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APPROXIMATELY QUINTIC AND SEXTIC MAPPINGS ON THE PROBABILISTIC NORMED SPACES

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Abstract. We prove the stability for the systems of quadratic-cubic and additive-quadratic-cubic functional equations with constant coefficients on the probabilistic normed spaces (briefly PN spaces).

1. Introduction and preliminaries

The stability of functional equations started with the following question concerning stability of group homomorphisms proposed by S. M. Ulam [80] during a talk before a Mathematical Colloquium at the University of Wisconsin, Madison, in 1940:

Let $(G_1, \cdot)$ be a group and let $(G_2, *)$ be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$, such that if a mapping $h : G_1 \longrightarrow G_2$ satisfies the inequality $d(h(x \cdot y), h(x) * h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \longrightarrow G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$?

In 1941, Hyers [40] gave a first affirmative answer to the question of Ulam for Banach spaces as follows:

If E and E' are Banach spaces and $f : E \longrightarrow E'$ is a mapping for which there is $\varepsilon > 0$ such that $\|f(x + y) - f(x) - f(y)\| \leq \varepsilon$ for all $x, y \in E$, then there is a unique additive mapping $L : E \longrightarrow E'$ such that $\|f(x) - L(x)\| \leq \varepsilon$ for all $x \in E$.

Hyers' Theorem was generalized by Aoki [3] for additive mappings and by Rassias [72] for linear mappings by considering an unbounded Cauchy difference. The paper of Rassias [73] has provided a lot of influence in the development of what we now call generalized Hyers–Ulam stability or as Hyers–Ulam–Rassias stability of functional equations. In 1982–1994, J. M. Rassias (see [64]-[71]) solved the Ulam problem for different mappings and for many Euler–Lagrange type quadratic mappings, by involving a product of different powers of norms. In addition, J. M. Rassias considered the mixed product–sum

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of powers of norms control function [75]. In 1994, a generalization of the Rassiastheorem was obtained by Găvruta [37] by replacing the unbounded Cauchy difference by a general control function in the spirit of Rassiasapproach. For more details about the results concerning such problems the reader is referred to [4, 6, 7, 24, 36, 41, 44, 45, 46] and [58]–[74].

Khodaei and Rassias [49] investigated the solution and stability of the n-dimensional additive functional equation such that in the special case $n = 2$,

$$f(ax + by) + f(ax - by) = 2af(x) \tag{1.1}$$

for $a, b \in \mathbb{Z}\backslash\{0\}$ with $a \neq \pm 1, \pm b$. The authors proved that additive equation is equivalent to the above equation.

The functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y) \tag{1.2}$$

is related to a symmetric bi-additive function [1, 47]. It is natural that this equation is called a quadratic functional equation. In particular, every solution of the quadratic equation (1.2) is said to be a quadratic function. It is well known that a function f between real vector spaces is quadratic if and only if there exists a unique symmetric bi-additive function B_1 such that $f(x) = B_1(x, x)$ for all x. The bi-additive function B_1 is given by $B_1(x, y) = \frac{1}{4}(f(x + y) - f(x - y))$. The Hyers–Ulam stability problem for the quadratic functional equation was solved by Skof [79]. In [6], Czerwik proved the Hyers–Ulam–Rassias stability of the equation (1.2). Eshaghi Gordji and Khodaei [25] obtained the general solution and the generalized Hyers–Ulam–Rassias stability of the following quadratic functional equation for $a, b \in \mathbb{Z}\backslash\{0\}$ with $a \neq \pm 1, \pm b$,

$$f(ax + by) + f(ax - by) = 2a^2 f(x) + 2b^2 f(y). \tag{1.3}$$

Jun and Kim [42] introduced the following cubic functional equation

$$f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x) \tag{1.4}$$

and they established the general solution and the generalized Hyers–Ulam–Rassias stability for the functional equation (1.4). They proved that a function f between two real vector spaces X and Y is a solution of (1.4) if and only if there exists a unique function $C : X \times X \times X \longrightarrow Y$ such that $f(x) = C(x, x, x)$ for all $x \in X$, moreover, C is symmetric for each fixed one variable and is additive for fixed two variables. The function C is given by $C(x, y, z) = \frac{1}{24}(f(x+y+z)+f(x-y-z)-f(x+y-z)-f(x-y+z))$ for all $x, y, z \in X$. Obviously, the function $f(x) = cx^3$ satisfies the functional equation (1.4), which is called the cubic functional equation. Jun et al. [43] investigated the solution and the Hyers-Ulam stability for the cubic functional equation

