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ON THE ERROR TERM IN THE PRIME GEODESIC THEOREM

Muharem Avdispahić and Dženan Gušić

Abstract. Taking the integrated Chebyshev-type counting function of the appropriate order, we improve the error term in Park's prime geodesic theorem for hyperbolic manifolds with cusps. The obtained estimate coincides with the best known result in the Riemann surfaces case.

Let Γ be a discrete co-finite torsion free subgroup of $G = \mathrm{SO}_0\,(d,1)$ satisfying the condition $\Gamma \cap P = \Gamma \cap N(P)$ for $P \in \mathfrak{P}_\Gamma$, where $\mathfrak{P}_\Gamma$ is the set of Γ-conjugacy classes of Γ-cuspidal parabolic subgroups in G and $N(P)$ is the unipotent part of P. Denote by K a maximal compact subgroup of G. The manifold $X_\Gamma = \Gamma\backslash G/K$ is a d-dimensional real hyperbolic manifold with cusps.

As usual, let $\pi_\Gamma(x)$ be the number of prime geodesics C_γ of length $l_\gamma \le \log x$ on X_Γ. Recall that prime geodesic C_γ corresponds to the conjugacy class γ of a primitive hyperbolic element with the norm $N(\gamma) = e^{l_\gamma}$.

The purpose of this short note is to prove that Park's refinement of the prime geodesic theorem, due to Gangolli [5] and DeGeorge [3] in the compact case and to Gangolli-Warner [6] in the finite volume case, can be further improved to obtain the theorem in the following form.

Theorem 1. *Let X_Γ be as above. Then*

$$\pi_\Gamma(x) = \sum_{\frac{3}{2}d_0 < s_n(k) \le 2d_0} (-1)^k \operatorname{li}\left(x^{s_n(k)}\right) + O\left(x^{\frac{3}{2}d_0}(\log x)^{-1}\right)$$

as $x \to +\infty$, where $d_0 = \frac{d-1}{2}$, $(s_n(k)-k)(2d_0-k-s_n(k))$ is a small eigenvalue in $\left[0, \frac{3}{4}d_0^2\right]$ of Δ_k on $\pi_{\sigma_k,\lambda_n(k)}$ with $s_n(k) = d_0 + i\lambda_n(k)$ or $s_n(k) = d_0 - i\lambda_n(k)$ in $\left(\frac{3}{2}d_0, 2d_0\right]$, Δ_k is the Laplacian acting on the space of k-forms over X_Γ and $\pi_{\sigma_k,\lambda_n(k)}$ is the principal series representation.

Proof. Let Γ_{h} (resp. $\mathrm{P}\Gamma_{\mathrm{h}}$) denote the set of the Γ-conjugacy classes of hyperbolic (resp. primitive hyperbolic) elements in Γ. Set $\Lambda(\gamma) = l_{\gamma_0}$, where

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$\gamma = \gamma_0^{j(\gamma)}$, $\gamma_0 \in \mathrm{P}\Gamma_{\mathrm{h}}$, $j(\gamma) \in \mathbb{N}$. It is well known that the assertion of the theorem can be easily deduced from the relation

$$\text{(1)} \qquad \psi_0(x) = \sum_{\gamma \in \Gamma_{\mathrm{h}}, N(\gamma) \leq x} \Lambda(\gamma) = \sum_{s_n(k) \in \left(\frac{3}{2}d_0, 2d_0\right]} \frac{(-1)^k x^{s_n(k)}}{s_n(k)} + O\left(x^{\frac{3}{2}d_0}\right).$$

We observe that J. Park [12] proved a variant of (1) with the error term $O\left(x^{\frac{3}{2}d_0} (\log x)^{\frac{1}{2}}\right)$ in place of $O\left(x^{\frac{3}{2}d_0}\right)$. The key role in his proof is played by the Ruelle zeta function replacing the Selberg zeta. This is in line with Parry-Pollicott [13]. The ingredients come from the results of Fried [4] and further investigations of the Ruelle zeta by Gon-Park [7]. The error term $O\left(x^{\frac{3}{2}d_0} (\log x)^{\frac{1}{2}}\right)$ stems from Park's modification of Hejhal's techniques [8], [9] and the use of a higher order counting function $\psi_{d-1}(x)$, where $\psi_n(x)$ is defined recursively by $\psi_n(x) = \int_0^x \psi_{n-1}(t)dt$ for $n \in \mathbb{N}$. Inspired by Randol's approach in the case of compact Riemann surfaces, we consider $\psi_d(x)$ instead of $\psi_{d-1}(x)$.

