

INSERTION-OF-FACTORS-PROPERTY ON NILPOTENT ELEMENTS

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ABSTRACT. We generalize the insertion-of-factors-property by setting nilpotent products of elements. In the process we introduce the concept of a *nil-IFP* ring that is also a generalization of an NI ring. It is shown that if Köthe's conjecture holds, then every nil-IFP ring is NI. The class of minimal noncommutative nil-IFP rings is completely determined, up to isomorphism, where the minimal means having smallest cardinality.

1. Nil-IFP rings

Throughout this note every ring is an associative ring with identity unless otherwise stated. Given a ring R , let $N(R)$, $N_*(R)$ and $N^*(R)$ denote the set of all nilpotent elements, the prime radical and the upper nilradical (*i.e.*, the sum of all nil ideals) of R , respectively. The n by n full (resp. upper triangular) matrix ring over R is denoted by $\text{Mat}_n(R)$ (resp. $U_n(R)$), and e_{ij} 's denote the matrix units. \mathbb{Z} and \mathbb{Z}_n denote the ring of integers and the ring of integers modulo n . The polynomial ring with an indeterminate x over R is denoted by $R[x]$.

A ring R is called *reduced* if $N(R) = 0$. Any reduced ring R satisfies, with the help of [20, Proposition 1], that $r_{\sigma(1)}r_{\sigma(2)} \cdots r_{\sigma(n)} = 0$ for any permutation σ of the set $\{1, 2, \dots, n\}$ when $r_1r_2 \cdots r_n = 0$ for any positive integer n and $r_i \in R$. We will use this fact freely. Due to Bell [5], a ring R is called to satisfy the *insertion-of-factors-property*, or simply called an *IFP* ring if $ab = 0$ implies $aRb = 0$ for $a, b \in R$. Narbonne [25] and Shin [27] used the terms *semicommutative* and *SI* for the IFP, respectively. Commutative rings are clearly IFP, and reduced rings are IFP by a simple computation. There exist many non-reduced commutative rings (e.g., \mathbb{Z}_{n^k} for $n, k \geq 2$), and many noncommutative reduced rings (e.g., direct products of noncommutative domains). A ring is usually

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called *abelian* if each idempotent is central. IFP rings are abelian by a simple computation.

In the present note we apply the IFP onto the set of all nilpotent elements in place of one-sided ideals. Consider the condition:

$$(*) \quad ab \in N(R) \text{ implies } aRb \subseteq N(R),$$

where R is a ring (possibly without identity) and $a, b \in R$. We first examine a kind of ring that does not satisfy the condition $(*)$. Every n by n full matrix ring cannot satisfy the condition $(*)$ over any ring when $n \geq 2$.

Example 1.1. Let A be any ring and $M = \text{Mat}_n(A)$ for $n \geq 2$. Consider two matrices $a = e_{11} + e_{21}$ and $b = e_{22}$ in M . Then $ab = 0$ and aMb contains $c = (e_{11} + e_{21})(e_{12} + e_{22})e_{22}$. However $c^k = e_{12} + e_{22}$ for all $k \geq 1$, entailing that $aMb \not\subseteq N(M)$.

In the following we see relations between IFP rings and rings satisfying the condition $(*)$.

Note. (1) IFP rings satisfy the condition $(*)$. Let R be an IFP ring and $a, b \in R$. Set $ab \in N(R)$ and say $(ab)^k = 0$. Since R is IFP, $0 = abab \cdots ab = aRbaRb \cdots aRb = (aRb)^k$ and so $aRb \subseteq N(R)$.

(2) Let $R = U_2(D)$ for a domain D and $A = (a_{ij}), B = (b_{ij}) \in R$. Assume $AB \in N(R)$. Then $a_{11}b_{11} = 0$ and $a_{22}b_{22} = 0$, entailing that $a_{11}Db_{11} = 0$ and $a_{22}Db_{22} = 0$. This yields $ARB \subseteq \begin{pmatrix} 0 & D \\ 0 & 0 \end{pmatrix}$. Thus R satisfies the condition $(*)$, but not IFP since it is non-abelian.

Based on the preceding note, a ring R will be called *nil-IFP* if R satisfies the condition $(*)$.

Next we consider another kind of nil-IFP ring. Marks [23] called a ring R (possibly without identity) *NI* if $N^*(R) = N(R)$. This definition implies that nil rings are NI and that NI rings are clearly nil-IFP. Hong et al. [12, Corollary 13] proved that a ring R is NI if and only if every minimal strongly prime ideal of R is completely prime. By the definition we have that a ring R is NI if and only if $N(R)$ forms an ideal if and only if $R/N^*(R)$ is reduced. Let $U = U_n(R)$ over a ring R . Then $N(U) = \{m = (m_{ij}) \in U \mid m_{ii} \in N(R) \text{ for all } i\}$ and $N^*(U) = \{m = (m_{ij}) \in U \mid m_{ii} \in N^*(R) \text{ for all } i\}$. So $U/N^*(U) \cong \bigoplus_{j=1}^n R_j$, where $R_i = R/N^*(R)$ for all j . This implies that R is NI if and only if so is U [14, Proposition 4.1(1)]. IFP rings are NI by a simple computation but the converse need not hold through $U_n(D)$ with D a domain and $n \geq 2$.

