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T-STRUCTURE AND THE YAMABE INVARIANT

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Abstract. The Yamabe invariant is a topological invariant of a smooth closed manifold, which contains information about possible scalar curvature on it. It is well-known that a product manifold $T^m \times B$ where T^m is the m-dimensional torus, and B is a closed spin manifold with nonzero $\hat{A}$-genus has zero Yamabe invariant.

We generalize this to various T-structured manifolds, for example T^m-bundles over such B whose transition functions take values in $Sp(m, \mathbb{Z})$ (or $Sp(m-1, \mathbb{Z}) \oplus \{\pm 1\}$ for odd m).

1. Introduction to Yamabe invariant

The Yamabe invariant is an invariant of a smooth closed manifold depending on its smooth topology.

Let M be a smooth closed manifold of dimension n. Given a smooth Riemannian metric g on it, the conformal class $[g]$ is defined as

$$[g] = \{\varphi g \mid \varphi : M \to \mathbb{R}^+ \text{ is smooth}\}.$$

The famous Yamabe problem ([13]) states that there exists a metric $\tilde{g}$ in $[g]$ which attains the minimum

$$\inf_{\tilde{g} \in [g]} \frac{\int_M s_{\tilde{g}} \, dV_{\tilde{g}}}{(\int_M dV_{\tilde{g}})^{\frac{n-2}{n}}},$$

where $s_{\tilde{g}}$ and $dV_{\tilde{g}}$ respectively denote the scalar curvature and the volume element of $\tilde{g}$.

It turns out that when $n \geq 3$, a unit-volume minimizer $\tilde{g}$ in $[g]$ has constant scalar curvature, which is equal to the above minimum value called the Yamabe constant of $[g]$ and denoted by $Y(M, [g])$.

It is known that the Yamabe constant of any n-manifold is bounded above by $Y(S^n, [g_0])$ where $[g_0]$ denotes a standard round metric. Thus following a

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min-max procedure we define the Yamabe invariant

$$Y(M) := \sup_{[g]} Y(M,[g])$$

of M.

The following facts are noteworthy.

- $Y(M) > 0$ if and only if M admits a metric of positive scalar curvature.
- If M is simply-connected and $\dim M \geq 5$, then $Y(M) \geq 0$. With the further assumption that M is spin, $Y(M) > 0$ if and only if the α-genus of M is 0.
- For $r \in [\frac{n}{2}, \infty]$,

  $$|Y(M,[g])| = \inf_{\tilde{g}\in[g]} (\int_M |s_{\tilde{g}}|^r d\mu_{\tilde{g}})^{\frac{1}{r}} (\mathrm{Vol}_{\tilde{g}})^{\frac{2}{n}-\frac{1}{r}},$$

  where the infimum is attained only by the Yamabe minimizers.
- When $Y(M,[g]) \leq 0$,

  $$Y(M,[g]) = -\inf_{\tilde{g}\in[g]} (\int_M |s^-_{\tilde{g}}|^r d\mu_{\tilde{g}})^{\frac{1}{r}} (\mathrm{Vol}_{\tilde{g}})^{\frac{2}{n}-\frac{1}{r}},$$

  where s^-_g is defined as $\min\{s_g, 0\}$.

  Therefore when $Y(M) \leq 0$,

  $$\begin{aligned} Y(M) &= -\inf_g (\int_M |s_g|^r d\mu_g)^{\frac{1}{r}} (\mathrm{Vol}_g)^{\frac{2}{n}-\frac{1}{r}} \\ &= -\inf_g (\int_M |s^-_g|^r d\mu_g)^{\frac{1}{r}} (\mathrm{Vol}_g)^{\frac{2}{n}-\frac{1}{r}}, \end{aligned}$$

  so that $Y(M)$ measures how much negative scalar curvature is inevitable on M.
- As an application of the above formula, if M has an F-structure which will be explained in a later section, M admits a sequence of metrics with volume form converging to zero while the sectional curvature are bounded below, so that $Y(M) \geq 0$ (See [14]).

2. Computation of Yamabe invariant

We now discuss how to compute the Yamabe invariant. When M is a closed oriented surface, by the Gauss-Bonnet theorem

$$Y(M) = 4\pi\chi(M),$$

where χ denotes the Euler characteristic.

When M is a closed oriented 3-manifold, the Ricci flow gave many answers by the proof of geometrization theorem due to G. Perelman (See [1]). For example, $Y(M) > 0$ if and only if M is a connected sum of $S^1 \times S^2$'s and finite quotients of S^3, and $Y(\mathbb{H}^3/\Gamma)$ is realized by the hyperbolic metric.

