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ON CONJUGACY OF p-GONAL AUTOMORPHISMS

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Abstract. In 1995 it was proved by González-Diez that the cyclic group generated by a p-gonal automorphism of a closed Riemann surface of genus at least two is unique up to conjugation in the full group of conformal automorphisms. Later, in 2008, Gromadzki provided a different and shorter proof of the same fact using the Castelnuovo-Severi theorem. In this paper we provide another proof which is shorter and is just a simple use of Sylow's theorem together with the Castelnuovo-Severi theorem. This method permits to obtain that the cyclic group generated by a conformal automorphism of order p of a handlebody with a Kleinian structure and quotient the three-ball is unique up to conjugation in the full group of conformal automorphisms.

1. p-gonal automorphisms of Riemann surfaces

In [3] it was proved by González-Diez that the cyclic group generated by a p-gonal automorphism of a closed Riemann surface of genus at least two is unique up to conjugation in the full group of conformal automorphisms. Later, Gromadzki [4] provided a different and shorter proof of the same fact using the Castelnuovo-Severi theorem. In this paper we provide another proof which is shorter and which is just a simple use of Sylow's theorem together with the Castelnuovo-Severi theorem. If S is a Riemann surface, then we denote by $\mathrm{Aut}(S)$ the group of conformal automorphisms of S.

Theorem 1 (González-Diez [3], Gromadzki [4]). *Let S be a given closed Riemann surface of genus $g \geq 2$ and let $\phi \in \mathrm{Aut}(S)$ be a conformal automorphism of order a prime p so that $S/\langle\phi\rangle$ is the Riemann sphere $\widehat{\mathbb{C}}$. Then the cyclic group $\langle\phi\rangle$ is unique up to conjugation in $\mathrm{Aut}(S)$.*

Proof. If $p = 2$, then ϕ is the hyperelliptic involution, which is known to be unique (see, for instance, [2]). So, we may assume $p \geq 3$. Let us consider a branched regular cover $\pi : S \to \widehat{\mathbb{C}}$ with deck group being $\langle\phi\rangle$. If we denote by k the number of fixed points of ϕ, then the Riemann-Hurwitz formula asserts

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that $g = (p-1)(k-2)/2$. The Castelnuovo-Severi theorem [1] asserts that if $g > (p-1)^2$, then $\langle \phi \rangle$ is unique in $\mathrm{Aut}(S)$. So, we may also assume that $g \leq (p-1)^2$; in particular $k \leq 2p$. If $\langle \phi \rangle$ is a p-Sylow subgroup of $\mathrm{Aut}(S)$, then Sylow's theorem asserts that $\langle \phi \rangle$ is unique up to conjugation in $\mathrm{Aut}(S)$. We assume from now on that this is not the case. In particular, by Sylow's theorem, there is a p-subgroup K of order p^2 (so an abelian group) containing $\langle \phi \rangle$ as a normal subgroup. The group K induces a conformal automorphism ψ of $\widehat{\mathbb{C}}$ (so a Möbius transformation) of order p that permutes the k branch values of π. As ψ has exactly two fixed points, then $k = a + lp$, where $a \in \{0, 1, 2\}$ and $l \in \{0, 1, \ldots\}$. As $0 < k \leq 2p$, there are exactly two cases to consider: (i) $a = 0$ and $l \in \{1, 2\}$ or (ii) $a \neq 0$ and $l \in \{0, 1\}$.

If $a \neq 0$, then it follows that ψ fixes some of the branch values of π. This asserts that there is a fixed point of ϕ which is also a fixed point of every element in K. In this way, $K = \langle \eta \rangle \cong \mathbb{Z}_{p^2}$ and $\phi = \eta^p$. If K is a p-Sylow subgroup, then $\langle \phi \rangle$ is unique up to conjugation. If K is not a p-Sylow subgroup, then there is a p-subgroup L, of order p^3, so that K is a normal subgroup of L, in particular, $\langle \phi \rangle$ is also a normal subgroup of L. This asserts that L induces a group H of order p^2 (so abelian) of conformal automorphisms of $\widehat{\mathbb{C}}$ (so a finite abelian group of Möbius transformations) keeping invariant the branch values of π. As the only abelian groups of Möbius transformations are either isomorphic to $\mathbb{Z}_2^2$ or to $\mathbb{Z}_n$ and $p \geq 3$, it follows that $H = \langle \rho \rangle \cong \mathbb{Z}_{p^2}$. As H contains ψ, we may also assume that $\rho^p = \psi$. It follows that the two fixed points of ρ are the same as for ψ. As the branch values of π are permuted by the rotation ρ of order p^2, at least one of the fixed points of η is also a branch value of π, and $k \leq p^2$, we must have $k = a$. As $a \in \{1, 2\}$, this contradicts the assumption that $g \geq 2$.