First we introduce a lemma which will make our approach to the nature of nil-IFP rings much easier.

Lemma 1.2. *For a ring R , the following conditions are equivalent:*

- (1) R is nil-IFP;
- (2) If $ab \in N(R)$, then $rbst \in N(R)$ and $rasbt \in N(R)$ for all $r, s, t \in R$;
- (3) If $a \in N(R)$, then $ras \in N(R)$ for all $r, s \in R$;

- (4) If $a \in N(R)$, then $ra \in N(R)$ for all $r \in R$;
- (5) If $a \in N(R)$, then $ar \in N(R)$ for all $r \in R$.

Proof. (1) \Rightarrow (2) Let R be nil-IFP. Then we have the following series of assertions: $ab \in N(R) \Rightarrow ba \in N(R) \Rightarrow bsa \in N(R)$ for all $s \in R \Rightarrow sab \in N(R) \Rightarrow satrb \in N(R)$ for all $r, t \in R \Rightarrow rbsat \in N(R)$. The same computation gives $rasbt \in N(R)$, using $ba \in N(R)$ in place of $ab \in N(R)$.

- (2) \Rightarrow (3) It's obvious.
- (3) \Rightarrow (4) and (4) \Rightarrow (5) are clear.

(5) \Rightarrow (1) If $ab \in N(R)$, then $ba \in N(R)$ and so $baR \subseteq N(R)$. Hence $aRb \subseteq N(R)$. □

Recall that the *center* of a ring R is $Z(R) = \{c \in R \mid cr = rc \text{ for all } r \in R\}$.

Proposition 1.3. (1) *The class of nil-IFP rings is closed under subrings (possibly without identity).*

(2) *If R is a ring such that $N(R) \subseteq Z(R)$, then R is nil-IFP.*

(3) *Let R be a nil-IFP ring. If $r \in R$ is invertible and $a_1, a_2, \dots, a_n \in N(R)$ for $n \geq 1$, then $r + a_1 + a_2 + \dots + a_n$ is invertible.*

Proof. (1) Let S be a subring of a nil-IFP ring R . Then $N(S) = N(R) \cap S$. Let $ab \in N(S)$. Then $ab \in N(R)$, so $aRb \subseteq N(R)$. This yields $aSb \subseteq N(S)$, entailing that S is nil-IFP.

(2) Let $a \in N(R)$. If $N(R) \subseteq Z(R)$, then $ar \in N(R)$ for each $r \in R$, entailing that R is nil-IFP by Lemma 1.2.

(3) Let R be nil-IFP. Note that if $a \in N(R)$, then $1 + a$ is invertible. Let $r \in R$ be invertible and e_0 be the inverse of r . Then $e_0(r + a_1 + a_2 + \dots + a_n) = 1 + e_0a_1 + e_0a_2 + \dots + e_0a_n$. Since R is nil-IFP, $e_0a_1 \in N(R)$ by Lemma 1.2. Say that $e_1 \in R$ is the inverse of $1 + e_0a_1$. Then $e_1(1 + e_0a_1 + e_0a_2 + \dots + e_0a_n) = 1 + e_1e_0a_2 + e_1e_0a_3 + \dots + e_1e_0a_n$. Next we consider the nilpotent $e_1e_0a_2$ (by Lemma 1.2) and the invertible $1 + e_1e_0a_2$. Then inductively we can obtain

$$\begin{aligned} & e_n e_{n-1} \cdots e_1 e_0 (r + a_1 + a_2 + \dots + a_n) \\ &= e_n e_{n-1} \cdots e_1 (1 + e_0 a_1 + e_0 a_2 + \dots + e_0 a_n) \\ &= \cdots = e_n (1 + e_{n-1} \cdots e_1 e_0 a_n) = 1, \end{aligned}$$

where $e_i \in R$ is the inverse of $1 + e_{i-1} \cdots e_1 e_0 a_i$ for $1 \leq i \leq n$, noting that $e_{i-1} \cdots e_1 e_0 a_i \in N(R)$ for all i by Lemma 1.2. Whence $r + a_1 + a_2 + \dots + a_n$ is invertible. □

The class of nil-IFP rings is closed under subrings by Proposition 1.3(1), but it is not closed under factor rings as follows. For example, let R be the ring of quaternions with integer coefficients. Then R is a domain and so nil-IFP. However for any odd prime integer q , the ring R/qR is isomorphic to $\text{Mat}_2(\mathbb{Z}_q)$ by the argument in [10, Exercise 2A]. But $\text{Mat}_2(\mathbb{Z}_q)$ is not nil-IFP by Example 1.1.

