Bull. Korean Math. Soc. **49** (2012), No. 2, pp. 373–379
http://dx.doi.org/10.4134/BKMS.2012.49.2.373

SOME HOMOGENEITY CLASSES OF POSETS OF HEIGHT 2

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Abstract. In this paper, we find the inclusion relation among four categories of posets, i.e., ideal-homogeneous, tower-homogeneous, quasi-complement-preserved, and complement-preserved posets.

1. Introduction

It is a well-known problem to give classifications of those which satisfy certain homogeneity conditions in graph theory [3], and a characterization of countable partially ordered sets was given in [5]. It is very natural to ask whether every isomorphism between finite substructures can be extendable to an automorphism of the whole structure. A few decades later, some results on the homogeneity for finite partially ordered sets are also given by G. Behrendt [1], and they make resume to investigate the relationship between the homogeneity conditions for finite partially ordered sets.

Suppose $(P, \leqslant)$ is a finite partially ordered set (simply called a finite poset) with a partial order relation $\leqslant$, which is simply denoted by P for convenience. If $Q \subset P$, we may refer to Q also as a poset, having in mind the subposet $(Q, \leqslant)$ whose order relation is the restriction of $(P, \leqslant)$'s.

A chain is said to be maximal if it is not a proper subposet of any other chain. A maximum chain is a maximal chain with the maximum cardinality. The *height* of a poset P, denoted by $ht(P)$, is the number of points in a maximum chain, and the *length* is one less than the height, denoted by $l(P)$. An element $x \in P$ is maximal if there is no element $y(\neq x) \in P$ such that $x \leqslant y$. For an element $x \in P$, the height $ht(x)$ is the maximal cardinality of chains in $\{y \in P |\ y \leqslant x\}$. For a positive integer n, let $H_n(P, \leqslant) = \{x \in P \mid ht(x) = n\}$.

We say that x is *covered* by y in P (also, y *covers* x in P and (x, y) is a *covering pair* in P) when $x \leqslant y$ and there is no $z \in P$ with $x \neq z \neq y$ such that $x \leqslant z \leqslant y$. For a covering pair (x, y), y is called a *up-cover* of x, and x is called a *down-cover* of y. Let $UCov(x)$ be the set of all up-covers of x. Define

Received November 8, 2010; Revised August 5, 2011.
2010 *Mathematics Subject Classification.* 06A07.
Key words and phrases. ideal-homogeneous, tower-homogeneous, quasi-complement-preserved poset, complement-preserved poset.
The first author was supported by Wonkwang University in 2011.

updeg(x) as $|UCov(x)|$. Similarly, let $DCov(x)$ be the set of all down-covers of x, and downdeg$(x) = |DCov(x)|$.

For a poset P and $x \in P$, let $U[x] = \{y \in P \mid y \geqslant x \text{ in } P\}$ and $D[x] = \{y \in P \mid y \leqslant x \text{ in } P\}$. Also, we let $U[A] = \cup_{x\in A} U[x]$ and $D[A] = \cup_{x\in A} D[x]$ for a subposet A of P. Let $U(x) = \{y \in P \mid y \geqslant x \text{ in } P\} \setminus \{x\}$ and $D(x) = \{y \in P \mid y \leqslant x \text{ in } P\} \setminus \{x\}$. Also, we let $U(A) = \cup_{x\in A} U(x) \setminus A$ and $D(A) = \cup_{x\in A} D(x) \setminus A$ for a subposet A of P.

The *dual* of a poset P, denoted by $P^d = (P, \leqslant)^d$, is defined as $(P, \leqslant^d)$ where $x \leqslant y$ in P if and only if $y \leqslant^d x$ in P^d. A poset P is *self-dual* if P is isomorphic to P^d.

A map $f : (P, \leqslant) \to (Q, \leqslant')$ of posets is *order-preserving* if $x \leqslant y$ implies $f(x) \leqslant' f(y)$ for all $x, y \in P$. If $x \leqslant y$ implies $f(y) \leqslant' f(x)$, the map is *order-reversing*. Two posets $(P, \leqslant)$ and $(Q, \leqslant')$ are *isomorphic* if there exists an order-preserving bijection $f : (P, \leqslant) \to (Q, \leqslant')$ such that f^{-1} is also order-preserving. A bijection $f : (P, \leqslant) \to (P, \leqslant)$ is an automorphism if f and f^{-1} are order-preserving, and an antiautomorphism if f and f^{-1} are order-reversing. We denote the set of all automorphisms of a poset P by Aut(P).