When $\dim M = 4$, the Seiberg-Witten theory enables us to compute the Yamabe invariant of Kähler surfaces through the Weitzenböck formula. LeBrun

[10, 11, 12] has shown that if M is a compact Kähler surface whose Kodaira dimension is not equal to $-\infty$, then

$$Y(M) = -4\sqrt{2}\pi\sqrt{2\chi(\tilde{M}) + 3\tau(\tilde{M})},$$

where τ denotes the signature and $\tilde{M}$ is the minimal model of M, and for $\mathbb{C}P^2$,

$$Y(\mathbb{C}P^2) = 4\sqrt{2}\pi\sqrt{2\chi(\mathbb{C}P^2) + 3\tau(\mathbb{C}P^2)}.$$

In higher dimensions, few examples have been computed so far, such as

$$Y(S^1 \times S^{n-1}) = Y(S^n) = n(n-1)(\mathrm{vol}(S^n(1)))^{\frac{2}{n}},$$

where $S^n(1)$ is the unit sphere in $\mathbb{R}^{n+1}$, and

$$Y(T^n) = Y(T^n \times H) = Y(T^n \times B) = 0,$$

where H is a closed Hadarmard-Cartan manifold, i.e., one with a metric of non-positive sectional curvature, and B is a closed spin manifold with nonzero $\hat{A}$-genus. These T^n-bundles have such property, because they admit a T-structure and never admit a metric of positive scalar curvature by Gromov-Lawson enlargeability method [5, 9].

We call a closed n-manifold M *enlargeable* if the following holds: for any $\epsilon > 0$ and any Riemannian metric g on M, there exists a Riemannian spin covering manifold $\tilde{M}$ of (M, g) and an ϵ-contracting map $f : \tilde{M} \to S^n(1)$, which is constant outside a compact subset of $\tilde{M}$ and of nonzero $\hat{A}$-degree defined as $\hat{A}(f^{-1}(\text{any regular value of } f))$.

Here a smooth map F is called ϵ-contracting if the norm of DF is less than ϵ. By using the Weitzenböck formula for an appropriate twisted Dirac operator, they showed that such manifolds never admit a metric of positive scalar curvature.

They also generalized this to so-called weakly-enlargeable manifolds, where "ϵ-contracting" is replaced by "ϵ-contracting on 2-forms" meaning that the induced map of DF on tangent bi-vectors, i.e., a section of $\Lambda^2(TM)$ has norm less than ϵ.

$\hat{A}$-genus of a closed spin manifold M is the integral over M of

$$\hat{A}(TM) := 1 - \frac{p_1}{24} + \frac{-4p_2 + 7p_1^2}{5760} + \cdots,$$

where $p_i \in H^{4i}(M, \mathbb{Z})$ is the i-th Pontryagin class of TM. An important fact is that a closed spin manifold with a metric of positive scalar curvature has zero $\hat{A}$-genus.

Then a natural question for us to explore is:

Question 2.1. Let M be a T^m-bundle over a closed spin manifold B with nonzero $\hat{A}$-genus. Is $Y(M)$ equal to zero?

3. *T*-structure

An *F-structure* which was introduced by Cheeger and Gromov [3, 4] generalizes an effective T^m-action for $m \in \mathbb{N}$.

Definition 3.1. An F-structure on a smooth manifold is given by data $(U_i, \hat{U}_i, T^{k_i})$ with the following conditions:

(1) $\{U_i\}$ is a locally finite open cover.
(2) Each $\pi_i : \hat{U}_i \mapsto U_i$ is a finite Galois covering with covering group Γ_i.
(3) Each torus T^{k_i} of dimension k_i acts effectively on $\hat{U}_i$ in a Γ_i-equivariant way, i.e., Γ_i also acts on T^{k_i} as an automorphism so that
$$\gamma(gx) = \gamma(g)\gamma(x)$$
for any $\gamma \in \Gamma_i$, $g \in T^{k_i}$, and $x \in \hat{U}_i$.
(4) If $U_i \cap U_j \neq \emptyset$, then there is a common covering of $\pi_i^{-1}(U_i \cap U_j)$ and $\pi_j^{-1}(U_i \cap U_j)$ such that it is invariant under the lifted actions of T^{k_i} and T^{k_j}, and they commute.

As a special case, a T-structure is an F-structure in which all the coverings π_i's are trivial.

Typical examples of T-structure are torus bundles.