Proposition 1.4. *Let R be a ring and I be a proper ideal of R . If both R/I and I (as a ring without identity) are nil-IFP, then so is R .*

Proof. Suppose that both R/I and I (as a ring without identity) are nil-IFP. Write $\bar{r} = r + I$ for $r \in R$. Let $a \in N(R)$. Then $\bar{a} \in N(R/I)$. Since R/I is nil-IFP, Lemma 1.2 implies that $\bar{r}\bar{a} \in N(R/I)$ for any $r \in R$. Hence $(ra)^k \in I$ for some positive integer k and so $(ra)^k r \in I$. Since $a \in N(R)$, $a^m = 0$ for some positive integer m . Then we have $(ra)^k r a^m (ra)^k = 0 \in N(I)$. Note that $(ra)^k r a^{m-1}, a(ra)^k \in I$. Since I is nil-IFP, we get

$$((ra)^k r a^{m-1})((ra)^k r)(a(ra)^k) \in N(I)$$

and

$$((ra)^k r a^{m-2})(a(ra)^k r)(a(ra)^k) = ((ra)^k r a^{m-2})(a(ra)^{2k+1}) \in N(I).$$

Note that $(ra)^k r a^{m-2}, a(ra)^{2k+1} \in I$. Since I is nil-IFP, we get

$$((ra)^k r a^{m-2})((ra)^k r)(a(ra)^{2k+1}) \in N(I)$$

and

$$((ra)^k r a^{m-3})(a(ra)^k r)(a(ra)^{2k+1}) = ((ra)^k r a^{m-3})(a(ra)^{3k+2}) \in N(I).$$

Repeating this process, we eventually obtain

$$((ra)^k r a)((ra)^k r)(a(ra)^{(m-1)k+(m-2)}) = (ra)^{(m+1)k+m} \in N(I).$$

Thus $ra \in N(I) \subseteq N(R)$ and so R is nil-IFP by Lemma 1.2. \square

Corollary 1.5. (1) *Let R be a ring and I be a nil ideal of R . Then R is nil-IFP if and only if so is R/I .*

(2) *Let e be a central idempotent of a ring R . Then R is nil-IFP if and only if eR and $(1-e)R$ are both nil-IFP.*

Proof. (1) Let R be nil-IFP and $a + I \in N(R/I)$. Then since I is nil, we have $a \in N(R)$. Next since R is nil-IFP, $ar \in N(R)$ for all $r \in R$. This yields $ar + I \in N(R/I)$ and so R/I is nil-IFP by Lemma 1.2. Conversely assume that R/I is nil-IFP. Since I is nil, I is nil-IFP clearly. Then R is nil-IFP by Proposition 1.4.

(2) The necessity comes from Proposition 1.3(1) since eR and $(1-e)R$ are subrings of R . For the converse, use $eR \cong R/(1-e)R$ and Proposition 1.4. \square

For the proof of the “if part” of Corollary 1.5(1), we can use directly the condition that I is nil, without using Proposition 1.4. Let $a \in N(R)$. Then since R/I is nil-IFP, $ar + I \in N(R/I)$ for all $r \in R$ by Lemma 1.2. But I is nil and so $ar \in N(R)$, concluding that R is nil-IFP by Lemma 1.2.

A ring R is called *directly finite* if $ab = 1$ implies $ba = 1$ for $a, b \in R$. Example 1.1 shows that there exists a directly finite ring but not nil-IFP. *NI* rings are directly finite by [14, Proposition 2.7].

Proposition 1.6. *Nil-IFP rings are directly finite.*

Proof. Let R be a nil-IFP ring. Suppose $ab = 1$ for $a, b \in R$. Then $aba = a$, so $a(ba - 1) = 0 \in N(R)$. Since R is a nil-IFP ring, $ab(ba - 1) = ba - 1 \in N(R)$ by Lemma 1.2. Say $(ba - 1)^n = 0$ for $n \geq 1$. Then

$$\begin{aligned} 0 &= (ba - 1)^n = (ba - 1)(ba - 1)^{n-1} \\ &= ba(ba - 1)^{n-1} - (ba - 1)^{n-1} = -(ba - 1)^{n-1} = \dots = (-1)^{n-1}(ba - 1), \end{aligned}$$

entailing $ba = 1$. Thus R is directly finite. □

We have another proof of Proposition 1.6 by applying [9, Proposition 5.5]. Let R be a nil-IFP ring and assume on the contrary that R is not directly finite. Then R contains an infinite set of matrix units

$$\{e_{11}, e_{12}, e_{13}, \dots, e_{21}, e_{22}, e_{23}, \dots\}$$

by [9, Proposition 5.5]. Take $a = e_{11} + e_{21}, b = e_{22} \in R$ as in Example 1.1, noting $ab = 0$. Since R is nil-IFP, $e_{12} + e_{22} = a(e_{12} + e_{22})b \notin N(R)$, a contradiction.

A ring R (possibly without identity) is called *2-primal* if $N_*(R) = N(R)$, according to Birkenmeier et al. [6]. It is obvious that R is 2-primal if and only if $R/N_*(R)$ is reduced. It is also easy to see that 2-primal rings are NI, but the converse need not hold by Marks [23, Example 2.2]. The *index of nilpotency* of a nilpotent element b in a ring R is the least positive integer n such that $b^n = 0$. The *index of nilpotency* of a subset S of R is the supremum of the indices of nilpotency of all nilpotent elements in S . If such a supremum is finite, then S is said to be *of bounded index of nilpotency*.

Proposition 1.7. *Suppose that a ring R is of bounded index of nilpotency. Then the following conditions are equivalent:*

- (1) R is nil-IFP;
- (2) R is 2-primal;
- (3) R is NI.

