

$$
\begin{aligned}
& \leq\left(\sum_{i=1}^{n} n^{-2 \alpha(r-1)} E\left|a_{n i} X_{i}\right|^{2(r-1)}\right)^{2} \\
& \ll n^{-4 \alpha(r-1)},
\end{aligned}
$$

by (3.3) and (3.4). Note that $X_{n i}^{(2)}>0$ by the definition of $X_{n i}^{(2)}$. Since $r-2-4 \alpha(r-1)<-1$ it follows from (3.11) that

$$
\begin{aligned}
I_{2} & =\sum_{n=1}^{\infty} n^{r-2} P\left(\sum_{i=1}^{n} X_{n i}^{(2)}>\frac{\epsilon}{4} n^{\frac{1}{2}}\right) \\
& \ll \sum_{n=1}^{\infty} n^{r-2-4 \alpha(r-1)}<\infty .
\end{aligned}
$$

Similarly, we have $X_{n i}^{(3)}<0$ and $I_{3}<\infty$.
It remains to prove that $I_{4}<\infty$. Let $Y=8 X / \epsilon$. By the definition of $X_{n i}^{(4)}$ and (3.1) we have

$$
\begin{aligned}
P\left(\max _{1 \leq k \leq n} S_{n k}^{(4)}>\frac{\epsilon}{4} n^{\frac{1}{2}}\right) & \leq P\left(\sum_{i=1}^{n} X_{n i}^{(4)}>\frac{\epsilon}{4} n^{\frac{1}{2}}\right) \\
& \leq P\left(\cup_{i=1}^{n}\left(a_{n i}\left|X_{i}\right|>\frac{\epsilon n^{\frac{1}{2}}}{8}\right)\right) \\
& \leq \sum_{i=1}^{n} P\left(a_{n i}\left|X_{i}\right|>\frac{\epsilon n^{\frac{1}{2}}}{8}\right) \\
& =\sum_{j=1}^{\infty} \sum_{(j+1)^{-1}<a_{n i}^{2}<j^{-1}} P\left(|Y|>(n j)^{\frac{1}{2}}\right) \\
& =\sum_{j=1}^{\infty}(N(n, j+1)-N(n, j)) P\left(l \leq|Y|^{2}<l+1\right) \\
& \approx \sum_{l=n}^{\infty}\left(\frac{l}{n}\right)^{r-1} P\left(l \leq|Y|^{2}<l+1\right),
\end{aligned}
$$

which yields

$$
\begin{aligned}
I_{4} & \approx \sum_{n=1}^{\infty} n^{r-2} \sum_{l=n}^{\infty}\left(\frac{l}{n}\right)^{r-1} P\left(l \leq|Y|^{2}<l+1\right) \\
& \approx \sum_{l=1}^{\infty} l^{r-1} \sum_{n=1}^{l} n^{r-2-r+1} P\left(l \leq|Y|^{2}<l+1\right) \\
& \approx \sum_{l=1}^{\infty} l^{r-1} \log l P\left(l \leq|Y|^{2}<l+1\right) \\
& \approx E\left(|X|^{2(r-1)} \log |X|\right)<\infty, \quad \text { by }(3.4)
\end{aligned}
$$

Now we prove (3.5) $\Rightarrow$ (3.4). Obviously (3.5) implies

$$
\begin{equation*}
\sum_{n=1}^{\infty} n^{r-2} P\left(\max _{1 \leq j \leq n}\left|a_{n j} X_{j}\right|>n^{\frac{1}{2}}\right)<\infty, \tag{3.12}
\end{equation*}
$$

which yields

$$
P\left(\max _{1 \leq j \leq n}\left|a_{n j} X_{j}\right|>n^{\frac{1}{2}}\right) \rightarrow 0, \quad \text { as } n \rightarrow \infty
$$

by the hypotheses of Theorem 2. Hence, for sufficiently large $n$,

$$
P\left(\max _{1 \leq j \leq n}\left|a_{n j} X_{j}\right|>n^{\frac{1}{2}}\right)<\frac{1}{2} .
$$

By Lemma $1\left\{a_{n j} X_{j}, 1 \leq j \leq n, n \geq 1\right\}$ is still a sequence of pairwise NQD random variables. By Lemma 3 and (3.1) we obtain

$$
\begin{equation*}
\sum_{k=1}^{n} P\left(\left|a_{n k} X_{k}\right|>n^{\frac{1}{2}}\right) \leq 8 P\left(\max _{1 \leq k \leq n}\left|a_{n k} X_{k}\right|>n^{\frac{1}{2}}\right) \tag{3.13}
\end{equation*}
$$

