

Influences of Dependence Degrees of a Component for the Mean Time to Failure of a System

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Abstract

This article considers the mean time to failure(MTTF) of a dependent parallel system. We study how the degree of dependency components influences the increase in the mean lifetime for this system. The results are illustrated by tables and figures.

Keywords: Dependent parallel system, Bivariate Weibull model, MTTF.

1. Introduction

The concepts of dependence did not receive sufficient attention in the statistical literature, since 1966 when the pioneering paper by E.L. Lehmann (1966) has appeared; however it now permeates throughout our daily life. There are many examples of interdependency in medicine, economic structures and reliability engineering.

In reliability literature, it is usually assumed that the component lifetimes are independent. However, components in the same system are used in the same environment (or share the same load,) and hence the failure of one component influences others (Esary and Proschan, 1970). In many real multi-component systems, the failure of a component affects the remaining components (Murthy and Nguyen, 1985), where we have the case of so-called common cause failure and components that might simultaneously fail. The dependence is usually difficult to describe, even for very similar components. From light bulbs in an overhead projector to engines in an airplane, we have dependence, and it is essential to study the effect of dependence for better reliability design and analysis.

There are many notions of bivariate and multivariate dependence. In literature these models have been proposed for the study of complex systems. The model proposed by Marshall and Olkin (1967) is based on the assumption that a two components system is distributed as a bivariate exponential(BVE) model with the interesting condition that these two components are not independent.

In reliability field an alternative distribution to the exponential, is the Weibull law (Johnson and Kots, 1970). The Weibull distribution is a versatile family of life distribution in view of its physical interpretation and its flexibility for empirical fit, and has been extensively applied to the analysis of life data concerning many types of manufactures components.

As the forms of dependence among the components in a two-components system, Lu and Bhat-tachrayya (1990) and Lu (1989) initially introduced some new construction of bivariate Weibull(BVW) distributions as the extension of the Freund (1961) and Marshall and Olkin's BVE distribution (1967).

The BVW distribution is crucial role that Weibull distribution plays in reliability as well as building models for various failure or life time distributions.

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In this paper, we drive the mean time to failure(MTTF) for systems with dependent parallel components and investigate how the degree of dependency will influence the increase in the mean lifetime. We assume that two components system follow a the BVW distribution that can be obtained from BVW model of Marshall-Olkin (1967).

In Section 2, we consider the on the Bivariate Weibull distribution of Marshal and Olkin. In Section 3, we drive the MTTF for Marshall-Olkin Bivariate Weibull(MOBVW) distribution. Numerical results are presented in Section 4 and finally conclude the paper in Section 5.

2. Bivariate Weibull(BVW) Distribution

The BVW model is studied under the assumption that a “fatal shock” could break the system. When a fatal shock comes, one of the components will fail.

There is extensive literature on the construction of bivariate exponential models, for instance, Gumbel (1960), Freund (1961), Marschall and Olkin (1967), and Clayton (1978). The models proposed by Freund and Marshall-Olkin have received the most attention in describing the statistical dependence of components in a 2-component system and in developing statistical inference procedures.

Suppose that the components of a two-component system fail after receiving a shock that is always fatal. Independent Poisson processes $Z_1(t; \lambda_1)$, $Z_2(t; \lambda_2)$, $Z_{12}(t; \lambda_{12})$ govern the occurrence of shocks. Events in the process $Z_1(t; \lambda_1)$ are shocks to component 1, events in the process $Z_2(t; \lambda_2)$ are shocks to component 2, and events in the process $Z_{12}(t; \lambda_{12})$ are shocks to both components. Thus if X_1 and X_2 denote the two remaining lifetime of the first and second components,

$$\begin{aligned}\bar{F}(x_1, x_2) &= P[X_1 > x_1, X_2 > x_2] \\ &= P\{Z_1(x_1; \lambda_1) = 0, Z_2(x_2; \lambda_2) = 0, Z_{12}(\max(x_1, x_2); \lambda_{12}) = 0\} \\ &= \exp[-(\lambda_1 x_1 + \lambda_2 x_2 + \lambda_{12} \max(x_1, x_2))].\end{aligned}\quad (2.1)$$

For convenience we say that X_1 and X_2 are BVE($\lambda_1, \lambda_2, \lambda_{12}$) if (2.1) holds. Also we refer to the distribution of (2.1) as the bivariate exponential, MOBVE($\lambda_1, \lambda_2, \lambda_{12}$) (Mashall and Olkin, 1967).