$$f(ax + by) + f(ax - by) = ab^2(f(x + y) + f(x - y)) + 2a(a^2 - b^2)f(x), \tag{1.5}$$

where $a, b \in \mathbb{Z}\backslash\{0\}$ with $a \neq \pm 1, \pm b$. For other cubic functional equations see [53]–[57]. Lee et al. [50] considered the following functional equation

$$f(2x + y) + f(2x - y) = 4f(x + y) + 4f(x - y) + 24f(x) - 6f(y). \tag{1.6}$$

In fact, they proved that a function f between two real vector spaces X and Y is a solution of (1.6) if and only if there exists a unique symmetric bi-quadratic function $B_2 : X\times X \longrightarrow Y$ such that $f(x) = B_2(x,x)$ for all x. The bi-quadratic function B_2 is given by $B_2(x,y) = \frac{1}{12}(f(x+y) + f(x-y) - 2f(x) - 2f(y))$. Obviously, the function $f(x) = cx^4$ satisfies the functional equation (1.6), which is called the quartic functional equation. For more functional equations see [8]–[35], [5, 38, 39, 48, 51, 52, 59, 78].

Ebadian, Najati and Eshaghi Gordji [9] considered the generalized Hyers-Ulam stability of the systems additive–quartic functional equations

$$\begin{cases} f(x_1+x_2,y) = f(x_1,y) + f(x_2,y), \\ f(x,2y_1+y_2) + f(x,2y_1-y_2) = 4f(x,y_1+y_2) + 4f(x,y_1-y_2) \\ \qquad + 24f(x,y_1) - 6f(x,y_2) \end{cases} \tag{1.7}$$

and the quadratic-cubic functional equations

$$\begin{cases} f(x,2y_1+y_2)+f(x,2y_1-y_2)=2f(x,y_1+y_2) + 2f(x,y_1-y_2) + 12f(x,y_1), \\ f(x,y_1+y_2) + f(x,y_1-y_2) = 2f(x,y_1) + 2f(x,y_2). \end{cases} \tag{1.8}$$

In this paper, we investigate the stability for the systems of quadratic-cubic functional equations

$$\begin{cases} f(ax_1+bx_2,y) + f(ax_1-bx_2,y) = 2a^2f(x_1,y) + 2b^2f(x_2,y), \\ f(x,ay_1+by_2) + f(x,ay_1-by_2) = ab^2(f(x,y_1+y_2) + f(x,y_1-y_2)) \\ \qquad + 2a(a^2-b^2)f(x,y_1) \end{cases} \tag{1.9}$$

and additive-quadratic-cubic functional equations

$$\begin{cases} f(ax_1+bx_2,y,z)+f(ax_1-bx_2,y,z)=2af(x_1,y,z), \\ f(x,ay_1+by_2,z)+f(x,ay_1-by_2,z)=2a^2f(x,y_1,z) + 2b^2f(x,y_2,z), \\ f(x,y,az_1+bz_2)+f(x,y,az_1-bz_2)=ab^2(f(x,y,z_1+z_2)+f(x,y,z_1-z_2)) \\ \qquad + 2a(a^2-b^2)f(x,y,z_1), \end{cases} \tag{1.10}$$

where $a, b \in \mathbb{Z}\backslash\{0\}$ with $a \neq \pm1, \pm b$. The function $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ given by $f(x,y) = cx^2y^3$ is a solution of (1.9). In particular, letting $y = x$, we get a quintic function $g : \mathbb{R} \to \mathbb{R}$ in one variable given by $g(x) := f(x,x) = cx^5$. Also, it is easy to see that the function $f : \mathbb{R}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ defined by $f(x,y,z) = cxy^2z^3$ is a solution of (1.10). In particular, letting $y = z = x$, we get a sextic function $h : \mathbb{R} \to \mathbb{R}$ in one variable given by $h(x) := f(x,x,x) = cx^6$.

The proof of the following propositions are evident and we omit the details.

Proposition 1.1. *Let X and Y be real linear spaces. If a function $f : X\times X \longrightarrow Y$ satisfies* (1.9)*, then $f(\lambda x, \mu y) = \lambda^2\mu^3 f(x,y)$ for all $x, y \in X$, and all rational numbers λ, μ.*

Proposition 1.2. *Let X and Y be real linear spaces. If a function $f : X \times X \times X \longrightarrow Y$ satisfies* (1.10), *then $f(\lambda x, \mu y, \eta z) = \lambda\mu^2\eta^3 f(x, y, z)$ for all $x, y, z \in X$, and all rational numbers λ, μ, η.*

PN spaces were first defined by Šerstnev in 1962 (see [77]). Their definition was generalized in [2]. We recall and apply the definition of probabilistic space briefly as given in [76], together with the notation that will be needed (see [76]).