Theorem 3.2. *Any T^m-bundle over a smooth manifold whose transition functions are $T^m \rtimes GL(m, \mathbb{Z})$-valued has a T-structure. In particular, any S^1 or T^2-bundle has a T-structure.*

Proof. Here T^m acts by translation, and hence the transition functions are affine maps at each fiber direction. Obviously the local T^m actions along the fiber are commutative on the intersections to give a global T-structure.

The second statement follows from the well-known fact that the diffeomorphism group of T^m for $m = 1, 2$ is homotopically equivalent to $T^m \rtimes GL(m, \mathbb{Z})$. Thus we may assume that the transition functions are $T^m \rtimes GL(m, \mathbb{Z})$-valued. □

Other typical examples are manifolds with a nontrivial smooth S^1 action. Such examples we will use are projective spaces such as $\mathbb{R}P^n, \mathbb{C}P^n, \mathbb{H}P^n$, and CaP^2 (For the case of the Cayley plane which actually has an S^3-action, see [2]). Or one can construct more examples by gluing T-structured manifolds. For example graph manifolds are obtained by gluing Seifert-fibred 3-manifolds along the toral boundaries, and also:

Theorem 3.3 (Paternain and Petean [14])**.** *Suppose X and Y are n-manifolds with $n > 2$, which admit a T-structure. Then $X \# Y$ also admits a T-structure.*

4. Main results

Motivated by Gromov-Lawson enlargeability technique, we prove:

Theorem 4.1. *Let B be a closed spin manifold of dimension $4d$ with nonzero $\hat{A}$-genus, and M be a T^m-bundle over B whose transition functions take values in $Sp(m, \mathbb{Z})$ (or $Sp(m-1, \mathbb{Z}) \oplus \{\pm 1\}$ for odd m). Then*

$$Y(M) = 0.$$

Proof. By Theorem 3.2, M has a T-structure so that $Y(M) \geq 0$.

We only have to show that M never admits a metric of positive scalar curvature. To the contrary, suppose that it admits such a metric h, and we will derive a contradiction. The basic idea is to apply the Bochner-type method to a twisted Spin^c bundle on M whose topological index is nonzero.

First, we consider the case when m is even, say $2k$. Let Λ denote a lattice in $\mathbb{R}^{2k}$ so that $T^{2k} = \mathbb{R}^{2k}/\Lambda$. Take an integer $n \gg 1$. There is an obvious covering map from $\mathbb{R}^{2k}/n\Lambda$ onto $\mathbb{R}^{2k}/\Lambda$ of degree n^{2k}, and we claim that this covering map can be extended to all the fibers in M to give a covering $p : M_n \to M$. The following lemma justifies this:

Lemma 4.2. *The same transition functions as $\mathbb{R}^{2k}/\Lambda$-bundle M give $\mathbb{R}^{2k}/n\Lambda$-bundle M_n with the covering projection p.*

Proof. For a transition map $g_{\alpha\beta} \in Sp(2k, \mathbb{Z})$ downstairs, the same transition map $g_{\alpha\beta}$ upstairs is the unique lifting map which satisfies $p \circ g_{\alpha\beta} = g_{\alpha\beta} \circ p$ and sends 0 to 0.

It only needs to be proved that the transition maps satisfy the axioms for the bundle, in particular the axiom $g_{\beta\gamma} \circ g_{\alpha\beta} = g_{\alpha\gamma}$. This follows from the uniqueness of the lifting map sending 0 to 0 (In fact, this cocycle condition holds without modulo $\mathbb{Z}$). □

We endow M_n with a metric $h_n := p^*h$.

Lemma 4.3. *There exists a closed 2-form ω on M_n such that $\omega^{k+1} = 0$ and it restricts to a generator of $H^2(T^{2k}, \mathbb{Z})$ at each fiber T^{2k}.*

Proof. For each $U \times T^{2k}$ where U is an open ball in B, take ω to be a standard symplectic form of T^{2k} representing a generator of $H^2(T^{2k}, \mathbb{Z})$. Since ω is invariant under $Sp(2k, \mathbb{Z})$, it is globally defined on M_n (Note that the transition functions are locally constant). Obviously $\omega^{k+1} = 0$ at each point. □

Let E be the complex line bundle on M_n whose first Chern class is $[\omega]$. Take a connection A^E of E whose curvature 2-form $R^E = dA^E$ is equal to $-2\pi i\omega$.

We claim that

$$|R^E|_{h_n} \to 0 \qquad \text{as} \;\; n \to \infty.$$

Lemma 4.4. $|R^E|_{h_n} = O(\frac{1}{n^2})$.

