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H-SLANT SUBMERSIONS

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Abstract. In this paper, we define the almost h-slant submersion and the h-slant submersion which may be the extended version of the slant submersion [11]. And then we obtain some theorems which come from the slant submersion's cases. Finally, we construct some examples for the almost h-slant submersions and the h-slant submersions.

1. Introduction

Given a C^∞-submersion F from a Riemannian manifold (M, g_M) onto a Riemannian manifold (N, g_N), there are several kinds of submersions according to the conditions on it: e.g. Riemannian submersion ([5], [10]), slant submersion ([3], [11]), almost Hermitian submersion [12], quaternionic submersion [6], etc. As we know, Riemannian submersions are related with physics and have their applications in the Yang-Mills theory ([2], [13]), Kaluza-Klein theory ([1], [7]), supergravity and superstring theories ([8], [9]), etc. And the quaternionic Kähler manifolds have applications in physics as the target spaces for nonlinear σ-models with supersymmetry [4]. The paper is organized as follows. In Section 2 we recall some notions needed for this paper. In Section 3 we give the definitions of the almost h-slant submersion and the h-slant submersion and obtain some interesting properties about them. In Section 4 we construct some examples for the almost h-slant submersions and the h-slant submersions.

2. Preliminaries

Let (M, E, g) be an almost quaternionic Hermitian manifold, where M is a $4n$-dimensional differentiable manifold, g is a Riemannian metric on M, and E is a rank 3 subbundle of $\mathrm{End}(TM)$ such that for any point $p \in M$ with its some neighborhood U, there exists a local basis $\{J_1, J_2, J_3\}$ of sections of E on U satisfying for all $\alpha \in \{1, 2, 3\}$

$$J_\alpha^2 = -id, \quad J_\alpha J_{\alpha+1} = -J_{\alpha+1} J_\alpha = J_{\alpha+2},$$

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$$g(J_\alpha X, J_\alpha Y) = g(X, Y) \quad \text{for all vector fields } X, Y \text{ on } M,$$

where the indices are taken from $\{1, 2, 3\}$ modulo 3. The above basis $\{J_1, J_2, J_3\}$ is said to be a *quaternionic Hermitian basis*. We call (M, E, g) a *quaternionic Kähler manifold* if there exist locally defined 1-forms $\omega_1, \omega_2, \omega_3$ such that for $\alpha \in \{1, 2, 3\}$

$$\nabla_X J_\alpha = \omega_{\alpha+2}(X) J_{\alpha+1} - \omega_{\alpha+1}(X) J_{\alpha+2} \quad \text{for any vector field } X \text{ on } M,$$

where the indices are taken from $\{1, 2, 3\}$ modulo 3. If there exists a global parallel quaternionic Hermitian basis $\{J_1, J_2, J_3\}$ of sections of E on M, then (M, E, g) is said to be *hyperkähler*. Furthermore, we call (J_1, J_2, J_3, g) a *hyperkähler structure* on M and g a *hyperkähler metric*. Let (M, g_M) and (N, g_N) be Riemannian manifolds and $F : M \mapsto N$ a C^∞-submersion. The map F is said to be *Riemannian submersion* if the differential F_* preserves the lengths of horizontal vectors [6]. Let (M, g_M, J) be an almost Hermitian manifold. A Riemannian submersion $F : (M, g_M, J) \mapsto (N, g_N)$ is called a *slant submersion* if the angle $\theta = \theta(X)$ between JX and the space $\ker(F_*)_p$ is constant for any nonzero $X \in T_pM$ and $p \in M$ [11]. We call θ a *slant angle*. For $X \in \Gamma(\ker F_*)$, we have

$$JX = \phi X + \omega X,$$

where ϕX and ωX are the vertical and horizontal components of JX, respectively. For $Z \in \Gamma((\ker F_*)^\perp)$, we get