Definition 1.1. A distance distribution function (briefly, a d.d.f.) is a non-decreasing function F from $\bar{\mathbb{R}}^+$ into $[0, 1]$ that satisfies $F(0) = 0$ and $F(+\infty) = 1$, and is left-continuous on $(0, +\infty)$; here as usual, $\bar{\mathbb{R}}^+ := [0, +\infty]$.

The space of d.d.f.'s will be denoted by Δ^+; and the set of all F in Δ^+ for which $\lim_{t\to+\infty^-} F(t) = 1$ by D^+. The space Δ^+ is partially ordered by the usual pointwise ordering of functions, i.e., $F \le G$ if and only if $F(x) \le G(x)$ for all x in $\bar{\mathbb{R}}^+$. For any $a \ge 0$, ε_a^+ is the d.d.f. given by

$$\varepsilon_a^+(t) = \begin{cases} 0, & \text{if} \quad t \le a, \\ 1, & \text{if} \quad t > a. \end{cases}$$

Definition 1.2. A triangle function is a binary operation on Δ^+, namely a function $\tau : \Delta^+ \times \Delta^+ \to \Delta^+$ that is associative, commutative, non-decreasing in each place and has ε_0 as identity, that is, for all F, G and H in Δ^+:

(TF1) $\tau\big(\tau(F, G), H\big) = \tau\big(F, \tau(G, H)\big)$;
(TF2) $\tau(F, G) = \tau(G, F)$;
(TF3) $F \le G \Rightarrow \tau(F, H) \le \tau(G, H)$;
(TF4) $\tau(F, \varepsilon_0) = \tau(\varepsilon_0, F) = F$.

Typical continuous triangle function is

$$\Pi_T(F, G)(x) = T\big(F(x), G(x)\big).$$

Here T is a continuous t-norm, i.e., a continuous binary operation on $[0, 1]$ that is commutative, associative, non-decreasing in each variable and has 1 as identity; For example

$$M(x, y) = \min(x, y)$$

for all x, y in $[0, 1]$, is a continuous and maximal t-norm, namely for any t-norm T, $M \ge T$.

Also, note that Π_M is a maximal triangle function, that is, for every triangle function τ, $\Pi_M \ge \tau$.

Definition 1.3. A *Šerstnev Probabilistic Normed space* (briefly, Šerstnev PN space) is a triple (X, ν, τ), where X is a real vector space, τ is a continuous triangle function and ν is a mapping (the *probabilistic norm*) from X into Δ^+, such that for every choice of p and q in X and a in $\mathbb{R}^+$, the following hold:

(N1) $\nu(p) = \varepsilon_0$, if and only if, $p = \theta$ (θ is the null vector in X);
(N2) $\nu(ap)(t) = \nu(p)(t/|a|)$;
(N3) $\nu(p + q) \ge \tau\big(\nu(p), \nu(q)\big)$;

Let (X, ν, τ) be a PN space let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is said to be convergent if there exists $x \in X$ such that

$$\lim_{n\to\infty} \nu(x_n - x)(t) = 1$$

for all $t > 0$. In this case x is called the limit of $\{x_n\}$.

The sequence x_n in (X, ν, τ) is called Cauchy if for each $\varepsilon > 0$ and $\delta > 0$, there exist some n_0 such that $\nu(x_n - x_m)(\delta) > 1 - \varepsilon$ for all $m, n \geq n_0$.

Clearly, every convergent sequence in a PN space is Cauchy. If each Cauchy sequence is convergent in a PN space (X, ν, τ), then (X, ν, τ) is called Probabilistic Banach space (briefly, PB space).

2. Approximation of quintic mappings on the PN spaces

In this section, we investigate the stability problem for system of functional equations (1.9) on the PN spaces.

Theorem 2.1. *Let $s \in \{-1, 1\}$ be fixed. Let G be a r-divisible group and (Y, ν, Π_T) be a Šerstnev PB space. Let $\phi, \psi : G \times G \times G \to D^+$ be functions such that*