Proof. Take a local coordinate $(x_1, \ldots, x_{4d}) \times (y_1, \ldots, y_{2k})$ of $B \times T^{2k}$ so that

$$\omega = dy_1 \wedge dy_2 + \cdots + dy_{2k-1} \wedge dy_{2k}.$$

We will show that $|dy_\mu|_{h_n} = O(\frac{1}{n})$ for all μ. First,

$$h_n(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) = h(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}),$$

$$h_n(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_\mu}) = nh(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_\mu}),$$

$$h_n(\frac{\partial}{\partial y_\mu}, \frac{\partial}{\partial y_\nu}) = n^2 h(\frac{\partial}{\partial y_\mu}, \frac{\partial}{\partial y_\nu})$$

for all i, j, μ, and ν. Thus

$$h_n = \begin{pmatrix} O(1) & O(n) \\ O(n) & O(n^2) \end{pmatrix},$$

where the block division is according to the division by x and y coordinates, and

$$\begin{aligned} (h_n)^{-1} &= \frac{1}{\det(h_n)} \mathrm{adj}(h_n) \\ &= \frac{1}{O(n^{4k})} \begin{pmatrix} O(n^{4k}) & O(n^{4k-1}) \\ O(n^{4k-1}) & O(n^{4k-2}) \end{pmatrix} \\ &= \begin{pmatrix} O(1) & O(\frac{1}{n}) \\ O(\frac{1}{n}) & O(\frac{1}{n^2}) \end{pmatrix}, \end{aligned}$$

which means $|dx_i|_{h_n} = O(1)$ and $|dy_\mu|_{h_n} = O(\frac{1}{n})$ for all i and μ, completing the proof. ☐

In order to use the Bochner argument, we need to show that M_n is spin^c. Using the orthogonal decomposition by h_n,

$$TM_n = V \oplus H = V \oplus \pi^*(TB),$$

where V and H respectively denote the vertical and horizontal space, and $\pi : M_n \to B$ be the torus bundle projection. Obviously H is spin, because B is spin. Since V is a symplectic $\mathbb{R}^{2k}$-vector bundle, it admits a compatible almost-complex structure, and hence it can be viewed as a $\mathbb{C}^k$-vector bundle. Thus

$$w_2(M_n) = w_2(V) + w_2(H) \equiv c_1(V) \mod 2$$

meaning that M_n is spin^c. Let S be the associated vector bundle to the Spin^c bundle over M_n obtained using $\sqrt{\det_{\mathbb{C}} V}$.

Consider a twisted spin^c Dirac operator D^E on $S \otimes E$ where E is equipped with a connection A^E. The Weitzenböck formula says that

$$(D^E)^2 = \nabla^* \nabla + \frac{1}{4} s_{h_n} + \mathfrak{R}^E.$$

Here $\mathfrak{R}^E(\sigma \otimes v) = \sum_{i<j} (e_i e_j \sigma) \otimes (R^E_{e_i, e_j} v)$ where $\{e_i\}$ is an orthonormal frame for (M_n, h_n). Note that

$$|\mathfrak{R}^E|_{h_n} \leq C |R^E|_{h_n},$$

where C is a positive constant depending on the dimension of M. By taking n sufficiently large, we can ensure that $s_{h_n} > |\mathfrak{R}^E|_{h_n}$ everywhere, and hence $\ker D^E = 0$. Thus the index of the operator

$$D_+^E : \Gamma(S_+ \otimes E) \to \Gamma(S_- \otimes E)$$

is

$$\dim \ker D^E|_{S_+\otimes E} - \dim \ker D^E|_{S_-\otimes E} = 0,$$

where $S_\pm$ respectively denotes the plus and negative spinor bundle.

On the other hand, we can also compute the index using the Atiyah-Singer index theorem [9]. Note that V has locally constant transition functions, and hence can be given a flat connection. This implies that $\hat{A}(V) = 1$ so that

$$\begin{aligned}
\mathrm{index}(D_+^E) &= \{ch(E) \cdot \hat{A}(TM_n)\}[M_n] \\
&= \{(1 + [\omega] + \cdots + \frac{1}{k!}[\omega]^k) \cdot \hat{A}(V) \cdot \hat{A}(\pi^*(TB))\}[M_n] \\
&= \{(1 + [\omega] + \cdots + \frac{1}{k!}[\omega]^k) \cdot \pi^*(\hat{A}(B))\}[M_n] \\
&= \{\frac{1}{k!}[\omega]^k \cdot \pi^*(\hat{A}_d(B))\}[M_n] \\
&= \int_{\pi^{-1}(PD[\hat{A}_d(B)])} \frac{1}{k!}[\omega]^k \\
&= (\hat{A}(B)[B]) \int_{\pi^{-1}(pt)} \frac{1}{k!}[\omega]^k \\
&\neq 0,
\end{aligned}$$

which yields a contradiction.