$$JZ = BZ + CZ,$$

where BZ and CZ are the vertical and horizontal components of JZ, respectively [11]. Let (M, E_M, g_M) and (N, E_N, g_N) be almost quaternionic Hermitian manifolds. A map $F : M \mapsto N$ is called a *(E_M, E_N)-holomorphic map* if given a point $x \in M$, for any $J \in (E_M)_p$ there exists $J' \in (E_N)_{f(x)}$ such that

$$F_* \circ J = J' \circ F_*.$$

A Riemannian submersion $F : M \mapsto N$ which is a (E_M, E_N)-holomorphic map is called a *quaternionic submersion*. Moreover, if (M, E_M, g_M) is a quaternionic Kähler manifold (or a hyperkähler manifold), then we say that F is a *quaternionic Kähler submersion* (or a *hyperkähler submersion*) [6].

Let (M, g_M) and (N, g_N) be Riemannian manifolds and $F : (M, g_M) \mapsto (N, g_N)$ a smooth map. The second fundamental form of F is given by

$$(\nabla F_*)(X, Y) := \nabla_{F_* X} F_* Y - F_*(\nabla_X Y) \quad \text{for } X, Y \in \Gamma(TM),$$

where we denote conveniently by ∇ the Levi-Civita connections of the metrics g_M and g_N. Recall that F is said to be *harmonic* if $\mathrm{trace}(\nabla F_*) = 0$ and F is called a *totally geodesic* map if $(\nabla F_*)(X, Y) = 0$ for $X, Y \in \Gamma(TM)$.

3. H-slant submersions

Definition 3.1. Let (M, E, g_M) be an almost quaternionic Hermitian manifold and (N, g_N) a Riemannian manifold. A Riemannian submersion $F : (M, E, g_M) \mapsto (N, g_N)$ is called an *almost h-slant submersion* if given a point $p \in M$ with its some neighborhood U, there exists a quaternionic Hermitian basis $\{I, J, K\}$ of sections of E on U such that for $R \in \{I, J, K\}$ the angle $\theta_R = \theta_R(X)$ between RX and the space $\ker(F_*)_q$ is constant for nonzero $X \in \ker(F_*)_q$ and $q \in U$.

We call such a basis $\{I, J, K\}$ an *almost h-slant basis*.

Definition 3.2. Let (M, E, g_M) be an almost quaternionic Hermitian manifold and (N, g_N) a Riemannian manifold. A Riemannian submersion $F : (M, E, g_M) \mapsto (N, g_N)$ is called a *h-slant submersion* if given a point $p \in M$ with its some neighborhood U, there exists a quaternionic Hermitian basis $\{I, J, K\}$ of sections of E on U such that for $R \in \{I, J, K\}$ the angle $\theta_R = \theta_R(X)$ between RX and the space $\ker(F_*)_q$ is constant for nonzero $X \in \ker(F_*)_q$ and $q \in U$, $\theta = \theta_I = \theta_J = \theta_K$.

We call such a basis $\{I, J, K\}$ a *h-slant basis* and the angle θ *h-slant angle*. Let $F : (M, E, g_M) \mapsto (N, g_N)$ be an almost h-slant submersion. Then for $X \in \Gamma(\ker F_*)$, we have

$$RX = \phi_R X + \omega_R X,$$

where $\phi_R X$ and $\omega_R X$ are the vertical and horizontal parts of RX, respectively, for $R \in \{I, J, K\}$. For $Z \in \Gamma((\ker F_*)^\perp)$, we get

$$RZ = B_R Z + C_R Z,$$

where $B_R Z$ and $C_R Z$ are the vertical and horizontal components of RZ, respectively, for $R \in \{I, J, K\}$.

Note that we denote the projection morphisms on the distributions $\ker F_*$ and $(\ker F_*)^\perp$ by $\mathcal{V}$ and $\mathcal{H}$, respectively. Define the tensor $\mathcal{T}$ and $\mathcal{A}$ by

$$\mathcal{A}_E F = \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F,$$
$$\mathcal{T}_E F = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F$$

for vector fields E, F on M, where ∇ is the Levi-Civita connection of g_M.