(2.1)
$$\begin{cases} \phi_n^s(x,0,y)(t) := \{\phi(a^{sn-(\frac{1+s}{2})}x, 0, a^{sn-(\frac{1+s}{2})}y)(2a^{5sn-(\frac{5s+1}{2})}t)\}, & n = 1, 2, \ldots \\ \psi_n^s(x,y,0)(t) := \{\psi(a^{sn+(\frac{1-s}{2})}x, a^{sn-(\frac{1+s}{2})}y, 0)(2a^{5sn+(\frac{5-5s}{2})}t)\}, & n = 1, 2, \ldots \end{cases}$$

$$\begin{cases} \Phi_1 := \Pi_T\{\phi_1^s(x,0,y)(t), \psi_1^s(x,y,0)(t)\} \\ \Phi_n := \Pi_T\{\Pi_T\{\phi_n^s(x,0,y)(t), \psi_n^s(x,y,0)(t)\}, \Phi_{n-1}\} \quad \textit{for} \quad n > 1 \end{cases}$$

for all $x, y \in G$ and

(2.2)
$$\lim_{n\to\infty} \phi(a^{sn}x_1, a^{sn}x_2, a^{sn}y)(a^{-5sn}t) = \lim_{n\to\infty} \psi(a^{sn}x, a^{sn}y_1, a^{sn}y_2)(a^{-5sn}t) = 1$$

for all $x, y, x_1, x_2, y_1, y_2 \in G$. Let $\hat{\Phi} = \lim_{n\to\infty} \Phi_n = 1$. If $f : G \times G \to Y$ is a function such that $f(0, y) = 0$ for all $y \in G$, and that

$$\begin{aligned} &\nu\left(f(ax_1 + bx_2, y) + f(ax_1 - bx_2, y) - 2a^2 f(x_1, y) - 2b^2 f(x_2, y)\right)(t) \\ &\geq \phi(x_1, x_2, y)(t), \end{aligned} \tag{2.3}$$

(2.4)
$$\begin{aligned} &\nu(f(x, ay_1 + by_2) + f(x, ay_1 - by_2) - ab^2 f(x, y_1 + y_2) - ab^2 f(x, y_1 - y_2) \\ &- 2a(a^2 - b^2) f(x, y_1))(t) \geq \psi(x, y_1, y_2)(t) \end{aligned}$$

for all $x, y, x_1, x_2, y_1, y_2 \in G$, then there exists a unique quintic function $T : G \times G \to Y$ satisfying (1.9) *and*

$$\nu\left(f(x, y) - T(x, y)\right)(t) \geq \hat{\Phi} \tag{2.5}$$

for all $x, y \in G$.

Proof. Putting $x_1 = 2x$ and $x_2 = 0$ and replacing y by $2y$ in (2.3), we get

$$\nu(f(2ax, 2y) - a^2 f(2x, 2y))(t) \geq \phi(2x, 0, 2y)(2t) \tag{2.6}$$

for all $x, y \in G$. Putting $y_1 = 2y$ and $y_2 = 0$ and replacing x by $2ax$ in (2.4), we get

$$\nu(f(2ax, 2ay) - a^3 f(2ax, 2y))(t) \geq \psi(2ax, 2y, 0)(2t) \tag{2.7}$$

for all $x, y \in G$. Thus

$$\nu(f(2ax, 2ay) - a^5 f(2x, 2y))(t) \geq \Pi_T\{\phi(2x, 0, 2y)(2a^{-3}t), \psi(2ax, 2y, 0)(2t)\} \tag{2.8}$$

for all $x, y \in G$. Replacing x, y by $\frac{x}{2}$, $\frac{y}{2}$ in (2.8), we have

$$\nu\left(f(ax, ay) - a^5 f(x, y)\right)(t) \geq \Pi_T\{\phi(x, 0, y)(2a^{-3}t), \psi(ax, y, 0)(2t)\} \tag{2.9}$$

for all $x, y \in G$. It follows from (2.9) that

$$\nu\left(a^{-5} f(ax, ay) - f(x, y)\right)(t) \geq \Pi_T\{\phi(x, 0, y)(2a^2 t), \psi(ax, y, 0)(2a^5 t)\} \tag{2.10}$$

and

$$\begin{aligned} &\nu\left(a^5 f(a^{-1}x, a^{-1}y) - f(x, y)\right)(t) \\ &\geq \Pi_T\{\phi(a^{-1}x, 0, a^{-1}y)(2a^{-3}t), \psi(ax, a^{-1}y, 0)(2t)\} \end{aligned} \tag{2.11}$$

for all $x, y \in G$. From the inequalities (2.10) and (2.11) we use iterative methods and induction on n and apply defined sequence in (2.1) to prove our next relation

$$\begin{aligned} &\nu\left(a^{-5sn} f(a^{sn}x, a^{sn}y) - f(x, y)\right)(t) \\ &\geq \Pi_T\{\Pi_T\{\phi_n^s(x, 0, y)(t), \psi_n^s(x, y, 0)(t)\}, \Phi_{n-1}\} \end{aligned} \tag{2.12}$$