Proof. It suffices to show (1) \Rightarrow (2). We apply the proof of [14, Proposition 1.4]. Let R be nil-IFP and $a \in N(R)$. Then $aR \subseteq N(R)$ by Lemma 1.2. Since R is of bounded index of nilpotency, either $aR = 0$ or aR contains a nonzero nilpotent ideal of R by Levitzki [11, Lemma 1.1] or Klein [18, Lemma 5]. Thus $aR \subseteq P$ for every prime ideal P of R , entailing $a \in N_*(R)$. This implies that R is 2-primal. □

The following is an immediate consequence of Proposition 1.7.

Corollary 1.8. *Every finite nil-IFP ring is 2-primal (hence NI).*

If a ring R is nil-IFP but not NI, then $N(R)$ is not closed under addition by Lemma 1.2. So if such a ring exists, then it must contain an infinite subring as follows (otherwise, R is NI by Proposition 1.7). In the following we also see a necessary condition for Köthe’s conjecture (i.e., the upper nilradical contains all nil left ideals) to hold.

Proposition 1.9. (1) *Let R be a nil-IFP ring but not NI. Then there exists a countably infinite subring of R which is nil-IFP but not NI.*

(2) *NI and nil-IFP properties are equivalent if and only if NI and nil-IFP properties are equivalent for countably infinite rings.*

(3) *If Köthe's conjecture holds, then every nil-IFP ring is NI.*

Proof. (1) Since R is nil-IFP but not NI, there exist two nilpotent elements a and b of R such that $a + b$ is not nilpotent. Consider the subring S of R generated by a and b . Then $S = \mathbb{Z}[a, b]$ with $a^m = b^m = 0$ for some $m \geq 2$. Since $a + b$ is not nilpotent, S is countably infinite. S is nil-IFP by Lemma 1.3(1), but not NI by the fact $a + b \notin N(S)$.

(2) is an immediate consequence of (1).

(3) Suppose that Köthe's conjecture holds. Let R be a nil-IFP ring and $a, b \in N(R)$. It suffices to show $a + b \in N(R)$. $ra, sb \in N(R)$ for all $r, s \in R$ by Lemma 1.2. Let $I = \{ra \mid r \in R\}$ and $J = \{sb \mid s \in R\}$. Then clearly I and J are left nil ideals of R . Since Köthe's conjecture holds, $I + J \subseteq N(R)$, so $a + b \in N(R)$. Thus, $N(R)$ is an ideal of R , entailing that R is NI. \square

As a contraposition of Proposition 1.9(3), we can say that if the implication from nil-IFP to NI is proper, then Köthe's conjecture need not hold.

In the following we see some information about the set of nilpotent elements in nil-IFP rings.

Proposition 1.10. *Let R be a nil-IFP ring and $a, b \in N(R)$ with $a^n = b^m = 0$ for some positive integers m, n .*

(1) *Suppose that i_1, i_2, j_1, j_2 are nonnegative integers such that $i_1 + i_2 \geq n$ and $j_1 + j_2 \geq m$. Then $(a^{i_1} + b^{j_1})(a^{i_2} + b^{j_2}) \in N(R)$.*

(2) *If $a^2 = b^2 = 0$, then $a + b \in N(R)$.*

(3) *$a^\alpha + b^\beta \in N(R)$ for $\alpha, \beta \in \mathbb{Z}$ such that $2\alpha \geq n$ and $2\beta \geq m$.*

(4) *Let $r \in Z(R)$ such that $rab + ba = 0$. Then $a + b \in N(R)$.*

Proof. (1) First we have $(a^{i_1} + b^{j_1})(a^{i_2} + b^{j_2}) = a^{i_1}b^{j_2} + b^{j_1}a^{i_2}$ from the condition that $i_1 + i_2 \geq n$ and $j_1 + j_2 \geq m$. Moreover

$$\left((a^{i_1} + b^{j_1})(a^{i_2} + b^{j_2})\right)^k = (a^{i_1}b^{j_2} + b^{j_1}a^{i_2})^k = (a^{i_1}b^{j_2})^k + (b^{j_1}a^{i_2})^k$$

for every positive integer k . Since $a, b \in N(R)$, $(a^{i_1}b^{j_2})^k$ and $(b^{j_1}a^{i_2})^k$ are nilpotent by Lemma 1.2. Thus we also have $\left((a^{i_1} + b^{j_1})(a^{i_2} + b^{j_2})\right)^k \in N(R)$ by Lemma 1.2.

(2) If $a^2 = b^2 = 0$, then $(a + b)^2 = (a + b)(a + b) \in N(R)$ by (1), and hence $a + b \in N(R)$.

(3) Since $a^n = b^m = 0$, $2\alpha \geq n$, and $2\beta \geq m$, it follows that $(a^\alpha + b^\beta)^2 = (a^\alpha + b^\beta)(a^\alpha + b^\beta) \in N(R)$ by (1). Hence $a^\alpha + b^\beta \in N(R)$.