The odd m case is reduced to the even case. If m is odd, consider an S^1-bundle over M with transition functions exactly equal to the transition functions $\{\pm 1\}$ of the last S^1-factor of T^m in M over B. Then M' is a T^{m+1}-bundle over B with transition functions taking values in $Sp(m+1, \mathbb{Z})$. We put a locally product metric on M'. Then it also has positive scalar curvature, yielding a contradiction from the above even case. □

Theorem 4.5. *Let B be a closed spin manifold of dimension $4d$ with nonzero $\hat{A}$-genus, and M be an S^1 or T^2-bundle over B whose transition functions take values in $GL(1, \mathbb{Z})$ or $GL(2, \mathbb{Z})$ respectively. Then*

$$Y(M) = 0.$$

Proof. Again by Theorem 3.2, M has a T-structure so that $Y(M) \geq 0$. It remains to show M never admits a metric of positive scalar curvature, and let's assume it does.

First, the case of S^1 bundle can be reduced to the case of T^2 bundle by considering a Riemannian product $M \times S^1$ which also has positive scalar curvature. From now on, we consider the case of T^2 bundle.

Secondly, we may also assume that M is orientable, i.e., the transition functions for the torus bundle are orientation-preserving. Otherwise, we consider $\bar{M}$ from the lemma below, which also admits a metric of positive scalar curvature by lifting the metric of M.

Lemma 4.6. *There exists a finite covering $\bar{M}$ of M such that $\bar{M}$ is an orientable T^2-bundle over a closed spin manifold of nonzero $\hat{A}$-genus.*

Proof. Let $\hat{B}$ be the universal cover of B, and $\hat{M}$ be the manifold obtained by lifting the torus bundle over B to $\hat{B}$. Since $\hat{B}$ is simply-connected, $\hat{M}$ is orientable, and $\pi_1(B)$ acts on $\hat{M}$ to give $M = \hat{M}/\pi_1(B)$.

Let G be a subset of $\pi_1(B)$, which consists of elements preserving orientation of the fiber torus. Then G is a subgroup of index 2. Thus $\hat{M}/G$ is an orientable T^2-bundle over $\hat{B}/G$ which is a double cover of B so that it is also spin with nonzero $\hat{A}$-genus. □

Now if M is orientable, its transition functions take values in $SL(2,\mathbb{Z}) = Sp(2,\mathbb{Z})$ so that the previous theorem can be applied to derive a contradiction. □

Remark 4.7. In fact, Theorem 4.1 holds for any T^m-bundle with $T^m \rtimes GL(m,\mathbb{Z})$-valued transition functions, which has a finite covering diffeomorphic to M as in Theorem 4.1.

Combining our results with the previous results in [16], we can compute more general T-structured manifolds:

Corollary 4.8. *Let M be a T^m-bundle in all the above so that $Y(M) = 0$. If* $\dim M = 4n$*, then*

$$Y(M \sharp\ k\ \mathbb{H}P^n \sharp\ l\ \overline{\mathbb{H}P^n}) = 0,$$

and if $\dim M = 16$*, then*

$$Y(M \sharp\ k\ \mathbb{H}P^4 \sharp\ l\ \overline{\mathbb{H}P^4} \sharp\ k'\ CaP^2 \sharp\ l'\ \overline{CaP^2}) = 0,$$

where k, l, k', and l' are nonnegative integers, and the overline denotes the reversed orientation.

In low dimensions such as 2 and 3, we understand all T-structured manifolds with zero Yamabe invariant. In dimension 4, we can compute the Yamabe invariant of some torus bundles by using the Seiberg-Witten theory.

Theorem 4.9. *Let B be a closed oriented manifold of dimension ≤ 3, and X be an S^1 or T^2-bundle over B. Suppose that $X \times T^m$ for $m = 4 - \dim X$ has a finite cover M with $b_2^+(M) > 1$ which is a T^2-bundle over an oriented surface whose transition functions take values in a discrete subset of $T^2 \rtimes SL(2,\mathbb{Z})$. Then*

$$Y(X) = 0.$$

Proof. It suffices to show that M never admits a metric of positive scalar curvature.

Using the 2-form ω on M which restricts to a standard symplectic form on each fiber, we have a symplectic form $\pi^*\sigma + \omega$ on M, where σ is a symplectic form of $\tilde{B}$, and $\pi : M \to \tilde{B}$ is the T^2-bundle projection.

Then the Seiberg-Witten invariant of the canonical Spin^c structure of M is ± 1 so that it never admits a metric of positive scalar curvature. □

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