for $n = 1, 2, \ldots$ and all $x, y \in G$. So

$$\begin{aligned} &\nu\left(a^{-5s(n+m)} f(a^{s(n+m)}x, a^{s(n+m)}y) - f(a^{sm}x, a^{sm}y)\right)(t) \\ &\geq \Pi_T\{\Pi_T\{\phi_{n+m}^s(x, 0, y)(t), \psi_{n+m}^s(x, y, 0)(t)\}, \Phi_{(n+m)-1}\} \end{aligned} \tag{2.13}$$

for all nonnegative integers n and m and for all $x, y \in G$. By the assumptions, (2.13) shows that the sequence $\{a^{-5sn} f(a^{sn}x, a^{sn}y)\}$ is a Cauchy sequence in Y for all $x, y \in G$. Since Y is a Banach space, it follows that the sequence $\{a^{-5sn} f(a^{sn}x, a^{sn}y)\}$ converges for all $x, y \in G$. We define the function $T : G \times G \to Y$ by

$$T(x, y) = \lim_{n\to\infty} a^{-5sn} f(a^{sn}x, a^{sn}y) \tag{2.14}$$

for all $x, y \in G$. It follows from (2.3) that

$$\begin{aligned} &\nu(T(ax_1 + bx_2, y) + T(ax_1 - bx_2, y) - 2a^2 T(x_1, y) - 2b^2 T(x_2, y))(t) \\ &= \lim_{n\to\infty} \nu(a^{-5sn} f(a^{sn}(ax_1 + bx_2), a^{sn}y) + a^{-5sn} f(a^{sn}(ax_1 - bx_2), a^{sn}y) \\ &\quad - 2a^{-5sn} a^2 f(a^{sn}x_1, a^{sn}y) - 2a^{-5sn} b^2 f(a^{sn}x_2, a^{sn}y))(t) \\ &\geq \lim_{n\to\infty} \phi(a^{sn}x_1, a^{sn}x_2, a^{sn}y)(a^{-5sn}t) = 1 \end{aligned}$$

for all $x_1, x_2, y \in G$. Also it follows from (2.4) that

$$\begin{aligned}
&\nu(T(x, ay_1 + by_2) + T(x, ay_1 - by_2) - ab^2(T(x, y_1 + y_2) \\
&\quad - T(x, y_1 - y_2)) - 2a(a^2 - b^2)T(x, y_1))(t) \\
&= \lim_{n\to\infty} \nu(a^{-5sn} f(a^{sn}x, a^{sn}(ay_1 + by_2)) + a^{-5sn} f(a^{sn}x, a^{sn}(ay_1 - by_2)) \\
&\quad - a^{-5sn}ab^2 f(a^{sn}x, a^{sn}(y_1 + y_2)) - a^{-5sn}ab^2 f(a^{sn}x, a^{sn}(y_1 - y_2)) \\
&\quad - 2a^{-5sn}a(a^2 - b^2) f(a^{sn}x, a^{sn}y_1))(t) \\
&\geq \lim_{n\to\infty} a^{-5sn}\psi(a^{sn}x, a^{sn}y_1, a^{sn}y_2)(t) = 1
\end{aligned}$$

for all $x, y_1, y_2 \in G$. This means that T satisfies (1.9) that is, T is quintic. Moreover, passing the limit $n \to \infty$ in (2.12), we get the inequality (2.5).

Now, let $T' : G \times G \to Y$ be another quintic function satisfying (1.9) and (2.5). By Proposition 1.1 we have $a^{-5sn}T'(a^{sn}x, a^{sn}y) = T'(x, y)$ for all $x, y \in G$. Therefore we conclude that

$$\begin{aligned}
&\nu(T(x, y), T'(x, y))(t) \\
&= \lim_{n\to\infty} \nu(a^{-5sn}T(a^{sn}x, a^{sn}y) - a^{-5sn}T'(a^{sn}x, a^{sn}y))(t) \\
&\geq \lim_{n\to\infty} \Pi_T\{\Pi_T\{\phi_n^s(a^{sn}x, 0, a^{sn}y)(a^{-5sn}t), \psi_n^s(a^{sn}x, a^{sn}y, 0)(a^{-5sn}t)\}, \Phi_{n-1}\}
\end{aligned}$$

which tends to 1 as $n \to \infty$ for all $x, y \in G$. So we can conclude that $T(x, y) = T'(x, y)$ for all $x, y \in G$. This proves the uniqueness of T. □

3. Approximation of sextic mappings on the PN spaces

In this section, we investigate the stability problem for system of functional equations (1.10) on the PN spaces.

