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A NOTE ON PSEUDO-RIEMANNIAN ASSOCIATIVE FERMIONIC NOVIKOV ALGEBRAS

Zhiqi Chen and Fuhai Zhu

Abstract. In this paper, we focus on pseudo-Riemannian associative fermionic Novikov algebras. We prove that the underlying Lie algebras of pseudo-Riemannian associative fermionic Novikov algebras are 2-step nilpotent and that pseudo-Riemannian associative fermionic Novikov algebras are 3-step nilpotent. Moreover, we construct a pseudo-Riemannian associative fermionic Novikov algebra in dimension 14, which is not a Novikov algebra. It implies that the inverse proposition of Corollary 2 in the paper "Pseudo-Riemannian Novikov algebras" [J. Phys. A: Math. Theor. **41** (2008), 315207] does not hold.

1. Underlying Lie algebras of pseudo-Riemannian associative fermionic Novikov algebras

Gel′fand and Dikii gave a bosonic formal variational calculus in [9, 10] and Xu gave a fermionic formal variational calculus in [15]. Moreover, motivated by the super-symmetric theory, a formal variational calculus of super-variables was given by Xu in [16] which combines the bosonic theory of Gel′fand-Dikii and the fermionic theory. Fermionic Novikov algebras are related to the Hamiltonian super-operator in terms of this theory. A fermionic Novikov algebra A is a vector space over a field $\mathbb{F}$ with a bilinear product $(x, y) \mapsto xy$ satisfying

$$(xy)z - x(yz) = (yx)z - y(xz), \tag{1.1}$$

$$(xy)z = -(xz)y \tag{1.2}$$

for any $x, y, z \in A$. It corresponds to the following Hamiltonian operator H of type 0 [16]:

$$H^0_{\alpha,\beta} = \sum_{\gamma\in I}(a^\gamma_{\alpha,\beta}\Phi_\gamma(2) + b^\gamma_{\alpha,\beta}\Phi_\gamma D), \quad a^\gamma_{\alpha,\beta}, b^\gamma_{\alpha,\beta} \in \mathbb{R}. \tag{1.3}$$

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Fermionic Novikov algebras are a special class of left-symmetric algebras which only satisfy equation (1.1). Left-symmetric algebras are a class of non-associative algebras arising from the study of affine manifolds, affine structures and convex homogeneous cones [4, 14]. The commutator of a left-symmetric A

$$[x, y] = xy - yx \tag{1.4}$$

defines a Lie algebra, which is called the underlying Lie algebra of A.

A pseudo-Riemannian connection is a pseudo-metric connection such that the torsion is zero and parallel translations preserve the bilinear form on the tangent spaces [13]. The corresponding structure on a fermionic Novikov algebra A is a non-degenerate symmetric bilinear form $f : A \times A \to \mathbb{F}$ such that

$$f(xy, z) + f(y, xz) = 0 \quad \text{for any } x, y, z \in A. \tag{1.5}$$

Such a fermionic Novikov algebra is called a pseudo-Riemannian fermionic Novikov algebra. It is given in [18] that the underlying Lie algebra of a pseudo-Riemannian fermionic Novikov algebra is a pseudo-Riemannian Lie algebra. A Lie algebra $\mathfrak{g}$ over a field $\mathbb{F}$ is called a pseudo-Riemannian Lie algebra if there is a bilinear product $(x, y) \mapsto xy$ such that, for any $x, y, z \in \mathfrak{g}$,

$$xy - yx = [x, y], \quad [xy, z] + [x, zy] = 0 \tag{1.6}$$

and a non-degenerate symmetric bilinear form $(\ ,\ )$ on $\mathfrak{g}$ such that

$$(xy, z) + (y, xz) = 0. \tag{1.7}$$

The notion of pseudo-Riemannian Lie algebras was introduced by Boucetta in [1], which are strongly related to pseudo-Riemannian Poisson manifolds (for more details see [2]).

In this note, we focus on pseudo-Riemannian associative fermionic Novikov algebras, which are pseudo-Riemannian fermionic Novikov algebras satisfying

$$(xy)z = x(yz) \quad \text{for any } x, y, z \in A. \tag{1.8}$$

It is proved in [6] that any pseudo-Riemannian Lie algebra is solvable if the characteristic of $\mathbb{F}$ is zero. For pseudo-Riemannian associative fermionic Novikov algebras, we have:

Theorem 1.1. *The underlying Lie algebra of any pseudo-Riemannian associative fermionic Novikov algebra is* 2*-step nilpotent.*

Proof. Let A be a pseudo-Riemannian associative fermionic Novikov algebra and f the corresponding bilinear form. Since $(xy)z = x(yz)$ for any $x, y, z \in A$, we can represent the product only by xyz. Furthermore for any $x, y, z, d \in A$,

$$f(xyz, d) = -f(yz, xd) = f(z, yxd) = -f(yxz, d).$$

It follows that $xyz = -yxz$ by the nondegeneracy of f. Then we have

$$xyz = yzx = zxy = -yxz = -zyx = -xzy. \tag{1.9}$$

By (1.9), we know that xy belongs to the center of the underlying Lie algebra. It follows that the underlying Lie algebra is 2-step nilpotent. □

2. Pseudo-Riemannian associative fermionic Novikov algebras and Novikov algebras

A Novikov algebra was introduced as a left-symmetric algebra with commutative right multiplication operators: an algebra is a Novikov algebra if its product satisfies equation (1.1) and

$$(xy)z = (xz)y. \tag{2.1}$$

It connects with the Poisson brackets of hydrodynamic type [7, 8] and Hamiltonian operators in the formal variational calculus [11, 17].