Theorem 3.1. *Let F be an almost h-slant submersion from an almost quaternionic Hermitian manifold (M, E, g_M) onto a Riemannian manifold (N, g_N). Then we get*

$$\phi_R^2 X = -\cos^2\theta_R X \quad \text{for } X \in \Gamma(\ker F_*) \text{ and } R \in \{I, J, K\},$$

where $\{I, J, K\}$ is an almost h-slant basis with the slant angles $\{\theta_I, \theta_J, \theta_K\}$.

Proof. By Theorem 3.1 of [11], we have the result. □

Lemma 3.1. *Let F be an almost h-slant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an almost h-slant basis with the slant angles $\{\theta_I, \theta_J, \theta_K\}$. If ω_R is parallel, then we have $\mathcal{T}_{\phi_R X}\phi_R X = -\cos^2\theta_R \mathcal{T}_X X$ for $X \in \Gamma(\ker F_*)$ and $R \in \{I, J, K\}$.*

Proof. By Lemma 3.1 of [11], we get the result. □

Lemma 3.2. *Let F be a h-slant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is a h-slant basis with the h-slant angle $\theta = 0$. Then we have*

$$\mathcal{H}\nabla_{RX}RX = -\mathcal{H}\nabla_X X \quad \text{and} \quad [RX, X] \in \Gamma(\ker F_*)$$

for $X \in \Gamma(\ker F_*)$ and $R \in \{I, J, K\}$.

Proof. Since $\theta = 0$, by Lemma 3.1, we easily get

$$\mathcal{H}\nabla_{RX}RX = -\mathcal{H}\nabla_X X$$

for $X \in \Gamma(\ker F_*)$ and $R \in \{I, J, K\}$. For $X \in \Gamma(\ker F_*)$, $Z \in \Gamma((\ker F_*)^\perp)$, and $R \in \{I, J, K\}$, we have

$$\begin{aligned} g_M(\mathcal{H}\nabla_{RX}RX, Z)Z &= -g_M(\nabla_{RX}X, RZ) \\ &= -g_M(\nabla_X RX + [RX, X], RZ) \\ &= g_M(-\mathcal{H}\nabla_X X + \mathcal{H}R[RX, X], Z) \end{aligned}$$

so that we obtain $R[RX, X] \in \Gamma(\ker F_*)$, which implies $[RX, X] \in \Gamma(\ker F_*)$ since $\theta = 0$. □

Remark 3.1. Let F be a hyperkähler submersion from a hyperkähler manifold (M, I, J, K, g_M) onto an almost quaternionic Hermitian manifold (N, E, g_N). Then it is easy to see the following [6]:

(a) the fibers are hyperkähler manifolds.
(b) the manifold (N, E, g_N) is also hyperkähler.
(c) the map F is a h-slant submersion with the h-slant angle $\theta = 0$.

Theorem 3.2. *Let F be an almost h-slant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an almost h-slant basis. If ω_R is parallel for some $R \in \{I, J, K\}$, then F is a harmonic map.*

Proof. We may assume that ω_I is parallel. Since $(\nabla F_*)(Z_1, Z_2) = 0$ for $Z_1, Z_2 \in \Gamma((\ker F_*)^\perp)$, we only need to show that $\sum_{i=1}^{2n}(\nabla F_*)(e_i, e_i) = 0$, where $\{e_i\}_{i=1}^{2n}$ is an orthonormal basis of $\ker F_*$.

Using Theorem 3.1, we can choose an orthonormal basis $\{e_i\}_{i=1}^{2n}$ of $\ker F_*$ such that $e_{2j} = \sec\theta_I \phi_I e_{2j-1}$ for $1 \le j \le n$.