It is well known that $\psi_d(x)$ can be represented in the form

$$\psi_d(x) = -\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{R'_\Gamma(s)}{R_\Gamma(s)} \frac{x^{s+d}}{s(s+1)\cdots(s+d)} ds$$

for some $c > 2d_0$. Here, R_Γ is the Ruelle zeta function defined by

$$R_\Gamma(s) = \prod_{\gamma_0 \in \mathrm{P}\Gamma_{\mathrm{h}}} \left(1 - e^{-s l_{\gamma_0}}\right)^{-1}, \ \mathrm{Re}\,(s) > 2d_0,$$

and meromorphically continued to the whole complex plane. Following [12], we apply the Cauchy residue theorem to the integrand of $\psi_d(x)$ over $R(T) = \{s \in \mathbb{C} \mid \ |s| \leq T, \mathrm{Re}\,(s) \leq d_0\} \ \cup \ \left\{s \in \mathbb{C} \mid d_0 \leq \mathrm{Re}\,(s) \leq c, \, -\widetilde{T} \leq \mathrm{Im}\,(s) \leq \widetilde{T}\right\}$, where $\widetilde{T} = \sqrt{T^2 - d_0^2}$ and $T \gg 0$ is such that there is no zero or pole of the integrand over the boundary of $R(T)$.

For a fixed $0 < \varepsilon < c - d_0$, the adjusted Park argumentation gives us

(2)

$$\begin{aligned}
&\frac{1}{2\pi i} \int_{c-i\widetilde{T}}^{c+i\widetilde{T}} \frac{R'_\Gamma(s)}{R_\Gamma(s)} \frac{x^{s+d}}{s(s+1)\cdots(s+d)} ds \\
= & -\frac{1}{2\pi i} \int_{C_T} + \frac{1}{2\pi i} \left(\int_{d_0+i\widetilde{T}}^{d_0+\varepsilon+i\widetilde{T}} + \int_{d_0+\varepsilon-i\widetilde{T}}^{d_0-i\widetilde{T}} \right) + \frac{1}{2\pi i} \left(\int_{d_0+\varepsilon+i\widetilde{T}}^{c+i\widetilde{T}} + \int_{c-i\widetilde{T}}^{d_0+\varepsilon-i\widetilde{T}} \right) \\
& + \sum_{z \in R(T)} \mathrm{Res}_{s=z} \left(\frac{R'_\Gamma(s)}{R_\Gamma(s)} \frac{x^{s+d}}{s(s+1)\cdots(s+d)} \right) \\
= & -\frac{1}{2\pi i} \int_{C_T} + O\left(x^{d+d_0+\varepsilon} T^{-2}\right) + O\left(\varepsilon^{-1} x^{c+d} T^{-2}\right)
\end{aligned}$$

$$+\sum_{z\in R(T)}\mathrm{Res}_{s=z}\left(\frac{R_\Gamma'(s)}{R_\Gamma(s)}\frac{x^{s+d}}{s(s+1)\cdots(s+d)}\right),$$

where C_T is the anti-clockwise oriented circular part of the boundary of $R(T)$.

Now, the first integral on the right hand side of (2) can be estimated by

$$\frac{1}{2\pi i}\int_{C_T}\frac{R_\Gamma'(s)}{R_\Gamma(s)}\frac{x^{s+d}}{s(s+1)\cdots(s+d)}ds=O\left(x^{d+d_0}T^{-1}\log T\right), \tag{3}$$

since

$$\begin{aligned}\left|\frac{1}{2\pi i}\int_{C_T}\frac{R_\Gamma'(s)}{R_\Gamma(s)}\frac{x^{s+d}}{s(s+1)\cdots(s+d)}ds\right| &\le C_1x^{d+d_0}T^{-(d+1)}\int_{C_T}\left|\frac{R_\Gamma'(s)}{R_\Gamma(s)}\right||ds|\\ &\le C_1x^{d+d_0}T^{-(d+1)}\int_{|s|=T}\left|\frac{R_\Gamma'(s)}{R_\Gamma(s)}\right||ds|\\ &\le C_2x^{d+d_0}T^{-1}\log T\end{aligned}$$

according to [4, p. 509, Prop. 7].