A pseudo-Riemannian Novikov algebra is a Novikov algebra with a non-degenerate symmetric bilinear form satisfying the equation (1.5). It is proved in [5] that pseudo-Riemannian Novikov algebras are fermionic Novikov algebras if the characteristic of $\mathbb{F}$ is not 2. By [3] or [12], the sets of pseudo-Riemannian Novikov algebras and pseudo-Riemannian fermionic Novikov algebras are same if $\mathbb{F} = \mathbb{R}$ and the bilinear forms are positive definite. By [18], pseudo-Riemannian fermionic Novikov algebras of dimensions up to 4 over $\mathbb{C}$ are Novikov algebras. Nevertheless,

Remark 2.1 ([5]). For dimensions greater than four, we could neither prove that pseudo-Riemannian fermionic Novikov algebras are Novikov algebras nor find a pseudo-Riemannian fermionic Novikov algebra which is not a Novikov algebra.

In the following, we will give a pseudo-Riemannian associative fermionic Novikov algebra which is not a Novikov algebra. Firstly, we establish a theorem.

Theorem 2.2. *Let A be a pseduo-Riemannian associative fermionic Novikov algebra over a filed $\mathbb{F}$. If the characteristic of $\mathbb{F}$ is not 2, then A is 3-step nilpotent.*

Proof. Let A be a pseudo-Riemannian associative fermionic Novikov algebra and f the corresponding bilinear form. Denote the product $(xy)z$ only by xyz. By the proof of Theorem 1.1, xy belongs to the center of the underlying Lie algebra. Then

$$xyzd = yzxd = -yzdx = -xyzd.$$

That is, $xyzd = 0$. Namely A is 3-step nilpotent. □

Example 2.3. Assume that A is a pseudo-Riemannian associative fermionic Novikov algebra, which is not a Novikov algebra. By [5], we must have $xyz \neq 0$ for some $x, y, z \in A$. By (1.9), we have $xxy = 0$ for any $x, y \in A$. If A is algebraically generated by x, y, z, then it is easy to see that $xyz \in A^{\perp}$ since $xxy = 0$ and $xyzd = 0$ for any $x, y, z, d \in A$. It follows that $xyz = 0$.

Assume that $xyz \neq 0$ for some $x, y, z \in A$. Then there exists another element d such that $f(xyz, d) = a \neq 0$. In the following, assume that A is algebraically generated by x, y, z, d. Without loss of generality, let $a = 1$. By the equation (1.5), we know that xyd, xzd, yzd are not zero and

$$f(xyz, d) = -f(xyd, z) = -f(yzd, x) = f(xzd, y) = 1. \tag{2.2}$$

Let V_1 be a subspace of A linearly generated by x, y, z, d. Furthermore, assume that $uu = 0$ for any $u \in V_1$. By the linearity of products, we have that

$$uv = -vu \quad \text{for any } u, v \in V_1.$$

It is easy to see that xy, xz, xd, yz, yd, zd are not zero. In fact, assume that $xy = 0$. Then

$$f(xyz, d) = f(zxy, d) = -f(xy, zd) = 0.$$

It is a contradiction. Similar to the others.

Moreover, $x, y, z, d, xy, xz, xd, yz, yd, zd, xyz, xyd, xzd, yzd$ are linearly independent. In fact, assume that there exist a_i for $1 \leq i \leq 14$ such that

$$\begin{aligned} &a_1x + a_2y + a_3z + a_4d \\ &\quad +a_5xy + a_6xz + a_7xd + a_8yz + a_9yd + a_{10}zd \\ &\quad\quad +a_{11}xyz + a_{12}xyd + a_{13}xzd + a_{14}yzd = 0. \end{aligned}$$

Multiplying xy on the left of the above equation, we have

$$a_3xyz + a_4xyd = 0.$$

It follows that

$$a_3 = f(a_3xyz, d) = f(a_3xyz + a_4xyd, d) = 0$$

since $f(xyd, d) = 0$. Similarly, we have $a_1 = a_2 = a_3 = a_4 = 0$. Then the equation is

$$\begin{aligned} &a_5xy + a_6xz + a_7xd + a_8yz + a_9yd + a_{10}zd \\ &\quad +a_{11}xyz + a_{12}xyd + a_{13}xzd + a_{14}yzd = 0. \end{aligned}$$

Multiplying x on the left of the above equation, we have

$$a_8xyz + a_9xyd + a_{10}xzd = 0.$$

It follows that

$$a_8 = f(a_8xyz, d) = f(a_8xyz + a_9xyd + a_{10}xzd, d) = 0.$$

Similarly, we have $a_5 = a_6 = a_7 = a_8 = a_9 = a_{10} = 0$. Then the equation is

$$a_{11}xyz + a_{12}xyd + a_{13}xzd + a_{14}yzd = 0.$$

It follows that

$$a_{11} = f(a_{11}xyz, d) = f(a_{11}xyz + a_{12}xyd + a_{13}xzd + a_{14}yzd, d) = 0.$$

Similarly, we have that $a_{11} = a_{12} = a_{13} = a_{14} = 0$. It proves the claim of the linear independence. Also it is easy to get that

$$f(xy, zd) = f(yz, xd) = -f(xz, yd) = -1. \tag{2.3}$$

In addition putting $f(u, v) = 0$ except the eqs. (2.2) and (2.3), we have constructed a pseudo-Riemannian associative fermionic Novikov algebra of dimension 14. It is not a Novikov algebra since $(xy)z = 0$ for any $x, y, z \in A$ if A is a pseudo-Riemannian Novikov algebra [5].

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Zhiqi Chen
School of Mathematical Sciences and LPMC
Nankai University
Tianjin 300071, P. R. China
E-mail address: chenzhiqi@nankai.edu.cn

Fuhai Zhu
School of Mathematical Sciences and LPMC
Nankai University
Tianjin 300071, P. R. China