(4) Since $rab + ba = 0$, $ba = -rab$. Then each term of expansion of $(a + b)^{n+m-1}$ can be simplified as $(-r)^\sigma a^k b^{n+m-1-k}$ where k is an integer such that $0 \leq k \leq n + m - 1$ and σ is some positive integer. If $k \geq n$, then $a^k = 0$. If

$k < n$, then $b^{n+m-1-k} = 0$. So each term of the expansion of $(a + b)^{n+m-1}$ is equal to 0, entailing $(a + b)^{n+m-1} = 0$. \square

Consider $e_{12}, e_{21} \in \text{Mat}_2(A)$ over a ring A . Then $e_{12}^2 = e_{21}^2 = 0$ but $e_{12} + e_{21}$ is not nilpotent. So the condition that R is a nil-IFP ring is not superfluous in Proposition 1.10.

Nil-IFP rings need not be abelian as can be seen by $U_2(R)$, over a reduced ring R , which is nil-IFP by Proposition 2.3 to follow. The next example illuminates that abelian rings also need not be nil-IFP. Due to Armendariz [3, Lemma 1], Rege et al. [26] called a ring R *Armendariz* if $a_i b_j = 0$ for all i and j whenever $f(x) = \sum_{i=0}^m a_i x^i, g(x) = \sum_{j=0}^n b_j x^j \in R[x]$ satisfy $f(x)g(x) = 0$. Armendariz rings are abelian by the proof of [1, Theorem 6].

Example 1.11. Let $S = \mathbb{C}\{a, b\}$ be the free algebra with non-commuting indeterminates a, b over \mathbb{C} , where \mathbb{C} is the field of complex numbers. Let I be an ideal of S generated by a^2 . Set $R = S/I$. We coincide a, b with their images in R for simplicity. Notice that $N(R)$ is the subring of R generated by

$$\{\alpha a, \beta a r a \mid \alpha, \beta \in \mathbb{C}, r \in R\}.$$

Since $baa \in N(R)$ and $baba \notin N(R)$, R is not nil-IFP. But R is Armendariz (hence abelian) by [2, Example 4.8] or [7, Examples 9.3].

A ring R is called (*von Neumann*) *regular* if for each $a \in R$ there exists $x \in R$ such that $a = axa$.

Proposition 1.12. *Let R be a regular ring. Then the following conditions are equivalent:*

- (1) R is reduced;
- (2) R is NI;
- (3) R is nil-IFP;
- (4) R is 2-primal;
- (5) R is abelian;
- (6) R is IFP.

Proof. Every regular ring is semiprimitive by [9, Corollary 1.2(c)], entailing that regular NI rings are reduced. Abelian regular rings are reduced by [9, Theorem 3.2(c)]. Let R be a nil-IFP ring and let $a \in N(R)$. Since R is regular, there exists $b \in R$ such that $a = aba$. Notice that ba is an idempotent element of R . But Lemma 1.2 implies $ba \in N(R)$ since R is nil-IFP and $a \in N(R)$. Therefore, ba must be zero. Thus, $a = aba = a0 = 0$, entailing that R is reduced. \square

A ring R is called π -*regular* if for each $a \in R$ there exists a positive integer n , depending on $a, b \in R$ such that $a^n = a^n b a^n$. The Jacobson radicals of π -regular rings are nil, comparing with that regular rings are semiprimitive. Regular rings are clearly π -regular. However the preceding result need not

hold on π -regular rings. $U_n(D)$ ($n \geq 2$ and D is a division ring) is π -regular and nil-IFP, but it is neither regular nor reduced.

In the following arguments, we characterize the class of minimal noncommutative nil-IFP rings for the cases of with identity and without identity. The term minimal means having the smallest cardinality. $|S|$ means the cardinality of a set S .

Proposition 1.13. *Let R be a ring with identity. If R is a minimal noncommutative nil-IFP ring, then R is of order 8 and is isomorphic to $U_2(\mathbb{Z}_2)$.*

Proof. Let R be a minimal noncommutative nil-IFP ring with identity. Then $|R| \geq 2^3$ by [8, Theorem]. If $|R| = 2^3$, then R is isomorphic to $U_2(\mathbb{Z}_2)$ by [8, Proposition]. But $U_2(\mathbb{Z}_2)$ is a nil-IFP ring by Proposition 2.3 to follow. This implies that R is of order 8 and is isomorphic to $U_2(\mathbb{Z}_2)$. \square

Next we observe the structure of minimal noncommutative nil-IFP rings without identity.

Example 1.14. Consider the subrings

$$R_1 = \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & 0 \end{pmatrix}, R_2 = \begin{pmatrix} 0 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{pmatrix}, \text{ and } R_3 = \begin{pmatrix} \mathbb{Z}_2 & 0 & 0 \\ 0 & 0 & \mathbb{Z}_2 \\ 0 & 0 & 0 \end{pmatrix}$$

of $U_2(\mathbb{Z}_2)$ and $U_3(\mathbb{Z}_2)$. Then R_i is nil-IFP for $i = 1, 2, 3$ by Proposition 1.3(1) and Proposition 2.3 to follow.

Consider the subring

$$R_4 = \mathbb{Z}_2 e_{ii} + \mathbb{Z}_2 e_{jk} \text{ with } i \neq j, i \neq k, j \neq k$$

of $U_l(\mathbb{Z}_2)$ for $l \geq 3$. Then R_3 is isomorphic to R_4 with $e_{11} \mapsto e_{ii}$ and $e_{23} \mapsto e_{jk}$.

Given a ring R , R^+ means the additive abelian group $(R, +)$. The Jacobson radical of a ring R is denoted by $J(R)$.