The obvious way of generating a BVW model is to make a power transformation of the marginals of the bivariate exponential model studied Marshall and Olkin (1967).

Consider the transformation of $(U_1, U_2) \equiv (X_1^{1/\alpha_1}, X_2^{1/\alpha_2})$. Then

$$\bar{F}_U(u_1, u_2) = \exp\left[-\left(\lambda_1 u_1^{\alpha_1} + \lambda_2 u_2^{\alpha_2} + \lambda_{12} \max(u_1^{\alpha_1}, u_2^{\alpha_2})\right)\right], \quad (2.2)$$

where $\lambda_1, \lambda_2, \lambda_{12} > 0$ are the scalar parameters they represent the failure rate, and $\alpha_1, \alpha_2 > 0$ are the shape parameters. We denote a Marshall-Olkin BVW distribution with parameters $\lambda_1, \lambda_2, \lambda_{12}, \alpha_1$ and α_2 as MOBVBW($\lambda_1, \lambda_2, \lambda_{12}, \alpha_1, \alpha_2$). The two marginal distributions are Weibull distributed with the following:

$$\bar{F}_{U_i}(u_i) = \exp\left[-(\lambda_i + \lambda_{12}) u_i^{\alpha_i}\right], \quad i = 1, 2. \quad (2.3)$$

(2.2) can also be independent in case that λ_{12} is null. The BVW distribution reduced to the BVE distribution when $\alpha_1 = \alpha_2 = 1$. If $\alpha_1 = \alpha_2 = \alpha$, then (2.2) distribution reduced a BVW distribution of Hanagal (1996) because of $\max(u_1^{\alpha_1}, u_2^{\alpha_2}) = \{\max(u_1, u_2)\}^\alpha$. There are many types of BVW distribution (Balakrishnan and Lai, 2009).

The usefulness of a BVW distribution can be visualized in many contexts, such as the times to first and second failures of a repairable device, the breakdown times of dual generators in a power plant, or

the survival times of the organs in a two-organ system, (such as lungs or kidneys,) in the human body. Moeschberger (1974) derived a competing risk model based on the BVW distribution of Marshal and Olkin. Mino *et al.* (2003) discussed the applications of a MOBVW in lifespan. Rachev *et al.* (1995) considered a MOBVW as a bivariate limiting the distribution of the tumor latency time.

3. MTTF for MOBVW

Let (U_1, U_2) be Marshall-Olkin BVW distribution with parameters $\lambda_1, \lambda_2, \lambda_{12} > 0, \alpha_1 > 0, \alpha_2 > 0$ if its survival function is of the form (2.2). The case $\lambda_{12} = 0$ leads to independence.

Let $T = \max(U_1, U_2)$. As we know, T denotes the system lifetime of a parallel system. The value of $E[T]$ for MOBVW distribution with Weibull components can be obtained as follow. The distribution function is

$$\begin{aligned}\bar{F}_T(t) &= 1 - F_T(t, t) \\ &= \exp\{-(\lambda_1 + \lambda_{12})t^{\alpha_1}\} + \exp\{-(\lambda_2 + \lambda_{12})t^{\alpha_2}\} - \exp\{-\lambda_1 t^{\alpha_1} - \lambda_2 t^{\alpha_2} - \lambda_{12} \max(t^{\alpha_1}, t^{\alpha_2})\}. \quad (3.1)\end{aligned}$$

In case $\alpha_1 \neq \alpha_2$, the value of $\max(t^{\alpha_1}, t^{\alpha_2})$ is equal to $t^{(\max(\alpha_1, \alpha_2))}$ if $t > 1$ or equal to $t^{(\min(\alpha_1, \alpha_2))}$ if $t < 1$. If $\alpha_1 > \alpha_2$,

$$\begin{aligned}\bar{F}_T(t) &= \\ \begin{cases} \bar{F}_1(t) = \exp\{-(\lambda_1 + \lambda_{12})t^{\alpha_1}\} + \exp\{-(\lambda_2 + \lambda_{12})t^{\alpha_2}\} - \exp\{-(\lambda_1 + \lambda_{12})t^{\alpha_1} - \lambda_2 t^{\alpha_2}\}, & t > 1, \\ \bar{F}_2(t) = \exp\{-(\lambda_1 + \lambda_{12})t^{\alpha_1}\} + \exp\{-(\lambda_2 + \lambda_{12})t^{\alpha_2}\} - \exp\{-\lambda_1 t^{\alpha_1} - (\lambda_2 + \lambda_{12})t^{\alpha_2}\}, & t < 1. \end{cases} \quad (3.2)\end{aligned}$$

