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GENERALIZED SCHWARZ LEMMAS FOR MEROMORPHIC FUNCTIONS

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Abstract. In this paper, we prove an analog of the generalized Schwarz lemma for meromorphic functions. Our results improve the classical generalized Schwarz lemma.

1. Introduction

The classical Schwarz lemma is one of the simplest results in all of the complex function theory. But there is hardly any result that has been quite so influential. Thanks in part to Lars Ahlfors' geometrization of the proof (he showed that the Schwarz lemma can be interpreted in terms of curvature), the Schwarz lemma has assumed a central and powerful role in complex geometry. A very simple and widely used general form of this lemma is as follows:

Theorem A (Generalized Schwarz Lemma)**.** *Let k be a positive integer and $f(z)$ be an analytic function defined on the unit disk $\mathbb{D}$. If $|f(z)| \leq 1$ for $z \in \mathbb{D}$ and $f(0) = f'(0) = \cdots = f^{(k-1)}(0) = 0$, then*

(a) $\frac{1}{k!}|f^{(k)}(0)| \leq 1$;
(b) $|f(z)| \leq |z|^k \quad for \quad z \in \mathbb{D}$.

Moreover, the equality $|f^{(k)}(0)| = k!$ or the equality $|f(z)| = |z|^k$ at a single point $z \neq 0$ holds if and only if $f(z) = cz^k$ with $|c| = 1$.

Remark 1. The proof of the generalized Schwarz lemma please refer to [2].

Obviously, there is a natural problem for meromorphic functions: Can we get an analog of the generalized Schwarz lemma for the meromorphic functions? In this paper, we give an affirmative answer to the above question giving a proof using winding numbers and homotopy, as in [4] where the case $k = 1$ was done. Furthermore, our method gives a new proof of the original generalized Schwarz lemma.

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Notational conventions. Through the paper, $\mathbb{D}$ will denote the unit disk $\{z : |z| < 1\}$. For a meromorphic function $f(z)$, we denote by $n(r, f)$ the number of poles of f in $|z| < r$, counted according to multiplicity. By this notation, the number of zeros of f in $|z| < r$ is thus given by $n(r, 1/f)$. We denote by $\mathrm{Wind}(\gamma, a)$ the winding number of the cycle γ around the point a.

Now, we state our results as follows.

Theorem 1. *Let k be a positive integer. Suppose that $f(z)$ is meromorphic on $\mathbb{D}$ and satisfies the following conditions*:

(1) $f(0) = f'(0) = \cdots = f^{(k-1)}(0) = 0$;
(2) *there exists a real number* $0 < r_0 < 1$ *such that* $|f(z)| \le 1$ *for* $z \in \{z : r_0 < |z| < 1\}$;
(3) *the function* $g(z) = z^k - (k!f(z))/f^{(k)}(0)$ *satisfies* $n(1, 1/g) - n(1, g) \neq k$, *provided that* $f^{(k)}(0) \neq 0$.

Then

$$\frac{1}{k!}|f^{(k)}(0)| \le 1.$$

Moreover, the equality $|f^{(k)}(0)| = k!$ *holds if and only if* $f(z) = cz^k$ *with* $|c| = 1$.

Remark 2. Note that 0 is a zero of $g(z)$ with multiplicity at least $k+1$ when the function $f(z)$ is holomorphic and $f^{(k)}(0) \neq 0$ in Theorem 1. Thus the assumption (3) of Theorem 1 automatically holds if the function $f(z)$ is holomorphic. Therefore, the conclusion (a) of Theorem A is a corollary of Theorem 1.

Theorem 2. *Let k be a positive integer. Let $f(z)$ be a meromorphic function on $\mathbb{D}$ and ω be a point, not a pole of f, in $\mathbb{D}$. Suppose that $f(z)$ satisfies the following conditions*:

(1) $f(0) = f'(0) = \cdots = f^{(k-1)}(0) = 0$;
(2) *there exists a real number* $0 < r_0 < 1$ *such that* $|f(z)| \le 1$ *for* $z \in \{z : r_0 < |z| < 1\}$;
(3) *the function* $g_\omega(z) = z^k - (\omega^k f(z))/f(\omega)$ *satisfies* $n(1, 1/g_\omega) - n(1, g_\omega) \neq k$, *provided that* $f(\omega) \neq 0$.