We note that (3) is to be compared to [12, relation (3.8)].

Furthermore,

$$\begin{aligned}&\sum_{z\in R(T)}\mathrm{Res}_{s=z}\left(-\frac{R_\Gamma'(s)}{R_\Gamma(s)}\frac{x^{s+d}}{s(s+1)\cdots(s+d)}\right)\\ &=\sum_{s_n(k)\in(d_0,2d_0]}(-1)^k\frac{x^{s_n(k)+d}}{s_n(k)(s_n(k)+1)\cdots(s_n(k)+d)}\\ &\quad+\sum_{-\widetilde{T}\le\lambda_n(0)\le\widetilde{T}}\frac{x^{s_n(0)+d}}{s_n(0)(s_n(0)+1)\cdots(s_n(0)+d)}\\ &\quad+\sum_{z\in R(T,d_0)}\mathrm{Res}_{s=z}\left(-\frac{R_\Gamma'(s)}{R_\Gamma(s)}\frac{x^{s+d}}{s(s+1)\cdots(s+d)}\right),\end{aligned} \tag{4}$$

where $R(T,d_0)=R(T)\cap\{s\in\mathbb{C}|\ \mathrm{Re}\,(s)<d_0\}$.

However,

$$\sum_{z\in R(T,d_0)}\mathrm{Res}_{s=z}\left(-\frac{R_\Gamma'(s)}{R_\Gamma(s)}\frac{x^{s+d}}{s(s+1)\cdots(s+d)}\right)=O\left(x^{d+d_0}T^{-1}\log T\right), \tag{5}$$

as opposed to [12, (3.17)].

Taking into account (3), (4), (5), as well as the fact that

$$\frac{1}{2\pi i}\int_{c-i\widetilde{T}}^{c+i\widetilde{T}}\frac{R_\Gamma'(s)}{R_\Gamma(s)}\frac{x^{s+d}}{s(s+1)\cdots(s+d)}ds=-\psi_d(x)+O\left(x^{c+d}T^{-d}\right)$$

and passing to the limit $T \to +\infty$ in (2), we end up with

$$\begin{aligned}\psi_d(x) = &\sum_{s_n(k)\in(d_0,2d_0]} (-1)^k \frac{x^{s_n(k)+d}}{s_n(k)(s_n(k)+1)\cdots(s_n(k)+d)} \\ &+ \sum_{s_n(0)=d_0\pm i\lambda_n(0)} \frac{x^{s_n(0)+d}}{s_n(0)(s_n(0)+1)\cdots(s_n(0)+d)}.\end{aligned} \tag{6}$$

To derive the asymptotics of $\psi_0(x)$ from the asymptotics of $\psi_d(x)$, one proceeds in the standard way introducing the functions

$$\Delta_d^+ f(x) = \int_x^{x+h} \int_{x_{2d_0}}^{x_{2d_0}+h} \cdots \int_{x_1}^{x_1+h} f^{(d)}(x_0)dx_0 \cdots dx_{2d_0}$$

and

$$\Delta_d^- f(x) = \int_{x-h}^{x} \int_{x_{2d_0}-h}^{x_{2d_0}} \cdots \int_{x_1-h}^{x_1} f^{(d)}(x_0)dx_0 \cdots dx_{2d_0}$$

for some constant h to be specified later on.