An *ideal* I of P is a non-empty subset of P such that if $x \leqslant y$ for $x \in P$ and $y \in I$, then $x \in I$. A poset P is *ideal-homogeneous*, provided that, for any ideals I and J with $I \cong_\sigma J$, there exists an automorphism $\sigma^* \in \operatorname{Aut} P$ such that $\sigma^*|_I = \sigma$. A poset P is *weakly ideal-homogeneous*, provided that for each I of P and $\sigma \in \operatorname{Aut}(I)$, there is $\sigma^* \in \operatorname{Aut}(P)$ such that $\sigma^*|_I = \sigma$.

A subset S of P is called a *tower* in P if for every $x \in S$ there exists a maximal chain C in $\{y \in P \mid y \leqslant x\}$ such that $C \subset S$. We call a poset P *tower-homogeneous* if for two isomorphic towers S_1 and S_2 with an isomorphism $\sigma : S_1 \to S_2$, there exists an automorphism β in Aut(P) such that $\sigma(x) = \beta(x)$ for all $x \in S_1$. We say that $(P, \leqslant)$ is *weakly tower-homogeneous* if for each tower S and each automorphism $\sigma \in \operatorname{Aut}(S)$, there exists an automorphism β in Aut(P) such that $\sigma(x) = \beta(x)$ for all $x \in S$. Throughout this paper, let $\mathbb{N}$, $\mathbb{Z}$, and $[n]$ denote the set of all natural numbers, the set of all integers, and $\{x \in \mathbb{Z} \,|\, 1 \le x \le n\}$, respectively.

The following two theorems, due to Behrendt [1], characterizes the (weakly) ideal-homogeneous posets and (weakly) tower-homogeneous posets of height 2, respectively.

Theorem 1.1 ([1]). *Let $(P, \leqslant)$ be a finite partially ordered set of height-two. The followings are equivalent.*

(i) *$(P, \leqslant)$ is ideal-homogeneous.*
(ii) *$(P, \leqslant)$ is weakly ideal-homogeneous.*
(iii) *There exist a positive integer n and a function $f : [n] \to \mathbb{N}$ such that there exists $i \in [n]$ with $f(i) \neq 0$ and $(P, \leqslant)$ is isomorphic to $(X, \leqslant)$ where*

$$X = [n] \cup \{(S, i) \mid \ \emptyset \neq S \subseteq [n], 1 \le i \le f(|S|)\}$$

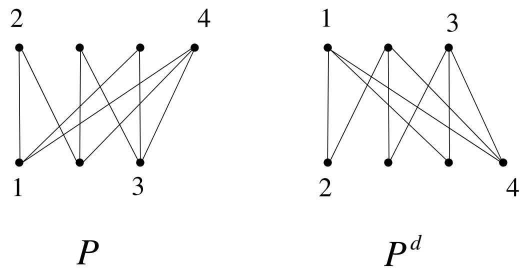

FIGURE 1. P is QCPP and ideal-homogeneous poset which is neither tower-homogeneous nor CPP

and for $k \in [n]$, $\emptyset \neq S \subseteq [n]$, $1 \leq i \leq f(|S|)$, *let*

$$k \leqslant (S, i) \quad \textit{if and only if} \quad k \in S.$$

For integers p, $q \geq 1$, let $C(p, q)$ be the linear sum $R_1 \oplus R_2$ of two disjoint antichains $R_1 = \{a_1, \ldots, a_p\}$ and $R_2 = \{b_1, \ldots, b_q\}$, i.e., all a_i's in R_1 are incomparable, and all b_j's in R_2 are also incomparable, and $a_i \leqslant b_j$ for all $a_i \in R_1$ and $b_j \in R_2$. For integers $n, k > 1$, let $A(n, k)$ be the disjoint sum of n copies of $C(1, k)$. For $n \geq 3$ let $B(n)$ be the poset consisting of 1-element subsets and $(n-1)$-element subsets of $\{1, \ldots, n\}$, ordered by set-theoretic inclusion.