Proposition 1.15. *Let R be a ring without identity. If R is a minimal noncommutative nil-IFP ring, then R is of order 4 and is isomorphic to R_i for some $i \in \{1, 2, 3\}$, where R_i 's are the rings in Example 1.14.*

Proof. Let R be a minimal noncommutative nil-IFP ring without identity. If $|R| \leq 3$, then R must be commutative, and so $|R| \geq 2^2$. Then $|R| = 2^2$ by considering the rings in Example 1.14. Assume that R is nil. Note that $J(R) = N^*(R) = R$ and R is nilpotent. If R^+ is cyclic, then R is commutative clearly. If R^+ be non-cyclic, then R is also commutative by [19, Theorem 2.3.3]. Thus R must be non-nil, entailing that $J(R) = 0$ or $|J(R)| = 2$.

If $J(R) = 0$, then R is commutative by applying the proof of [15, Theorem 3.3]. Consequently we get to the only case of $|J(R)| = 2$. We also apply the proof of [15, Theorem 3.3] in this case. $|J(R)| = 2$ implies $R/J(R) \cong \mathbb{Z}_2$. By [21, Proposition 3.6.2], there exists an idempotent $e \in R$ such that $1 + J(R) = e + J(R)$. Then there exists $0 \neq b \in J(R)$ such that $R = \{0, e, b, e + b\}$ and

$J(R) = \{0, b\}$, where $b^2 = 0$ (if $b^2 \neq 0$, then $b^2 = b$, a contradiction). Here we have the following three cases:

$$(eb = b, be = 0), (be = b, eb = 0), \text{ and } (eb = 0, be = 0),$$

noting that if $be = b = eb$, then R is commutative. Therefore R is isomorphic to one of the rings

$$R_1 = \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{pmatrix}, \text{ and } \begin{pmatrix} \mathbb{Z}_2 & 0 & 0 \\ 0 & 0 & \mathbb{Z}_2 \\ 0 & 0 & 0 \end{pmatrix}$$

in Example 1.14, through one of the maps

$$(e \mapsto e_{11}, b \mapsto e_{12}), (e \mapsto e_{22}, b \mapsto e_{12}), \text{ and } (e \mapsto e_{11}, b \mapsto e_{23}). \quad \square$$

Note that $U_2(\mathbb{Z}_2)$ and the rings in Example 1.14 are all NI rings. So we also obtain the following by Propositions 1.13 and 1.15.

Corollary 1.16. *Let R be a ring (possibly without identity). Then R is a minimal noncommutative nil-IFP ring if and only if R is a minimal NI ring if and only if R is a minimal 2-primal ring.*

2. Examples of nil-IFP rings

Let R be an algebra over a commutative ring S . Recall that the *Dorroh extension* of R by S is the ring $R \times S$ with operations $(r_1, s_1) + (r_2, s_2) = (r_1 + r_2, s_1 + s_2)$ and $(r_1, s_1)(r_2, s_2) = (r_1r_2 + s_1r_2 + s_2r_1, s_1s_2)$, where $r_i \in R$ and $s_i \in S$.

Proposition 2.1. (1) *Let $R = \sum_{i \in I} R_i$ be a direct sum of rings R_i . R is nil-IFP if and only if R_i is nil-IFP for each $i \in I$.*

(2) *Let R be an algebra over a commutative ring S . Then the Dorroh extension D of R by S is nil-IFP if and only if so is R .*

(3) *Let R be a ring without identity. Attach an identity to R by considering the ring $R \times \mathbb{Z}$ with the same operations as in Dorroh extension. Then $R \times \mathbb{Z}$ is nil-IFP if and only if so is R .*

(4) *The class of nil-IFP rings is closed under direct limits.*

Proof. (1) Let $a = (a_i) \in N(R)$ and $r = (r_i) \in R$. Then there exist positive integers m, n such that $a_k = 0$ for all $k > m$ and $a_i^n = 0$ for all $i \in I$. Since each R_i is nil-IFP, $a_i r_i \in N(R_i)$ for all $i = 1, \dots, m$ by Lemma 1.2. Then there exists some positive integer h such that $(a_i r_i)^h = 0$ for all $i \in I$. Hence, $ar \in N(R)$ and so R is nil-IFP by Lemma 1.2.

Conversely, let R be nil-IFP and $k \in I$. Suppose that $a_k \in N(R_k)$ and $r_k \in R_k$. Take $a = (a_i), r = (r_i) \in R$ such that $a_i = 0$ for all $i \in I \setminus \{k\}$. Then obviously $a \in N(R)$. By Lemma 1.2, $ar \in N(R)$ and this yields $a_k r_k \in N(R_k)$. Thus R_k is nil-IFP by Lemma 1.2.

(2) Let R be a nil-IFP ring. Note $R \cong R \times 0$, so the proper ideal $R \times 0$ of D is nil-IFP as a ring without identity. Since $D/(R \times 0) \cong S$ and S is nil-IFP, D

is also nil-IFP by Proposition 1.4. Conversely, if D is nil-IFP, then the subring R of D is nil-IFP by Proposition 1.3(1).

(3) The proof is essentially the same as in (2).