Theorem 3.1. *Let $s \in \{-1, 1\}$ be fixed. Let G be a r-divisible group and (Y, ν, Π_T) be a Šerstnev PB space. Let $\Phi, \Psi, \Upsilon : G \times G \times G \to [0, \infty)$ be functions such that*

(3.1)

$$\begin{cases}
\Phi_n^s(x, 0, y, z)(t) := \{\Phi(a^{-sn+\frac{3s-1}{2}}x, 0, a^{-sn+\frac{3s-1}{2}}y, a^{-sn+\frac{3s-1}{2}}z)(2a^{(9s+8)-6sn}t)\}, \\
\Psi_n^s(x, y, 0, z)(t) := \{\Psi(a^{-sn+\frac{3s+1}{2}}x, a^{-sn+\frac{3s-1}{2}}y, 0, a^{-sn+\frac{3s-1}{2}}z)(2a^{(9s+6)-6sn}t)\}, \\
\Upsilon_n^s(x, y, z, 0)(t) := \{\Upsilon(a^{-sn+\frac{3s+1}{2}}x, a^{-sn+\frac{3s+1}{2}}y, a^{-sn+\frac{3s-1}{2}}z, 0)(2a^{(9s+3)-6sn}t)\},
\end{cases}$$

$n = 1, 2, \ldots,$

$$\begin{cases}
\Theta_1 := \Pi_T\{\Upsilon_1^s(x, y, z, 0)(t), \Pi_T\{\Psi_1^s(x, y, 0, z)(t), \Phi_1^s(x, 0, y, z)(t)\}\} \\
\Theta_n := \Pi_T\{\Pi_T\{\Upsilon_n^s(x, y, z, 0)(t), \Pi_T\{\Psi_n^s(x, y, 0, z)(t), \Phi_n^s(x, 0, y, z)(t)\}\}, \Theta_{n-1}\}
\end{cases}$$

for $n > 1$,

for all $x, y, z \in G$ and

$$
\begin{aligned}
&\lim_{n\to\infty} \Phi(a^{sn}x_1, a^{sn}x_2, a^{sn}y, a^{sn}z)(a^{-6sn}t) \\
=&\lim_{n\to\infty} \Psi(a^{sn}x, a^{sn}y_1, a^{sn}y_2, a^{sn}z)(a^{-6sn}t) \\
=&\lim_{n\to\infty} \Upsilon(a^{sn}x, a^{sn}y, a^{sn}z_1, a^{sn}z_2)(a^{-6sn}t) = 1
\end{aligned} \tag{3.2}
$$

for all $x, y, x_1, x_2, y_1, y_2, z_1, z_2 \in G$. Let $\hat{\Theta} = \lim_{n\to\infty} \Theta_n = 1$. If $f : G \times G \times G \to Y$ is a function such that $f(x, 0, z) = 0$ for all $x, z \in G$ and that

$$
\begin{aligned}
&\nu \left(f(ax_1 + bx_2, y, z) + f(ax_1 - bx_2, y, z) - 2af(x_1, y, z)\right)(t) \\
&\geq \Phi(x_1, x_2, y, z)(t),
\end{aligned} \tag{3.3}
$$

$$
\begin{aligned}
&\nu \left(f(x, ay_1 + by_2, z) + f(x, ay_1 - by_2, z) - 2a^2 f(x, y_1, z) - 2b^2 f(x, y_2, z)\right)(t) \\
&\geq \Psi(x, y_1, y_2, z)(t),
\end{aligned} \tag{3.4}
$$

$$
\begin{aligned}
&\nu(f(x, y, az_1 + bz_2) + f(x, y, az_1 - bz_2) - ab^2(f(x, y, z_1 + z_2) + f(x, y, z_1 - z_2)) \\
&- 2a(a^2 - b^2)f(x, y, z_1))(t) \geq \Upsilon(x, y, z_1, z_2)(t)
\end{aligned} \tag{3.5}
$$

for all $x, y, x_1, x_2, y_1, y_2, z_1, z_2 \in G$, then there exists a unique quintic function $T : G \times G \times G \to Y$ satisfying (1.9) and

$$\nu \left(f(x, y, z) - T(x, y, z)\right)(t) \geq \hat{\Theta} \tag{3.6}$$

for all $x, y, z \in G$.