Theorem 1.2 ([1])**.** *Let* $(P, \leqslant)$ *be a finite height-two ordered set. The following are equivalent.*

(i) $(P, \leqslant)$ *is tower-homogeneous.*
(ii) $(P, \leqslant)$ *is weakly tower-homogeneous.*
(iii) $(P, \leqslant)$ *is isomorphic to* $C(p, q)$ *for some* $p, q \geq 1$*, or to* $A(n, k)$ *for some* $n, k \geq 1$ *or to* $B(n)$ *for* $n \geq 3$.

Definition 1 ([4])**.** Let I and J be ideals of a poset P. Then P is called a *quasi-complement-preserved poset* (*QCPP*) if $I \cong J$ in P, then $I^c \cong J^c$ in P^d.

It looks like that, for two isomorphic ideals I and J of P^d, if P is a QCPP, then $I^c \cong J^c$ in P. However, in Figure 1, $I = \{2\}$ and $J = \{4\}$ are isomorphic in P^d, but I^c and J^c are not isomorphic in P while P is a QCPP. Hence we can define a type of poset which is a QCPP satisfying the converse of Definition 1, as follows.

Definition 2. A poset P is a *complement-preserved poset* (*CPP*) if P and P^d are QCPPs.

In this paper, we find the relationship among these four homogeneity classes of posets of height 2.

2. Ideals in a quasi-complement-preserved poset

Lemma 2.1. *For a poset P, if I is an ideal in P, then I^c is an ideal in P^d.*

Proof. Let $x \in I^c$ and $y \leqslant x$ in P^d. Then $x \leqslant y$ in P. Since $x \in I^c$, we have $x \notin I$. Since $x \leqslant y$ in P, we have $y \notin I$, i.e., $y \in I^c$. Therefore I^c is an ideal in P^d. □

Lemma 2.2. *If P is a CPP and $x,\ y \in H_1(P, \leqslant)$, then* $\operatorname{updeg} x = \operatorname{updeg} y$.

Proof. Suppose $I = \{x\}$ and $J = \{y\}$ for any elements $x,\ y$ in $H_1(P, \leqslant)$. Then $I \cong J$. Since P is a CPP, we have $I^c \cong J^c$. Let m be the number of edges in P when, temporarily, we regard P as a graph, and m' be the number of edges in I^c and m'' be the number of edges in J^c. Since $I = \{x\}$, we have $\operatorname{updeg} x = \deg x = m - m'$ and, similarly, $m - m'' = \deg y = \operatorname{updeg} y$. Since $I^c \cong J^c$, we have

$$m - m' = m - m''.$$

Therefore

$$\operatorname{updeg} x = \operatorname{updeg} y$$

for every $x,\ y$ in $H_1(P, \leqslant)$. □

Consequently, we have the following result from the duality of a CPP.

Corollary 2.3. *If P is a CPP and x, y are maximal elements, then*

$$\operatorname{downdeg}(x) = \operatorname{downdeg}(y).$$

Lemma 2.4. *Let P be a CPP. Suppose I and J are ideals in P such that $I \cong_\sigma J$. Then*

$$\sum_{x \in H_1(P,\leqslant)\cap I} \operatorname{updeg} x = \sum_{\sigma(x) \in H_1(P,\leqslant)\cap J} \operatorname{updeg} \sigma(x).$$

Proof. It is clear by Lemma 2.2. □

Let I be an ideal of P and $T(I) = \{x \in P \mid DCov(x) = I\}$. In fact, if P is a poset of height 2, we have $T(I)$ is a subset of $H_2(P, \leqslant)$. The following lemma shows $T(I)$ is not empty if I is an ideal in $H_1(P, \leqslant)$.

Lemma 2.5. *Let P be a CPP of height 2 and* $\operatorname{downdeg}(x) = r$ *for all $x \in H_2(P, \leqslant)$. Let I be an ideal in $H_1(P, \leqslant)$ with $|I| = r$. Then $T(I)$ is not an empty set, i.e., $|T(I)| \geq 1$.*

Proof. Since $\operatorname{downdeg}(x) = r$ for any $x \in H_2(P, \leqslant)$, there is an r-element ideal I_0 in $H_1(P, \leqslant)$ such that $I_0 = DCov(x)$. On the other hand, assume that there exists an r-element ideal I in $H_1(P, \leqslant)$ such that $T(I) = \emptyset$. Then I_0^c and I^c are not isomorphic in P^d, while I_0 and I are isomorphic in P. Hence, this contradicts the hypothesis that P is a CPP. Therefore, $T(I) \neq \emptyset$. □

Proposition 2.6. *Let P be a CPP of height* 2 *and* $\text{downdeg}(x) = r$ *for all* $x \in H_2(P, \leqslant)$. *Suppose that I and J are ideals in* $H_1(P, \leqslant)$ *with* $|I| = |J| = r$. *Then* $|T(I)| = |T(J)|$.