From (3.12) and (3.13) it follows that

$$
\begin{equation*}
\sum_{n=1}^{\infty} n^{r-2} \sum_{k=1}^{n} P\left(\left|a_{n k} X_{k}\right|>n^{\frac{1}{2}}\right)<\infty . \tag{3.14}
\end{equation*}
$$

So, by the process of proof of $I_{4}<\infty$ from (3.14) we obtain

$$
\begin{aligned}
E\left(|X|^{2(r-1)} \log |X|\right) & \approx \sum_{n=1}^{\infty} n^{r-2} \sum_{k=1}^{n} P\left(\left|a_{n k} X_{k}\right|>n^{\frac{1}{2}}\right) \\
& \leq 8 \sum_{n=1}^{\infty} n^{r-2} P\left(\max _{1 \leq k \leq n}\left|a_{n k} X_{k}\right|>n^{\frac{1}{2}}\right) \\
& <\infty .
\end{aligned}
$$

Hence, the proof of Theorem 2 is complete.
Remark 1. Theorem 2 shows that for identically distributed pairwise NQD random variables the equivalence between complete convergence of series (3.5) and the existence of moment (3.4).

Moreover, we consider the sequence without identical distribution restriction and prove the following result.

Theorem 3. Let $\left\{X_{n}, n \geq 1\right\}$ be a sequence of zero mean pairwise NQD random variables and let $\left\{a_{n k}, 1 \leq k \leq n, n \geq 1\right\}$ be an array of real numbers satisfying the conditions, for some $0<\delta<1$

$$
\begin{equation*}
\sum_{k=1}^{n} a_{n k}^{2}=O\left(n^{\delta}\right), \quad \text { as } n \rightarrow \infty \quad\left|a_{n k}\right|=O(1), 1 \leq k \leq n, n \geq 1 \tag{3.15}
\end{equation*}
$$

If

$$
\begin{equation*}
\sup _{k \geq 1} E X_{k}^{2}<\infty, \tag{3.16}
\end{equation*}
$$

then

$$
\sum_{n=1}^{\infty} \frac{1}{n} P\left(\max _{1 \leq k \leq n}\left|\sum_{i=1}^{k} a_{n i} X_{i}\right|>\epsilon n^{\frac{1}{2}}\right)<\infty, \quad \forall \epsilon>0
$$

Proof: Without loss of generality we assume that $a_{n k} \geq 0$. By Lemma 1, (3.15), (3.16) and Chebyshev's inequality we obtain

$$
\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n} P\left(\max _{1 \leq k \leq n}\left|\sum_{i=1}^{k} a_{n i} X_{i}\right|>\epsilon n^{\frac{1}{2}}\right) & <\sum_{n=1}^{\infty} \frac{1}{n^{2}}\left(\log _{2} n\right)^{2} \sum_{i=1}^{n} a_{n i}^{2} E X_{i}^{2} \\
& \ll \sum_{n=1}^{\infty} n^{-(2-\delta)}\left(\log _{2} n\right)^{2}<\infty .
\end{aligned}
$$

From Lemma 3 we obtain the following result.

Corollary 1. Let $\left\{X_{n}, n \geq 1\right\}$ be a sequence of pairwise NQD random variables and let $\left\{a_{n k}, 1 \leq k \leq\right.$ $n, n \geq 1\}$ be an array of real numbers. Assume that for $\delta>0$ small enough,

$$
P\left(\max _{1 \leq j \leq n}\left|a_{n j} X_{j}\right|>\epsilon n^{\frac{1}{2}}\right)<\delta, \quad \forall \epsilon>0
$$

for sufficiently large $n$. Then

$$
\sum_{j=1}^{n} P\left(\left|a_{n j} X_{j}\right|>\epsilon n^{\frac{1}{2}}\right) \ll P\left(\max _{1 \leq i \leq n}\left|a_{n i} X_{i}\right|>\epsilon n^{\frac{1}{2}}\right), \quad \forall \epsilon>0
$$

for sufficiently large $n$.

## Remark 2.

1. Theorem 2 is an extension of Theorem 2.1(I) of Liang and Su (1999) to the case of pairwise NQD random variables without extra conditions.
2. Theorem 3 and Corollary 1 show that Theorem 2.2 and Corollary 3.1 of Liang and Su (1999) still hold under the pairwise NQD assumption with restriction $p=2$, respectively.
3. As an application, we can prove the complete convergence of linear processes under pairwise NQD assumption by Theorem 2.

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