Proof. Putting $x_1 = 2x$ and $x_2 = 0$ and replacing y, z by $2y, 2z$ in (3.3), we get

$$\nu(f(2ax, 2y, 2z) - af(2x, 2y, 2z))(\frac{1}{2}t) \geq \Phi(2x, 0, 2y, 2z)(t) \tag{3.7}$$

for all $x, y, z \in G$. Putting $y_1 = 2y$ and $y_2 = 0$ and replacing x, z by $2ax, 2z$ in (3.4), we get

$$\nu(f(2ax, 2ay, 2z) - a^2 f(2ax, 2y, 2z))(\frac{1}{2}t) \geq \Psi(2ax, 2y, 0, 2z)(t) \tag{3.8}$$

for all $x, y, z \in G$. Putting $z_1 = 2z$ and $z_2 = 0$ and replacing x, y by $2ax, 2ay$ in (3.5), we get

$$\nu(f(2ax, 2ay, 2az) - a^3 f(2ax, 2ay, 2z))(\frac{1}{2}t) \geq \Upsilon(2ax, 2ay, 2z, 0)(t) \tag{3.9}$$

for all $x, y, z \in G$. Thus

$$
\begin{aligned}
&\nu(f(2ax, 2ay, 2az) - a^6 f(2x, 2y, 2z))(t) \\
&\geq \Pi_T\{\Upsilon(2ax, 2ay, 2z, 0)(2t), \Pi_T\{\Psi(2ax, 2y, 0, 2z)(2a^3t), \Phi(2x, 0, 2y, 2z)(2a^5t)\}\}
\end{aligned} \tag{3.10}
$$

for all $x, y, z \in G$. Replacing x, y and z by $\frac{x}{2}$, $\frac{y}{2}$ and $\frac{z}{2}$ in (3.10), we have

$$\begin{aligned}(3.11)\qquad &\nu(f(ax, ay, az) - a^6 f(x, y, z))(t) \\ &\geq \Pi_T\{\Upsilon(ax, ay, z, 0)(2t), \Pi_T\{\Psi(ax, y, 0, z)(2a^3t), \Phi(x, 0, y, z)(2a^5t)\}\}\end{aligned}$$

for all $x, y, z \in G$. It follows from (3.11) that

(3.12)

$$\begin{aligned}&\nu(a^{-6} f(ax, ay, az) - f(x, y, z))(t) \\ &\geq \Pi_T\{\Upsilon(ax, ay, z, 0)(2a^6t), \Pi_T\{\Psi(ax, y, 0, z)(2a^9t), \Phi(x, 0, y, z)(2a^{11}t)\}\}\end{aligned}$$

and

(3.13)

$$\begin{aligned}&\nu(a^6 f(a^{-1}x, a^{-1}y, a^{-1}z) - f(x, y, z))(t) \\ &\geq \Pi_T\{\Upsilon(x, y, a^{-1}z, 0)(2at), \Pi_T\{\Psi(x, a^{-1}y, 0, a^{-1}z)(2a^3t), \Phi(a^{-1}x, 0, a^{-1}y, a^{-1}z)(2a^5t)\}\}\end{aligned}$$

for all $x, y, z \in G$. From the inequalities (3.12) and (3.13) we use iterative methods and induction on n and apply defined sequence in (3.1) to prove our next relation

(3.14)

$$\begin{aligned}&\nu\left(a^{-6sn} f(a^{sn}x, a^{sn}y, a^{sn}z) - f(x, y, z)\right)(t) \\ &\geq \Pi_T\{\Pi_T\{\Upsilon_n^s(x, y, , z, 0)(t), \Pi_T\{\Phi_n^s(x, 0, y, z)(t), \Psi_n^s(x, y, 0, z)(t)\}\}, \Theta_{n-1}\}\end{aligned}$$

for all $x, y, z \in G$. So

(3.15)

$$\begin{aligned}&\nu\left(a^{-6s(n+m)} f(a^{s(n+m)}x, a^{s(n+m)}y, a^{s(n+m)}z) - f(a^{sm}x, a^{sm}y, a^{sm}z)\right)(t) \\ &\geq \Pi_T\{\Pi_T\{\Upsilon_{n+m}^s(x, y, , z, 0)(t), \Pi_T\{\Phi_{n+m}^s(x, 0, y, z)(t), \Psi_{n+m}^s(x, y, 0, z)(t)\}\}, \Theta_{(n+m)-1}\}\end{aligned}$$

for all nonnegative integers n and m and for all $x, y, z \in G$. By the assumptions, (3.15) shows that the sequence $\{a^{-6sn} f(a^{sn}x, a^{sn}y, a^{sn}z)\}$ is a Cauchy sequence in Y for all $x, y, z \in G$. Since Y is a Banach space, it follows that the sequence $\{a^{-6sn} f(a^{sn}x, a^{sn}y, a^{sn}z)\}$ converges for all $x, y, z \in G$. We define the function $T : G \times G \times G \to Y$ by

$$(3.16)\qquad T(x, y, z) = \lim_{n\to\infty} a^{-6sn} f(a^{sn}x, a^{sn}y, a^{sn}z)$$