Proof. Suppose that I and J are ideals in $H_1(P, \leqslant)$ with $|I| = |J| = r$. Let $S_1 = D(I)$ and $S_2 = D(J)$ in P^d. Then clearly, $T(I) \subseteq S_1$ and $T(J) \subseteq S_2$. Since I and J are isomorphic in a CPP P, we have $I^c \cong J^c$ in P^d. Since I^c has no element of I, I^c has exactly $|T(I)|$ isolated elements from the definition of $T(I)$. Similarly, J^c has exactly $|T(J)|$ isolated elements. Therefore, $|T(I)| = |T(J)|$. □

In Proposition 2.6, we emphasize the existence of an invariant s which is the number of elements of $H_2(P, \leqslant)$, each of which covers every element of any r-element ideal where $r = \text{downdeg}(x)$ for $x \in H_2(P, \leqslant)$.

3. Relationship among four categories of posets

An ideal of a poset P is clearly a tower of P. It is clear that if a poset P is tower-homogeneous, then it is ideal-homogeneous. Therefore, we have the following proposition.

Proposition 3.1. *Every tower-homogeneous poset is ideal-homogeneous.*

However, the converse is not always true (See Figure 1). As a matter of fact, in Figure 1, the towers $I = \{1, 2\}$ and $J = \{3, 4\}$ are isomorphic with isomorphism α where $\alpha(1) = 3$ and $\alpha(2) = 4$. It can be easily seen that α cannot be extended to any automorphism of P. Hence, P is not tower-homogeneous. Moreover, P^d is not a QCPP as stated previously.

According to Theorem 1.2, a tower-homogeneous poset of height 2 is one of the following cases:

(i) $C(p, q)$ for some $p, q \geq 1$.
(ii) $A(n, k)$ for some $n, k \geq 1$.
(iii) $B(n)$ for $n \geq 3$.

Then, from the definitions of these posets, $C(p, q)$ and $B(n)$ are CPPs, and $A(1, k)$ is a CPP. However, $A(n, k)$ is not a CPP but a QCPP for $n \geq 2$. That is, the converse of Theorem 1.2 is not true.

Theorem 3.2. *Every CPP of height* 2 *is tower-homogeneous.*

Proof. For a given CPP P of height 2 and $x \in H_2(P, \leqslant)$, let $\text{downdeg}(x) = r$ and $|H_1(P, \leqslant)| = m \geq r$. For every r-element subset I of $H_1(P, \leqslant)$, by Proposition 2.6, there is the fixed number $|T(I)|$(denoted by t)≥ 1 of elements of $H_2(P, \leqslant)$ which cover all elements of I. In order to show the result, we only have to check the following five cases; (1) $r = m$, (2) $r = m - 1$ and $t = 1$, (3) $r = 1$, (4) $1 < r < m - 1$, and (5) $r = m - 1$ and $t > 1$.

(1) If $r = m$, then P is isomorphic to $C(r, t)$.

(2) If $r = m - 1$ and $t = 1$, then P is isomorphic to $B(m)$.

(3) If $r = 1$, then P is isomorphic to $A(m,t)$. If $m \geq 2$, then $A(m,t)$ is not a CPP so that we exclude all other possibilities for $m \geq 2$.

(4) Suppose $1 < r < m-1$. Let $x_0 \in H_1(P,\leqslant)$ and $u \in H_2(P,\leqslant)$ with $x_0 < u$. Since $r < m-1$, there exist distinct $y, z \in H_1(P,\leqslant)$ incomparable with u. Let S_1 be the set of elements of $H_1(P,\leqslant)$ covered by u. Since $r > 1$, there exists $x_1 \in S_1 \setminus \{x_0\}$. Then there exists $v \in H_2(P,\leqslant)$ such that $D(v) = (S_1 \setminus \{x_0\}) \cup \{y\}$, say S_2, in P. And there exists $w \in H_2(P,\leqslant)$ such that $D(w) = (S_1 \setminus \{x_0, x_1\}) \cup \{y, z\}$, say S_3, in P. Let $I = \{u, v\}$ and $J = \{u, w\}$, then $I \cong J$ in P^d. We count the decreasing numbers of updeg(x) for $x \in H_1(P,\leqslant)$ in I^c and J^c, respectively.