Then

$$|f(\omega)| \le |\omega|^k.$$

Moreover, the equality $|f(\omega)| = |\omega|^k$ *holds at a non-zero point* ω *if and only if* $f(z) = cz^k$ *with* $|c| = 1$.

Remark 3. We note that 0 is a zero of $g_\omega(z)$ with multiplicity at least $k + 1$ when the function $f(z)$ is holomorphic and $f(\omega) \neq 0$ in Theorem 2. Thus the assumption (3) of Theorem 2 automatically holds if the function $f(z)$ is holomorphic. Therefore, the conclusion (b) of Theorem A is a corollary of Theorem 2.

Example 1. Define the meromorphic function

$$f(z) = \frac{6z^k}{(8z - 1)} \quad \text{on} \quad \mathbb{D}.$$

It is clear that $f(0) = f'(0) = \cdots = f^{(k-1)}(0) = 0$ and $|f(z)| \leq 1$ for $z \in \{z : 7/8 < |z| < 1\}$. By a simple computation, we get

$$g(z) = z^k - \frac{k!f(z)}{f^{(k)}(0)} = \frac{8z^{k+1}}{8z-1}.$$

Then $n(1, 1/g) - n(1, g) = k$. But $\frac{1}{k!}|f^{(k)}(0)| = 6 > 1$. So, this example illustrates that the assumption (3) in Theorem 1 is necessary.

Example 2. Define the meromorphic function

$$f(z) = \frac{z^k}{20(z-1/2)^2} \quad \text{on} \quad \mathbb{D}.$$

It is obvious that $f(0) = f'(0) = \cdots = f^{(k-1)}(0) = 0$ and $|f(z)| \leq 1$ for $z \in \{z : 3/4 < |z| < 1\}$. At the same time, it is not difficult to find out

$$g(z) = z^k - \frac{k!f(z)}{f^{(k)}(0)} = \frac{z^{k+1}(z-1)}{(z-1/2)^2}.$$

Noting $n(1, 1/g) = k+1$, then $n(1, 1/g) - n(1, g) = k - 1 \neq k$. Clearly, we have $\frac{1}{k!}|f^{(k)}(0)| = 1/5 < 1$. Therefore, this example shows that there exist the meromorphic functions satisfying all assumptions of Theorem 1.

Example 3. Define the meromorphic function

$$f(z) = \frac{z^k}{16(z-1/2)^2} \quad \text{on} \quad \mathbb{D}.$$

It is clear that $f(0) = f'(0) = \cdots = f^{(k-1)}(0) = 0$ and $|f(z)| \leq 1$ for $z \in \{z : 3/4 < |z| < 1\}$. For the point $\omega = 3/8$, we obtain

$$g_\omega(z) = \frac{z^k(z-5/8)(z-3/8)}{(z-1/2)^2}.$$

Then $n(1, 1/g_\omega) - n(1, g_\omega) = k$. But $\left|f(\frac{3}{8})\right| > \left|\frac{3}{8}\right|^k$. This fact shows that the assumption (3) of Theorem 2 is necessary. Furthermore, for the point $\omega = -\frac{1}{4}$, we have

$$g_\omega(z) = \frac{z^k(z-5/4)(z+1/4)}{(z-1/2)^2}.$$

Then $n(1, 1/g_\omega) - n(1, g_\omega) = k - 1 \neq k$. Simultaneously, it is easy to see

$$\left|f(-\frac{1}{4})\right| \leq \left|-\frac{1}{4}\right|^k.$$

Hence, this fact shows that there exist the meromorphic functions satisfying all assumptions of Theorem 2.

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2. Proof of the theorems

Proof of Theorem 1. Let $\lambda = \frac{1}{k!}f^{(k)}(0)$.