for all $x, y, z \in G$. It follows from (3.3), (3.4) and (3.5) that

(3.17)

$$\begin{aligned}&\nu(T(ax_1 + bx_2, y, z) + T(ax_1 - bx_2, y, z) - 2aT(x_1, y, z))(t) \\ &= \lim_{n\to\infty} \nu(a^{-6sn} f(a^{sn}(ax_1 + bx_2), a^{sn}y, a^{sn}z) + a^{-6sn} f(a^{sn}(ax_1 - bx_2), \\ &\quad a^{sn}y, a^{sn}z) - 2aa^{-6sn} f(a^{sn}x_1, a^{sn}y, a^{sn}z))(t) \\ &\geq \lim_{n\to\infty} \Phi(a^{sn}x_1, a^{sn}x_2, a^{sn}y, a^{sn}z)(a^{-6sn}t) = 1,\end{aligned}$$

(3.18)
$$\begin{aligned}
&\nu(T(x,ay_1+by_2,z)+T(x,ay_1-by_2,z)-2a^2T(x,y_1,z)-2b^2T(x,y_2,z))(t)\\
&= \lim_{n\to\infty}\nu(a^{-6sn}f(a^{sn}x,a^{sn}(ay_1+by_2),a^{sn}z)+f(a^{sn}x,a^{sn}ay_1-a^{sn}by_2,a^{sn}z)\\
&\quad -2a^2a^{-6sn}f(a^{sn}x,a^{sn}y_1,a^{sn}z)+2b^2a^{-6sn}f(a^{sn}x,a^{sn}y_2,a^{sn}z))(t)\\
&\geq \lim_{n\to\infty}\Psi(a^{sn}x,a^{sn}y_1,a^{sn}y_2,a^{sn}z)(a^{-6sn}t)=1,
\end{aligned}$$

(3.19)
$$\begin{aligned}
&\nu(T(x,y,az_1+bz_2)+T(x,y,az_1-bz_2)-ab^2(T(x,y,z_1+z_2)\\
&\quad -T(x,y,z_1-z_2))-2a(a^2-b^2)T(x,y,z_1))(t)\\
&= \lim_{n\to\infty}\nu(a^{-6sn}f(a^{sn}x,a^{sn}y,a^{sn}(az_1+bz_2))+a^{-6sn}f(a^{sn}x,a^{sn}y,a^{sn}(az_1-bz_2))\\
&\quad -ab^2a^{-6sn}(f(a^{sn}x,a^{sn}y,a^{sn}(z_1+z_2))+f(a^{sn}x,a^{sn}y,a^{sn}(z_1-z_2)))\\
&\quad -2aa^{-6sn}(a^2-b^2)f(a^{sn}x,a^{sn}y,a^{sn}z_1))(t)\\
&\geq \lim_{n\to\infty}\Upsilon(a^{sn}x,a^{sn}y,a^{sn}z_1,a^{sn}z_2)(a^{-6sn}t)=1
\end{aligned}$$

for all $x, y, x_1, x_2, y_1, y_2, z_1, z_2 \in G$. This means that T satisfies (1.10), that is, T is sextic. Moreover, passing the limit $n \to \infty$ in (3.15), we get the inequality (3.6). Now, let $T' : G \times G \times G \to Y$ be another sextic function satisfying (1.10) and (3.6). By Proposition 1.2 we have $a^{-6sn}T'(a^{sn}x, a^{sn}y, a^{sn}z) = T'(x, y, z)$ for all $x, y, z \in G$. Therefore we conclude that

(3.20)
$$\begin{aligned}
&\nu(T(x,y,z),T'(x,y,z))(t)\\
&= \lim_{n\to\infty}\nu(a^{-6sn}f(a^{sn}x,a^{sn}y,a^{sn}z)-a^{-6sn}T'(a^{sn}x,a^{sn}y,a^{sn}z))(t)\\
&\geq \lim_{n\to\infty}\Pi_T\{\Pi_T\{\Upsilon_n^s(a^{sn}x,a^{sn}y,a^{sn}z,0)(a^{-6sn}t),\Pi_T\{\Phi_n^s(a^{sn}x,0,a^{sn}y,a^{sn}z)\\
&\quad (a^{-6sn}t),\Psi_n^s(a^{sn}x,a^{sn}y,0,a^{sn}z)(a^{-6sn}t)\}\},\Theta_{n-1}\}
\end{aligned}$$

which tends to 1 as $n \to \infty$ for all $x, y \in G$. So we can conclude that $T(x, y, z) = T'(x, y, z)$ for all $x, y, z \in G$. This proves the uniqueness of T. □

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