Also when $\alpha_1 < \alpha_2$,

$$\begin{aligned}\bar{F}_T(t) &= \\ \begin{cases} \bar{F}_3(t) = \exp\{-(\lambda_1 + \lambda_{12})t^{\alpha_1}\} + \exp\{-(\lambda_2 + \lambda_{12})t^{\alpha_2}\} - \exp\{-\lambda_1 t^{\alpha_1} - (\lambda_2 + \lambda_{12})t^{\alpha_2}\}, & t > 1, \\ \bar{F}_4(t) = \exp\{-(\lambda_1 + \lambda_{12})t^{\alpha_1}\} + \exp\{-(\lambda_2 + \lambda_{12})t^{\alpha_2}\} - \exp\{-(\lambda_1 + \lambda_{12})t^{\alpha_1} - \lambda_2 t^{\alpha_2}\}, & t < 1. \end{cases} \quad (3.3)\end{aligned}$$

Then,

$$E(T) = \begin{cases} \int_1^\infty \bar{F}_1(t)dt + \int_0^1 \bar{F}_2(t)dt, & \alpha_1 > \alpha_2, \\ \int_1^\infty \bar{F}_3(t)dt + \int_0^1 \bar{F}_4(t)dt, & \alpha_1 < \alpha_2. \end{cases} \quad (3.4)$$

But, the integral part of (3.4) has not closed form. In case $\alpha_1 = \alpha_2 = \alpha$, situation that frequently we can observe in applicative fields, an simple distribution is obtained :

$$\bar{F}_T(t) = \exp\{-(\lambda_1 + \lambda_{12})t^\alpha\} + \exp\{-(\lambda_2 + \lambda_{12})t^\alpha\} - \exp\{-(\lambda_1 + \lambda_2 + \lambda_{12})t^\alpha\}. \quad (3.5)$$

Therefore, the mean time to failure(MTTF) for system with dependent parallel components is

$$\begin{aligned}E[T] &= \int_0^\infty \bar{F}_T(t)dt \\ &= \alpha^{-1} \left[(\lambda_1 + \lambda_{12})^{-\frac{2}{\alpha}} + (\lambda_2 + \lambda_{12})^{-\frac{2}{\alpha}} + (\lambda_1 + \lambda_2 + \lambda_{12})^{-\frac{2}{\alpha}} \right] \Gamma\left(\frac{2}{\alpha}\right). \quad (3.6)\end{aligned}$$

To see the relationships between the components dependency and MTTF, we consider the correlation coefficient between U_1 and U_2 in (2.2).

$$\rho(U_1, U_2) = \frac{\text{Cov}(U_1, U_2)}{\sigma_{U_1} \sigma_{U_2}}. \quad (3.7)$$

$\text{Cov}(U_1, U_2)$ is obtained by Hoeffding's formula as follows:

$$\text{Cov}(U_1, U_2) = \iint [F(u_1, u_2) - F(u_1)F(u_2)] du_1 du_2. \quad (3.8)$$

After the tedious calculations, we obtain as below:

$$\text{Cov}(U_1, U_2) = \int_0^\infty \left[\frac{\exp\{-(\lambda_2 + \lambda_{12})s\}}{\alpha s^{1-\frac{1}{\alpha}}} (A) + \frac{\exp\{-(\lambda_2 + \lambda_{12}) - \exp(-\lambda_2 s)\}}{\alpha s^{1-\frac{1}{\alpha}}} (B) \right] ds, \quad (3.9)$$

where $A = \int_0^s \{\lambda_1 \exp(-\lambda_1 t) - (\lambda_1 + \lambda_{12}) \exp\{-(\lambda_1 + \lambda_{12})t\}\} t^{1/\alpha} dt$ and $B = \int_s^\infty (\lambda_1 + \lambda_{12}) \exp\{-(\lambda_1 + \lambda_{12})t\} t^{1/\alpha} dt$.

The result of (3.9) is calculated under the same condition $\alpha_1 = \alpha_2 = \alpha$. But we cannot calculate the integral parts of A and B .