Step 1. We first show that $|\lambda| \le 1$. If not, then $|\lambda| > 1$. Thus we define the function

$$g_\lambda(z) = z^k - \frac{f(z)}{\lambda}.$$

If $g_\lambda(z) \equiv 0$ in $\mathbb{D}$, then $f(z) \equiv \lambda z^k$. Since $|\lambda| > 1$, we know that

$$|f(z)| > 1 \quad \text{for} \quad (1/|\lambda|)^{\frac{1}{k}} < |z| < 1,$$

which contradicts the assumption (2) of Theorem 1. Hence, $g_\lambda(z) \not\equiv 0$ in $\mathbb{D}$.

Step 1.1. We prove that there exists a positive real number r satisfying:

(1) $g_\lambda(z)$ has no zeros and poles on $|z| = r$;
(2) $\max\{(1/|\lambda|)^{\frac{1}{k}}, r_0\} < r < 1$;
(3) $n(r, 1/g_\lambda) - n(r, g_\lambda) \neq k$.

It is clear that the set of poles of $g_\lambda(z)$ is the same as the set of poles of $f(z)$ in $\mathbb{D}$. Therefore, by assumption (2) of Theorem 1, we know that the total number of poles of $g_\lambda(z)$ in $\mathbb{D}$ is finite and

$$n(\rho, g_\lambda) = n(r_0, g_\lambda) \quad \text{for} \quad r_0 < \rho < 1. \tag{2.1}$$

If the total number of the zeros of $g_\lambda(z)$ in $\mathbb{D}$ is finite, then we can choose a real number r such that all zeros and poles of $g_\lambda(z)$ in $\mathbb{D}$ are contained in $|z| < r$ and $\max\{(1/|\lambda|)^{\frac{1}{k}}, r_0\} < r < 1$. It is clear that $g_\lambda(z)$ has no zeros and poles on $|z| = r$. Since $n(r, 1/g_\lambda) = n(1, 1/g_\lambda)$ and $n(r, g_\lambda) = n(1, g_\lambda)$, combining the assumption (3) of Theorem 1, it follows that $n(r, 1/g_\lambda) - n(r, g_\lambda) \neq k$.

If the total number of the zeros of $g_\lambda(z)$ in $\mathbb{D}$ is infinite, then, combing (3.1), we can choose a real number r such that $n(r, 1/g_\lambda) > n(r, g_\lambda) + k$, $\max\{(1/|\lambda|)^{\frac{1}{k}}, r_0\} < r < 1$ and $g_\lambda(z)$ has no zeros and poles on $|z| = r$.

Hence, the conclusion of Step 1.1 is proved.

Step 1.2. We show that $n(r, 1/g_\lambda) - n(r, g_\lambda) = k$.

Let γ and Γ be two Jordan curves in $\mathbb{D}$ with parametric equations being

$$\gamma(t) = re^{2\pi it}, t \in [0, 1];$$
$$\Gamma(t) = r^k e^{i2\pi kt}, t \in [0, 1].$$

It is clear that

$$\text{Wind}(\Gamma, 0) = k. \tag{2.2}$$

From the definition of winding number, it is easy to see that

$$\text{Wind}(g_\lambda \circ \gamma, 0) = \frac{1}{2\pi i}\int_\gamma \frac{g'_\lambda(z)}{g_\lambda(z)}dz.$$

Since $g_\lambda(z)$ has no zeros and poles on γ,

$$\begin{aligned}\mathrm{Wind}(g_\lambda\circ\gamma,0) &= \frac{1}{2\pi i}\int_\gamma \frac{g'_\lambda(z)}{g_\lambda(z)}dz\\ &= \sum_j \mathrm{Wind}(\gamma,a_j)-\sum_l \mathrm{Wind}(\gamma,b_l)\\ &= n(r,1/g_\lambda)-n(r,g_\lambda),\end{aligned}\tag{2.3}$$

where a_j and b_l are the zeroes and poles of $g_\lambda(z)$, repeated according to multiplicity, *cf.*, [1, p. 152, Th. 18].

In the following, we will show that $\mathrm{Wind}(g_\lambda\circ\gamma,0)=\mathrm{Wind}(\Gamma,0)$.