(4) Let $D = \{R_i, \alpha_{ij}\}$ be a direct system of nil-IFP rings R_i for $i \in I$ and ring homomorphisms $\alpha_{ij} : R_i \rightarrow R_j$ for each $i \leq j$ satisfying $\alpha_{ij}(1) = 1$, where I is a directed partially ordered set. Set $R = \varinjlim R_i$ the direct limit of D with $\iota_i : R_i \rightarrow R$ and $\iota_j \alpha_{ij} = \iota_i$. Let $a, b \in R$. Then $a = \iota_i(a_i)$, $b = \iota_j(b_j)$ for some $i, j \in I$ and there is $k \in I$ such that $i \leq k, j \leq k$. Define $a + b = \iota_k(\alpha_{ik}(a_i) + \alpha_{jk}(b_j))$ and $ab = \iota_k(\alpha_{ik}(a_i)\alpha_{jk}(b_j))$, where $\alpha_{ik}(a_i)$ and $\alpha_{jk}(b_j)$ are in R_k . Then R forms a ring with $0 = \iota_i(0)$ and $1 = \iota_i(1)$.

Next suppose that $a \in N(R)$ and consider ar for $r \in R$. Then $a = \iota_i(a_i)$ and $r = \iota_j(r_j)$ for some $i, j \in I$. Further, there exists $k \in I$ such that $\alpha_{ik}(a_i) \in N(R_k)$ and $\iota_k(\alpha_{ik}(a_i)) = \iota_i(a_i)$, $\iota_k(\alpha_{jk}(r_j)) = \iota_j(r_j)$. Since R_k is nil-IFP, $\alpha_{ik}(a_i)\alpha_{jk}(r_j) \in N(R_k)$ and so

$$ar = \iota_i(a_i)\iota_j(r_j) = \iota_k(\alpha_{ik}(a_i))\iota_k(\alpha_{jk}(r_j)) = \iota_k(\alpha_{ik}(a_i)\alpha_{jk}(r_j)) \in N(R).$$

So R is nil-IFP by Lemma 1.2. □

Observing Proposition 2.1(1), one may suspect that the direct product of nil-IFP rings is also nil-IFP. However, this is not the case as follows.

Example 2.2. The construction and computation are according to [13, Examples 1.6], [14, Example 2.5], and [24, Remark, p. 508]. Let K be a field and define $D_n = K\{x_n\}$, a free algebra generated by x_n , with a relation $x_n^{n+2} = 0$ for each nonnegative integer n . Then clearly $D_n \cong K[x]/(x^{n+2})$, where (x^{n+2}) is the ideal of $K[x]$ generated by x^{n+2} . Next let $R_n = \begin{pmatrix} D_n & x_n D_n \\ x_n D_n & D_n \end{pmatrix}$ be a subring of the 2 by 2 matrix ring over D_n . Then every R_n is 2-primal (hence nil-IFP) by the computation in [13, Examples 1.6]. Set $R = \prod_{n=0}^{\infty} R_n$ and consider $(x_n e_{12}) \in R$. Then $(x_n e_{12}) \in N(R)$ but $(x_n e_{12})(x_n e_{21}) = (x_n^2 e_{11}) \notin N(R)$. Hence, R is not nil-IFP by Lemma 1.2.

The n by n lower triangular matrix ring over a ring R is denoted by $L_n(R)$.

Proposition 2.3. For a ring R and an integer $n \geq 2$, the following conditions are equivalent:

- (1) R is nil-IFP;
- (2) $U_n(R)$ is nil-IFP;
- (3) $L_n(R)$ is nil-IFP;

$$(4) V_n(R) = \left\{ \left(\begin{array}{cccccc} a_1 & a_2 & a_3 & \cdots & a_{n-1} & a_n \\ 0 & a_1 & a_2 & \cdots & a_{n-2} & a_{n-1} \\ 0 & 0 & a_1 & \cdots & a_{n-3} & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_1 & a_2 \\ 0 & 0 & 0 & \cdots & 0 & a_1 \end{array} \right) \mid a_i \in R \right\} \text{ is a nil-IFP}$$

ring;

(5) $R[x]/\langle x^n \rangle$ is a nil-IFP ring, where $\langle x^n \rangle$ is the ideal of $R[x]$ generated by x^n .

Proof. Note that $I = \{b \in U_n(R) \mid \text{the diagonal entries of } b \text{ are all zero}\}$ is a nil ideal of $U_n(R)$. Moreover $U_n(R)/I$ is isomorphic to the direct sum of n copies of R . Then the equivalence between the conditions (1) and (2) is obtained by Corollary 1.5 (1) and Proposition 2.1. The proof of the equivalence between the conditions (1) and (3) is similar to the preceding case.

(2) \Rightarrow (4) and (4) \Rightarrow (1) are shown by Proposition 1.3(1).

The equivalence between the conditions (4) and (5) are obtained from the well-known fact that $V_n(R) \cong R[x]/\langle x^n \rangle$. \square

Proposition 2.4. *Let R and S be rings and ${}_R M_S$ an (R, S) -bimodule. Then $E = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$ is nil-IFP if and only if R and S are both nil-IFP.*

Proof. Note that $J = \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix}$ is a nil ideal of E and $E/J \cong R \oplus S$. So the proof is obtained by Corollary 1.5(1) and Proposition 2.1. \square

In fact, we do not know any example of a nil-IFP ring such that $R[x]$ is not nil-IFP.