Notice that

$$h^{-d}\Delta_d^+ \frac{x^{s_n(k)+d}}{s_n(k)(s_n(k)+1)\cdots(s_n(k)+d)} = \frac{x^{s_n(k)}}{s_n(k)} + O\left(h^{s_n(k)}\right) \tag{7}$$

and

$$\begin{aligned}&h^{-d}\Delta_d^+ \frac{x^{s_n(0)+d}}{s_n(0)(s_n(0)+1)\cdots(s_n(0)+d)} \\ &= O\left(\min\left(x^{d_0}\left|s_n(0)\right|^{-1}, h^{-d}\left|s_n(0)\right|^{-(d+1)} x^{d+d_0}\right)\right).\end{aligned} \tag{8}$$

The second bound in (8) is straightforward from the representation

$$\Delta_d^+ f(x) = \sum_{i=0}^{d} (-1)^i \binom{d}{i} f\left(x + (d-i)\,h\right).$$

The relation (7) and the first bound in (8) are obtained by application of the mean value theorem (Cf. [8, p. 114]), [14, p. 246]).

Following [11, pp. 463–464] and using (8) we deduce

$$\begin{aligned}&h^{-d}\Delta_d^+ \sum_{s_n(0)=d_0\pm i\lambda_n(0)} \frac{x^{s_n(0)+d}}{s_n(0)(s_n(0)+1)\cdots(s_n(0)+d)} \\ &= O\left(x^{d_0}\int_{d_0}^{M} t^{-1}dN(t)\right) + O\left(h^{-d}x^{d+d_0}\int_{M}^{+\infty} t^{-(d+1)}dN(t)\right) \\ &= O\left(x^{d_0}M^{2d_0}\right) + O\left(h^{-d}x^{d+d_0}M^{-1}\right)\end{aligned} \tag{9}$$

for some $M > 2d_0$, where $N(t) = O\left(t^d\right)$ denotes the counting function of $s_n(0) = d_0 + i\lambda_n(0)$. Thus, (6), (7) and (9) imply

$$\begin{aligned} h^{-d}\Delta_d^+\psi_d(x) = & \sum_{s_n(k)\in(d_0,2d_0]} \frac{(-1)^k x^{s_n(k)}}{s_n(k)} + O\left(h^{2d_0}\right) \\ & + O\left(x^{d_0}M^{2d_0}\right) + O\left(h^{-d}x^{d+d_0}M^{-1}\right). \end{aligned} \tag{10}$$

Substituting $M = x^{\frac{1}{4}}$, $h = x^{\frac{3}{4}}$ into (10), we get

$$\psi_0(x) \le h^{-d}\Delta_d^+\psi_d(x) = \sum_{s_n(k)\in\left(\frac{3}{2}d_0,2d_0\right]} \frac{(-1)^k x^{s_n(k)}}{s_n(k)} + O\left(x^{\frac{3}{2}d_0}\right). \tag{11}$$

Reasoning in an analogous way, one proves

$$\sum_{s_n(k)\in\left(\frac{3}{2}d_0,2d_0\right]} \frac{(-1)^k x^{s_n(k)}}{s_n(k)} + O\left(x^{\frac{3}{2}d_0}\right) = h^{-d}\Delta_d^-\psi_d(x) \le \psi_0(x). \tag{12}$$

Combining (11) and (12), we finally obtain

$$\psi_0(x) = \sum_{s_n(k)\in\left(\frac{3}{2}d_0,2d_0\right]} \frac{(-1)^k x^{s_n(k)}}{s_n(k)} + O\left(x^{\frac{3}{2}d_0}\right).$$

This completes the proof. □

Final Remarks. It is easily seen that taking $\psi_n(x)$ with $n > d$ does not yield a better result. The obtained error term $O\left(x^{\frac{3}{2}d_0}(\log x)^{-1}\right)$ in the prime geodesic theorem is in accordance with the known estimate in the case of Riemann surfaces that can be achieved through several different approaches (see, e.g., [1], [2], [14]). Actually, in the Concluding Remark of [14], Randol noted that it would be interesting to determine the extent to which his methods are applicable to more general symmetric spaces. Theorem 1 can be interpreted as an answer to this query in our setting.

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Muharem Avdispahić
Department of Mathematics
University of Sarajevo
Zmaja od Bosne 35, 71000 Sarajevo, Bosnia and Herzegovina
E-mail address: mavdispa@pmf.unsa.ba

Dženan Gušić
Department of Mathematics
University of Sarajevo
Zmaja od Bosne 35, 71000 Sarajevo, Bosnia and Herzegovina
E-mail address: dzenang@yahoo.com