Let us now assume $a = 0$. In this case, we have that $k \in \{p, 2p\}$. If $k = p$, then $g = (p-1)(p-2)/2$ and we may assume that $\psi(z) = e^{2\pi i/p} z$ and that the k branch points are given by the p-roots of unity. In this case, S is given by the Fermat curve $y^p + x^p + z^p = 0$ and $\phi(x, y, z) = (x, e^{2\pi i/p} y, z)$. It is well known that for this surface $\mathrm{Aut}(S) \cong \mathbb{Z}_p^2 \rtimes \mathfrak{S}_3$ ($\mathfrak{S}_3$ is the symmetric group of order 6), where $\mathbb{Z}_p^2$ is generated by ϕ and $\tau(x, y, z) = (e^{2\pi i/p} x, y, z)$ and $\mathfrak{S}_3$ is generated by $\rho(x, y, z) = (y, x, z)$ and $\eta(x, y, z) = (y, z, x)$. The cyclic groups or order p generated by an automorphism with fixed points are either $\langle \phi \rangle$, $\langle \tau \rangle$ or $\langle \phi\tau \rangle$. These three subgroups are conjugated by $\mathfrak{S}_3$.

If $k = 2p$, then $g = (p-1)^2$ and again we may assume that $\psi(z) = e^{2\pi i/p} z$ and that the k branch points are given by the p-roots of unity and the p-roots of some $a^p \neq 0, 1$. In this case, S is given by the algebraic curve $y^p = (x^p - 1)(x^p - a^p)^r$, where $r \in \{1, \ldots, p-1\}$, and $\phi(x, y) = (x, e^{2\pi i/p} y)$. The group $H \cong \mathbb{Z}_p^2 \rtimes L$, where $\mathbb{Z}_p^2$ is generated by ϕ and $\tau(x, y) = (e^{2\pi i/p} x, y)$, and that L is either isomorphic to either the trivial group or $\mathbb{Z}_2^2$ or D_4 depending on the value of r, is a subgroup of $\mathrm{Aut}(S)$ (see also [8]). The quotient orbifold S/H has either signature $(2, 2, 2, p)$ or $(2, 4, 2p)$. By Singerman's list of not finitely maximal Fuchsian groups [7], we may see that the above two signatures are maximal ones. In particular, it asserts that $\mathrm{Aut}(S) = H$. In this case, the cyclic groups of order p generated by automorphisms with fixed points are either $\langle \phi \rangle$ or $\langle \tau \rangle$. If $r < p-1$, then it follows (from the Riemann-Hurwitz formula) that $S/\langle \tau \rangle$ has genus greater than zero. If $r = p-1$, then $\langle \phi \rangle$ and $\langle \tau \rangle$ are conjugated by the automorphism $\alpha(x,y) = \big((x^p - a^p)/y, x^{p-1}(1-a^p)/(x^p-1)\big)$ [3]. □

2. p-gonal automorphisms of handlebodies

A Kleinian group is a discrete group of Möbius transformations and its region of discontinuity is the set of points on $\widehat{\mathbb{C}}$ on which the group acts discontinuously. By the Poincaré extension property, every Kleinian group acts on the hyperbolic 3-space $\mathbb{H}^3$. If Γ is a Kleinian group, with region of discontinuity Ω, then $M = (\mathbb{H}^3 \cup \Omega)/\Gamma$ is called a Kleinian 3-orbifold uniformized by Γ. The conformal boundary $S = \Omega/\Gamma$ is a Riemann orbifold and the interior $M^0 = \mathbb{H}^3/\Gamma$ is an hyperbolic 3-orbifold. If Γ is torsion free, then M is a Kleinian 3-manifold, S is a Riemann surface and M^0 is an hyperbolic 3-manifold.

If Γ is torsion free, then a conformal automorphism of M is a self-homeomorphism $\phi : M \to M$ so that its restriction $\phi : M^0 \to M^0$ is a hyperbolic isometry (the restriction $\phi : S \to S$ is a conformal automorphism). We denote by $\mathrm{Aut}(M)$ the group of conformal automorphisms of M. If $H < \mathrm{Aut}(M)$, then, by lifting to the universal cover space $\mathbb{H}^3 \cup \Omega$, one obtains a Kleinian group G containing Γ as a normal subgroup so that $H = G/\Gamma$.

If Γ has torsion, then one can also define conformal automorphisms of M, but in this case one has to ensure the self-homeomorphism ϕ to preserve the cone set of M (the projection of points with non-trivial Γ-stabilizers) in order to ensure that it can be lifted to obtain a Kleinian group G containing Γ as a normal subgroup so that $G/\Gamma = \langle \phi \rangle$ (this is similar to the case of Riemann orbifolds).

Let us assume M is a handlebody of genus $g \geq 2$. It is well known that in this case Γ is a Schottky group of rank g. As a handlebody is a retraction body, one can notice that if $\phi_1, \phi_2 \in \mathrm{Aut}(M)$ are so that $\phi_1|_S = \phi_2|_S$, then $\phi_1 = \phi_2$. In this way, there is a natural one-to-one homomorphism $\theta : \mathrm{Aut}(M) \to \mathrm{Aut}(S)$ given by restriction. In general, this homomorphism is not surjective. Necessary and sufficient conditions for an element of $\mathrm{Aut}(S)$ to be in the image of θ can be found in [5].