The up-degrees of x_0 and y in I^c are one less than those of x and y in P, and the up-degree of $a \in S_1 \cap S_2$ in I^c is reduced by two. The up-degrees of others are not changed in I^c. On the other hand, the up-degrees of x_0, x_1, y, z in J^c are one less than those of x_0, x_1, y, z in P, and the up-degree of $b \in S_1 \cap S_3$ in J^c is reduced by two in J^c. The up-degrees of others are not changed in J^c. Note that $|S_1 \cap S_2| = r-1$ and $|S_1 \cap S_3| = r-2$. That is, in I^c, there are $r-1$ elements in $H_1(P,\leqslant)$ whose up-degrees are reduced by two; however, in J^c, there are $r-2$ elements in $H_1(P,\leqslant)$ whose up-degrees are reduced by two, which means that $I^c \ncong J^c$ in P. Therefore, P is not a CPP.

(5) Suppose $r = m-1$ and $t > 1$ and $m \geq 3$. Then there exist distinct $x_0, y, z \in H_1(P,\leqslant)$. Note that P is a CPP. From Lemma 2.2 and its corollary, there exist $u, v, w \in H_2(P,\leqslant)$ such that $D(u) = D(v) = H_1(P,\leqslant) \setminus \{y\}$ and $D(w) = H_1(P,\leqslant) \setminus \{z\}$. Now let $I = \{u, v\}$ and $J = \{u, w\}$ then $I \cong J$ in P^d. The up-degree of $a \in H_1(P,\leqslant)$ is reduced by two in I^c, and that of y is not changed in I^c. On the other hand, the up-degree of $b \in D(u) \setminus D(w)$ is reduced by one, and that of $c \in D(u) \cap D(w)$ is reduced by two, and that of $d \in D(w) \setminus D(u)$ is reduced by one. Note that $D(u) \setminus D(w) = \{z\}$, $D(w) \setminus D(u) = \{y\}$ and $H_1(P,\leqslant) = D(u) \cup \{y\} = D(w) \cup \{z\}$. This implies that $|D(u) \cap D(w)| = r-1$. Consequently, in I^c, there are r elements in $H_1(P,\leqslant)$ whose up-degrees are reduced by two; however, in J^c, there are $r-1$ elements whose up-degrees are reduced by two, which implies that $I^c \ncong J^c$ in P. Therefore, P is not a CPP.

Through the cases, from (1) to (5), if P is a CPP of height 2, then P is one of $A(1,t)$, $B(m)$, and $C(m,t)$. Therefore, P is a tower-homogeneous poset. □

If a poset P is ideal-homogeneous, then, for ideals I and J with $I \cong_\sigma J$, there exists an automorphism $\sigma^* \in \mathrm{Aut}P$ such that $\sigma^*|_I = \sigma$. The restriction of σ^* to I^c induces an isomorphism from I^c to J^c. Therefore P is a QCPP as stated:

Proposition 3.3. *Every ideal-homogeneous poset of height* 2 *is a QCPP.*

For the converse of Proposition 3.3, we give a conjecture as follows.

Conjecture 3.4. *Every QCPP of height* 2 *is ideal-homogeneous.*

The results are summarized in Figure 2.

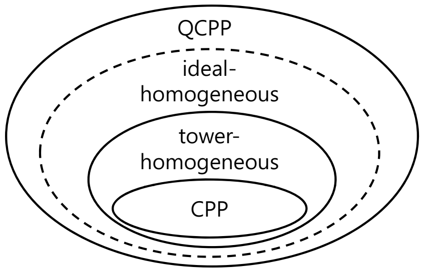

Figure 2. A relationship between the homogeneity classes of posets of height 2

Acknowledgment. The authors are grateful to the anonymous referees whose comments improved the clarity of this paper as well as Jong-Yoon Hyun for giving us the idea on a quasi-complement preserved poset and a complement preserved poset.

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