However, we can the marginal variances of U_1 and U_2 . The results are:

$$\sigma_{U_1}^2 = \left[\Gamma\left(1 + \frac{2}{\alpha}\right) - \Gamma\left(1 + \frac{1}{\alpha}\right) \right]^2 (\lambda_1 + \lambda_{12})^{-\frac{2}{\alpha}}$$

and

$$\sigma_{U_2}^2 = \left[\Gamma\left(1 + \frac{2}{\alpha}\right) - \Gamma\left(1 + \frac{1}{\alpha}\right) \right]^2 (\lambda_2 + \lambda_{12})^{-\frac{2}{\alpha}}.$$

Because we cannot evaluate (3.9), calculation of (3.7) leaves afterward. As an alternative, we want to investigate the relationship the dependency parameter λ_{12} and $E[T]$ in the next section.

4. Numerical Results

The simulation considers the situation where the components of the system are similar ($\alpha_1 = \alpha_2 = \alpha$) as stated above. The value of α has a marked effect on the failure rate of the Weibull distribution whether the value of α is less than, equal to, or greater than one.

We investigate the mean times as dependency of components is increasing when the value of α is less than, equal to, or greater than one, respectively. As a matter of convenience, we assume that $\lambda_1 = \lambda_2 = 1$.

The figures in the Table show that $E[T]$ decreases as dependency parameter (λ_{12}) increases irrespective of any aging assumption of components (Figure 1).

5. Conclusion

In this paper, we have considered the MOBVW distribution and drive the MTTF for systems with dependent parallel components. Using the MTTF, we investigate how the degree of dependency will influence the increase in the mean lifetime of a system. We know that $E[T]$ decreases as dependency parameter (λ_{12}) increases irrespective of any aging assumption of components. In the future, we wish to study a related research for different BVW distributions.

Table 1: MTTF and dependence degree

λ_{12}	$\alpha = 2$	$\alpha = 1$	$\alpha = 1/2$
		MTTF	
0.0	1.250	2.250	24.750
0.1	1.147	1.880	17.009
0.2	1.061	1.596	12.086
0.3	0.987	1.372	8.832
0.4	0.923	1.194	6.609
0.5	0.867	1.049	5.048
0.6	0.817	0.929	3.925
0.7	0.773	0.829	3.099
0.8	0.734	0.745	2.481
0.9	0.699	0.673	2.011
1.0	0.667	0.611	1.648
2.0	0.458	0.258	0.343
3.0	0.350	0.165	0.113
4.0	0.283	0.108	0.048
5.0	0.238	0.076	0.024

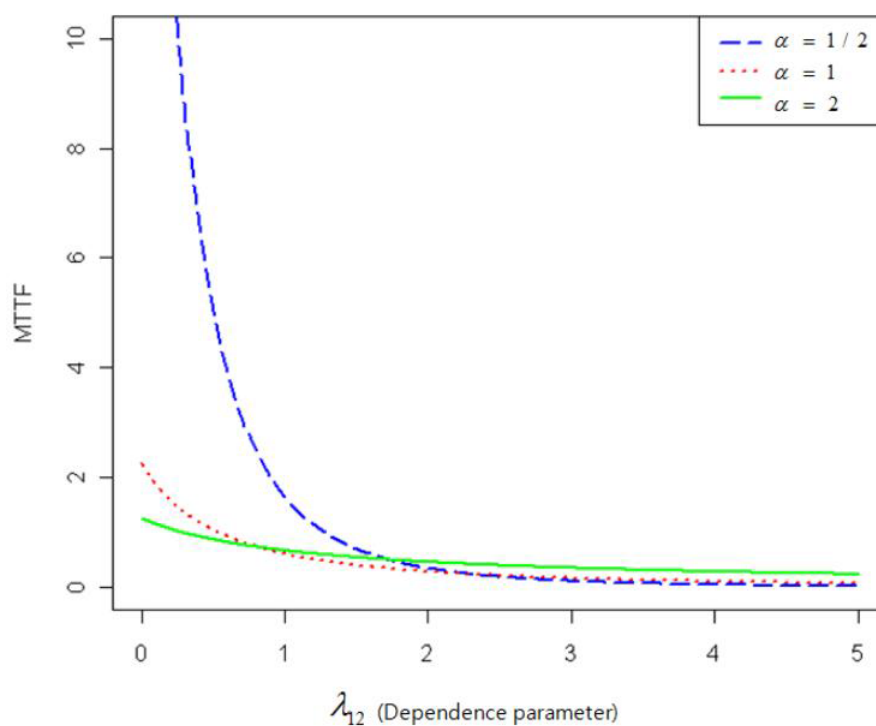


Figure 1: Plot of MTTF versus dependence degree

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