Let $\phi_1, \phi_2 \in \mathrm{Aut}(M)$ be a conformal automorphisms of order a prime p so that $M/\langle \phi_j \rangle$ is the 3-ball. If we consider the restrictions to S, we known from González-Diez's result the existence of some $\tau \in \mathrm{Aut}(S)$ so that $\tau \langle \phi_1 \rangle \tau^{-1} = \langle \phi_2 \rangle$. As θ is not surjective in general, it may be that τ is not the restriction of an element of $\mathrm{Aut}(M)$. Nevertheless, using the same ideas of the proof above, we notice that González-Diez's result still valid at the level of handlebodies.

Theorem 2. *Let M be a given handlebody with a Kleinian structure of genus $g \geq 2$ and let $\phi \in \mathrm{Aut}(M)$ be a conformal automorphism of order a prime p so* *that $M/\langle\phi\rangle$ is the* 3*-ball. Then the cyclic group $\langle\phi\rangle$ is unique up to conjugation in* $\mathrm{Aut}(M)$*.*

Proof. Let $M = (\mathbb{H}^3 \cup \Omega)/\Gamma$ be a handlebody of genus $g \geq 2$, uniformized by the Schottky group Γ, and let $\phi \in \mathrm{Aut}(M)$ of order p prime so that $B = M/\langle\phi\rangle$ is the 3-ball. By lifting ϕ to the universal cover space, we obtain a Kleinian group G containing Γ as a normal subgroup of index p and $G/\Gamma = \langle\phi\rangle$. Clearly, $B = (\mathbb{H}^3 \cup \Omega)/G$.

We may now follow the same proof as in the previous case by working at the level of the restriction of ϕ to $S = \Omega/\Gamma$. We may assume $p > 2$.

First, we notice that the injectivity of θ asserts that if $\langle\phi\rangle$ is unique in $\mathrm{Aut}(S)$, then $\langle\phi\rangle$ is unique in $\mathrm{Aut}(M)$. So, by Castelnuovo-Severi's theorem, we may assume $g \geq (p-1)^2$. Again, we may also assume that $\langle\phi\rangle$ is a not a p-Sylow subgroup of $\mathrm{Aut}(M)$. So, we have $K < \mathrm{Aut}(M)$ of order p^2 so that $\langle\phi\rangle$ is a normal subgroup of K. Again working at the level of S, one obtains that $k = a + lp \leq 2p$, where either (i) $a = 0$ and $l \in \{1,2\}$ or (ii) $a \neq 0$ and $l \in \{0,1\}$. In the case $a \neq 0$ we again obtain that $K \cong \mathbb{Z}_{p^2}$ must be a p-Sylow subgroup of $\mathrm{Aut}(M)$ and, in particular, that $\langle\phi\rangle$ is unique up to conjugation.

If $a = 0$, then we have that $k \in \{p, 2p\}$. But, the number k of fixed points of ϕ on S is necessarily even [5]. It follows that $k = 2p$. Moreover, the cone set of B is given by p pairwise disjoint simple arcs, which are cyclically permuted by the automorphism ψ (again induced by K). In this case, we have that S is given by the algebraic curve $y^p = (x^p - 1)(x^p - a^p)^r$, where $r \in \{1, \dots, p-1\}$, and $\phi(x,y) = (x, e^{2\pi i/p} y)$. We have, as already seen, that the only case to consider is $k = p-1$. But in this case, the quotient orbifold M/A, where $A = \langle\phi,\tau\rangle \cong \mathbb{Z}_p^2$, is the 3-ball with exactly two disjoint simple arcs. We only need to ensure that the automorphism $\alpha(x,y) = \big((x^p - a^p)/y, x^{p-1}(1-a^p)/(x^p-1)\big) \in \mathrm{Aut}(S)$ belongs to the image of θ. For it, let us assume the end points of the two cone arcs of M/A are ∞, 0 (for the first one) and 1 and $\lambda \in \mathbb{C} - \{0\}$ (for the second one), where we have identified the conformal boundary of M/A with $\widehat{\mathbb{C}}$. We notice that the Möbius transformation $\beta(z) = \lambda(z-1)/(z-\lambda)$ induces a conformal automorphism of M/A. This automorphism necessarily lifts to M to a conformal automorphism of order two that conjugates $\langle\phi\rangle$ to $\langle\tau\rangle$. More explicitly, the Kleinian group K_1 uniformizing the orbifold $(M/A)/\langle\beta\rangle$ is a free product (in the sense of the Klein-Maskit combination theorems [6]) of a cyclic group generated by an elliptic transformation γ of order p and a cyclic group generated by an elliptic transformation η of order two. The index two subgroup $K_2 = \langle\gamma, \eta\gamma\eta\rangle \cong \mathbb{Z}_p * \mathbb{Z}_p$ uniformizes M/A. The group Γ is the smallest normal subgroup of K_2 containing the commutator $[\gamma, \eta\gamma\eta]$. The transformation γ induces ϕ, and $\eta\gamma\eta$ induces τ. It can be seen by direct computation that Γ is also a normal subgroup of K_1. In this way, the transformation η induces a conformal involution on M conjugating $\langle\phi\rangle$ to $\langle\tau\rangle$. □

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