Proposition 2.5. *Suppose that a ring R is of bounded index of nilpotency. Then the following conditions are equivalent:*

- (1) R is nil-IFP;
- (2) $R[x]$ is 2-primal;
- (3) $R[x]$ is NI;
- (4) $R[x]$ is nil-IFP.

Proof. It suffices to show (1) \Rightarrow (2) by Proposition 1.3(1). Let R be of bounded index of nilpotency. Then, by Proposition 1.7, R is 2-primal if and only if R is nil-IFP. But R is 2-primal if and only if $R[x]$ is 2-primal by [6, Proposition 2.6]. \square

Next we consider the nil-IFP condition of Ore extensions. For a ring R , a ring endomorphism $\sigma : R \rightarrow R$ and a σ -derivation $\delta : R \rightarrow R$, the Ore extension $R[x; \sigma, \delta]$ of R is the ring obtained by giving $R[x]$ the multiplication $xr = \sigma(r)x + \delta(r)$ for all $r \in R$. If $\delta = 0$, we write $R[x; \sigma]$ for $R[x; \sigma, 0]$ and is called an Ore extension of endomorphism type (also called a skew polynomial ring). While if $\sigma = 1$, we write $R[x; \delta]$ for $R[x; 1, \delta]$ and is called an Ore extension of derivation type (also called a differential polynomial ring).

Example 2.6. (1) There exists a nil-IFP ring over which the skew polynomial ring need not be nil-IFP. For a domain D let $R = D \oplus D$, then R is reduced (hence nil-IFP). Consider the automorphism σ of R defined by $\sigma((s, t)) = (t, s)$. Let $R[x; \sigma]$ be the skew polynomial ring over R by σ . Consider $(1, 0)x$ and $(0, 1)x$. Then $((1, 0)x(1, 0)) = 0 \in N(R[x; \sigma])$ but $(0, 1)xx(0, 1) = (0, 1)x^2 \notin N(R[x; \sigma])$, entailing that $R[x; \sigma]$ is not nil-IFP.

(2) There exists a nil-IFP ring over which the differential polynomial ring need not be nil-IFP. We use the ring and argument in [4, Example 11]. Let $\mathbb{Z}_2[t]$ be the polynomial ring with an indeterminate t over \mathbb{Z}_2 . Then $R = \mathbb{Z}_2[t]/(t^2)$ is commutative (hence nil-IFP), where (t^2) is the ideal of $\mathbb{Z}_2[t]$ generated by t^2 . Define a derivation δ on R by $\delta(t + (t^2)) = 1 + (t^2)$. Then $R[x; \delta] \cong \text{Mat}_2(\mathbb{Z}_2[y^2])$ which is not nil-IFP by example 1.1.

According to Antoine [2, Definition 2.3], a ring R is called *nil-Armendariz* if $ab \in N(R)$ for every coefficient a of $f(x)$ and every coefficient b of $g(x)$ whenever $f(x), g(x) \in R[x]$ satisfy $f(x)g(x) \in N(R)[x]$.

Lemma 2.7. *Given a ring R , the following conditions are equivalent:*

- (1) R is nil-Armendariz and nil-IFP;
- (2) R is an NI ring.

Proof. (1) \Rightarrow (2) Suppose that R is nil-Armendariz and nil-IFP. Let $a \in N(R)$. Every element of RaR is of the form $\sum_{\text{finite}} ras$ with $r, s \in R$. Every ras is nilpotent by Lemma 1.2 since R is nil-IFP, and so $\sum_{\text{finite}} ras \in N(R)$ by [2, Theorem 3.2] since R is nil-Armendariz. This yields RaR is nil and $a \in N^*(R)$, entailing that R is an NI ring.

(2) \Rightarrow (1) Let R be NI. Then R is nil-IFP clearly, and moreover R is nil-Armendariz by [2, Proposition 2.1]. \square

If Köthe's conjecture does not hold, then one should find a nil-IFP ring but not NI out of the class of nil-Armendariz rings, considering Proposition 1.9(3) and Lemma 2.7.

Proposition 2.8. *Let R be an Armendariz ring. Then the following conditions are equivalent:*

- (1) R is nil-IFP;
- (2) R is NI;
- (3) R is 2-primal;
- (4) $R[x]$ is nil-IFP;
- (5) $R[x]$ is NI;
- (6) $R[x]$ is 2-primal.

Proof. Armendariz rings are nil-Armendariz by [2, Proposition 2.7]. Hence R is NI by Lemma 2.7 if R is nil-IFP. If R is Armendariz and NI, then $R[x]$ is NI by [16, Proposition 20]. If R is Armendariz, then $N_*(R) = N^*(R)$ by [17, Lemma 2.3(5)], but since R is NI we have $N_*(R) = N^*(R) = N(R)$. This implies that R is 2-primal, and so $R[x]$ is 2-primal by [6, Proposition 2.6]. The proof of the remainder is obtained by Lemma 1.3(1). \square

The condition Armendariz in Proposition 2.8 is not superfluous since there exist many NI rings but not 2-primal by [14, Example 1.2], Marks [23, Example 2.2] and Smoktunowicz [28].

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