Consider the homotopy

$$\begin{aligned}h(t,s) &= \gamma^k(t)-s\frac{f(\gamma(t))}{\lambda}\\ &= \Gamma(t)-s\frac{f(\gamma(t))}{\lambda}\quad\text{for}\quad s\in[0,1],\ t\in[0,1],\end{aligned}$$

from the curve Γ to the curve $g_\lambda\circ\gamma$. Note that $r^k>1/|\lambda|$, $|f(\gamma(t))|\le 1$ and $|\Gamma(t)|=r^k$ for $t\in[0,1]$. Then

$$\begin{aligned}|h(t,s)| &\ge r^k-\frac{|f(\gamma(t))|}{|\lambda|}\\ &\ge r^k-\frac{1}{|\lambda|}>0.\end{aligned}$$

So the two closed curves Γ and $g_\lambda\circ\gamma$ are homotopic in $\mathbb{C}-\{0\}$. Then, we deduce that

$$\mathrm{Wind}(g_\lambda\circ\gamma,0)=\mathrm{Wind}(\Gamma,0)=k.$$

Thus, according to (2.3), we get

$$n(r,1/g_\lambda)-n(r,g_\lambda)=k.$$

This fact contradicts $n(r,1/g_\lambda)-n(r,g_\lambda)\ne k$. Therefore, this contradiction shows that $|\lambda|\le 1$, that is, $\frac{1}{k!}|f^{(k)}(0)|\le 1$.

Step 2. We prove that $f(z)\equiv\lambda z^k$ if $|\lambda|=1$. Recall that

$$g_\lambda(z)=z^k-\frac{f(z)}{\lambda},$$

where $|\lambda|=1$.

Suppose that $g_\lambda(z)\not\equiv 0$ in $\mathbb{D}$.

With entirely similar arguments as Step 1.1, we can find a positive real number $\rho>r_0$ satisfying:

$$n(\sigma,1/g_\lambda)-n(\sigma,g_\lambda)\ne k\quad\text{for}\quad 1>\sigma>\rho.\tag{2.4}$$

Since $g_\lambda(z)\not\equiv 0$ in $\mathbb{D}$, it follows that there exists a positive number r such that $\rho<r<1$ and g_λ has no zeros and poles on $|z|=r$. So, we have

$$\eta=\inf_{|z|=r}|g_c(z)|>0.$$

Choose a complex number λ' such that $1/r > |\lambda'| > 1$ and $|\lambda' - \lambda| < \eta$. Then

$$|g_{\lambda'}(z) - g_\lambda(z)| = |f(z)| \frac{|\lambda - \lambda'|}{|\lambda| \cdot |\lambda'|} \leq |\lambda - \lambda'| < \eta \leq |g_\lambda(z)| \quad \text{on } |z| = r.$$

By the general form of Rouche's theorem, we obtain that

$$n(r, 1/g_{c'}) - n(r, g_{c'}) = n(r, 1/g_c) - n(r, g_c).$$

Combining (2.4), we have $n(r, 1/g_{\lambda'}) - n(r, g_{\lambda'}) \neq k$.

Now, with entirely similar arguments as Step 1.2, we can get that

$$n(r, 1/g_{\lambda'}) - n(r, g_{\lambda'}) = k.$$

This equation contradicts $n(r, 1/g_{\lambda'}) - n(r, g_{\lambda'}) \neq k$. Therefore, this contradiction shows that $g_\lambda(z) \equiv 0$ in $\mathbb{D}$, that is, $f(z) \equiv \lambda z^k$ where $|\lambda| = 1$. Hence, Theorem 1 is proved. □

Proof of Theorem 2. If $\omega = 0$, the inequality $|f(\omega)| \leq |\omega|^k$ automatically holds as $f(0) = 0$. So, without loss the generality, we may assume that $\omega \neq 0$.

Let

$$\lambda = \frac{f(\omega)}{\omega^k} \quad \text{and} \quad g_\lambda(z) = z^k - \frac{f(z)}{\lambda}.$$

Now, with entirely similar arguments as the proof of Theorem 1, we can prove Theorem 2. Here we omit the details, which are left to the readers. □

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