Hence,

$$\sum_{i=1}^{2n}(\nabla F_*)(e_i, e_i) = -\sum_{i=1}^{2n} F_*(\mathcal{T}_{e_i}e_i)$$

$$= -\sum_{j=1}^{n} F_*(\mathcal{T}_{e_{2j-1}} e_{2j-1} + \mathcal{T}_{\sec\theta_I \phi_I e_{2j-1}} \sec\theta_I \phi_I e_{2j-1}).$$

Since ω_I is parallel, by Lemma 3.1, we obtain the result. □

Corollary 3.1. *Let F be an almost h-slant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an almost h-slant basis with the slant angles $\{\theta_I, \theta_J, \theta_K\}$ not all non-zeroes. Then F is a harmonic map.*

Proof. We may assume $\theta_I = 0$. Then it implies $\omega_I = 0$ so that ω_I is parallel. By Theorem 3.2, we get the result. □

Theorem 3.3. *Let F be an almost h-slant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an almost h-slant basis with the slant angles $\{\theta_I, \theta_J, \theta_K\}$ all non-zeroes. Then the following conditions are equivalent*:

(a) *the distribution* $\ker F_*$ *defines a totally geodesic foliation on* M,
(b) $g_M(\mathcal{H}\nabla_X \omega_I \phi_I Y, Z) = g_M(\mathcal{H}\nabla_X \omega_I Y, C_I Z) + g_M(\mathcal{T}_X \omega_I Y, B_I Z)$ *for* X, $Y \in \Gamma(\ker F_*)$ *and* $Z \in \Gamma((\ker F_*)^\perp)$,
(c) $g_M(\mathcal{H}\nabla_X \omega_J \phi_J Y, Z) = g_M(\mathcal{H}\nabla_X \omega_J Y, C_J Z) + g_M(\mathcal{T}_X \omega_J Y, B_J Z)$ *for* X, $Y \in \Gamma(\ker F_*)$ *and* $Z \in \Gamma((\ker F_*)^\perp)$,
(d) $g_M(\mathcal{H}\nabla_X \omega_K \phi_K Y, Z) = g_M(\mathcal{H}\nabla_X \omega_K Y, C_K Z) + g_M(\mathcal{T}_X \omega_K Y, B_K Z)$ *for* X, $Y \in \Gamma(\ker F_*)$ *and* $Z \in \Gamma((\ker F_*)^\perp)$.

Proof. Given a complex structure $R \in \{I, J, K\}$, for $X, Y \in \Gamma(\ker F_*)$ and $Z \in \Gamma((\ker F_*)^\perp)$, we have

$$\begin{aligned} g_M(\nabla_X Y, Z) &= g_M(\nabla_X \phi_R Y, RZ) + g_M(\nabla_X \omega_R Y, RZ) \\ &= -g_M(\nabla_X \phi_R^2 Y, Z) - g_M(\nabla_X \omega_R \phi_R Y, Z) \\ &\quad + g_M(\nabla_X \omega_R Y, B_R Z) + g_M(\nabla_X \omega_R Y, C_R Z). \end{aligned}$$

Using Theorem 3.1, we obtain

$$\begin{aligned} g_M(\nabla_X Y, Z) = {} & \cos^2\theta_R g_M(\nabla_X Y, Z) - g_M(\mathcal{H}\nabla_X \omega_R \phi_R Y, Z) \\ & + g_M(\mathcal{T}_X \omega_R Y, B_R Z) + g_M(\mathcal{H}\nabla_X \omega_R Y, C_R Z) \end{aligned}$$

so that

$$\begin{aligned} \sin^2\theta_R g_M(\nabla_X Y, Z) = {} & -g_M(\mathcal{H}\nabla_X \omega_R \phi_R Y, Z) + g_M(\mathcal{T}_X \omega_R Y, B_R Z) \\ & + g_M(\mathcal{H}\nabla_X \omega_R Y, C_R Z) \end{aligned}$$

Hence, we get

$$\text{(a)} \Leftrightarrow \text{(b)}, \ \text{(a)} \Leftrightarrow \text{(c)}, \ \text{and (a)} \Leftrightarrow \text{(d)}.$$

Therefore, we have the result. □

Theorem 3.4. *Let F be an almost h-slant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an almost h-slant basis with the slant angles $\{\theta_I, \theta_J, \theta_K\}$ all non-zeroes. Then the following conditions are equivalent*:

(a) *the distribution $(\ker F_*)^\perp$ defines a totally geodesic foliation on M,*
(b) $g_M(\mathcal{H}\nabla_{Z_1}Z_2, \omega_I\phi_I X) = g_M(\mathcal{A}_{Z_1}B_I Z_2 + \mathcal{H}\nabla_{Z_1}C_I Z_2, \omega_I X)$ *for $X \in \Gamma(\ker F_*)$ and $Z_1, Z_2 \in \Gamma((\ker F_*)^\perp)$,*
(c) $g_M(\mathcal{H}\nabla_{Z_1}Z_2, \omega_J\phi_J X) = g_M(\mathcal{A}_{Z_1}B_J Z_2 + \mathcal{H}\nabla_{Z_1}C_J Z_2, \omega_J X)$ *for $X \in \Gamma(\ker F_*)$ and $Z_1, Z_2 \in \Gamma((\ker F_*)^\perp)$,*
(d) $g_M(\mathcal{H}\nabla_{Z_1}Z_2, \omega_K\phi_K X) = g_M(\mathcal{A}_{Z_1}B_K Z_2 + \mathcal{H}\nabla_{Z_1}C_K Z_2, \omega_K X)$ *for $X \in \Gamma(\ker F_*)$ and $Z_1, Z_2 \in \Gamma((\ker F_*)^\perp)$.*

Proof. For $X \in \Gamma(\ker F_*)$ and $Z_1, Z_2 \in \Gamma((\ker F_*)^\perp)$, we have

$$\begin{aligned} g_M(\nabla_{Z_1}Z_2, X) &= g_M(\nabla_{Z_1}(IZ_2), IX) \\ &= g_M(\nabla_{Z_1}(IZ_2), \phi_I X) + g_M(\nabla_{Z_1}(IZ_2), \omega_I X) \\ &= \cos^2\theta_I \cdot g_M(\nabla_{Z_1}Z_2, X) - g_M(\nabla_{Z_1}Z_2, \omega_I\phi_I X) \\ &\quad + g_M(\mathcal{A}_{Z_1}B_I Z_2 + \mathcal{H}\nabla_{Z_1}C_I Z_2, \omega_I X) \end{aligned}$$

so that

$$\begin{aligned} \sin^2\theta_I \cdot g_M(\nabla_{Z_1}Z_2, X) = &- g_M(\mathcal{H}\nabla_{Z_1}Z_2, \omega_I\phi_I X) \\ &+ g_M(\mathcal{A}_{Z_1}B_I Z_2 + \mathcal{H}\nabla_{Z_1}C_I Z_2, \omega_I X). \end{aligned}$$

Hence, we get (a) $\Leftrightarrow$ (b). Similarly, we can obtain (a) $\Leftrightarrow$ (c) and (a) $\Leftrightarrow$ (d). Therefore, we get the result. □

Theorem 3.5. *Let F be an almost h-slant submersion from a hyperkähler manifold (M, I, J, K, g_M) onto a Riemannian manifold (N, g_N) such that (I, J, K) is an almost h-slant basis with the slant angles $\{\theta_I, \theta_J, \theta_K\}$ all non-zeroes. Then the following conditions are equivalent*:

(a) *F is totally geodesic,*
(b) $g_M(\mathcal{T}_X\omega_I Y, B_I Z_1) + g_M(\mathcal{H}\nabla_X\omega_I Y, C_I Z_1) = g_M(\mathcal{H}\nabla_X\omega_I\phi_I Y, Z_1)$, $g_M(\mathcal{A}_{Z_1}B_I Z_2 + \mathcal{H}\nabla_{Z_1}C_I Z_2, \omega_I X) = -g_M(\mathcal{H}\nabla_{Z_1}\omega_I\phi_I X, Z_2)$ *for $X, Y \in \Gamma(\ker F_*)$ and $Z_1, Z_2 \in \Gamma((\ker F_*)^\perp)$,*
(c) $g_M(\mathcal{T}_X\omega_J Y, B_J Z_1) + g_M(\mathcal{H}\nabla_X\omega_J Y, C_J Z_1) = g_M(\mathcal{H}\nabla_X\omega_J\phi_J Y, Z_1)$, $g_M(\mathcal{A}_{Z_1}B_J Z_2 + \mathcal{H}\nabla_{Z_1}C_J Z_2, \omega_J X) = -g_M(\mathcal{H}\nabla_{Z_1}\omega_J\phi_J X, Z_2)$ *for $X, Y \in \Gamma(\ker F_*)$ and $Z_1, Z_2 \in \Gamma((\ker F_*)^\perp)$,*
(d) $g_M(\mathcal{T}_X\omega_K Y, B_K Z_1) + g_M(\mathcal{H}\nabla_X\omega_K Y, C_K Z_1) = g_M(\mathcal{H}\nabla_X\omega_K\phi_K Y, Z_1)$, $g_M(\mathcal{A}_{Z_1}B_K Z_2 + \mathcal{H}\nabla_{Z_1}C_K Z_2, \omega_K X) = -g_M(\mathcal{H}\nabla_{Z_1}\omega_K\phi_K X, Z_2)$ *for $X, Y \in \Gamma(\ker F_*)$ and $Z_1, Z_2 \in \Gamma((\ker F_*)^\perp)$.*

Proof. Given a complex structure $R \in \{I, J, K\}$, for $X, Y \in \Gamma(\ker F_*)$ and $Z, Z_1, Z_2 \in \Gamma((\ker F_*)^\perp)$, we have

$$g_N((\nabla F_*)(X, Y), F_*Z) = g_M(\nabla_X R\phi_R Y, Z) - g_M(\nabla_X \omega_R Y, RZ)$$

$$
\begin{aligned}
&= g_M(\nabla_X \phi_R^2 Y, Z) + g_M(\nabla_X \omega_R \phi_R Y, Z) \\
&\quad - g_M(\nabla_X \omega_R Y, B_R Z) - g_M(\nabla_X \omega_R Y, C_R Z) \\
&= -\cos^2 \theta_R g_M(\nabla_X Y, Z) + g_M(\mathcal{H}\nabla_X \omega_R \phi_R Y, Z) \\
&\quad - g_M(\mathcal{T}_X \omega_R Y, B_R Z) - g_M(\mathcal{H}\nabla_X \omega_R Y, C_R Z)
\end{aligned}
$$

so that

$$
\begin{aligned}
\sin^2 \theta_R g_N((\nabla F_*)(X, Y), F_* Z) &= g_M(\mathcal{H}\nabla_X \omega_R \phi_R Y, Z) - g_M(\mathcal{T}_X \omega_R Y, B_R Z) \\
&\quad - g_M(\mathcal{H}\nabla_X \omega_R Y, C_R Z).
\end{aligned}
$$

Similarly, we obtain

$$
\begin{aligned}
\sin^2 \theta_R g_N((\nabla F_*)(X, Z_1), F_* Z_2) = &- g_M(\mathcal{H}\nabla_{Z_1} \omega_R \phi_R X, Z_2) \\
&- g_M(\mathcal{A}_{Z_1} B_R Z_2 + \mathcal{H}\nabla_{Z_1} C_R Z_2, \omega_R X).
\end{aligned}
$$

Hence, we get

$$
\text{(a)} \Leftrightarrow \text{(b)}, \ \text{(a)} \Leftrightarrow \text{(c)}, \ \text{and (a)} \Leftrightarrow \text{(d)}.
$$

Therefore, we obtain the result. □

Remark 3.2. Let F be a h-slant submersion from a $4n$-dimensional hyperkähler manifold (M, I, J, K, g_M) onto a $3n$-dimensional Riemannian manifold (N, g_N) such that (I, J, K) is a h-slant basis with the h-slant angle $\theta = \frac{\pi}{2}$. Since

$$
R(\ker F_*) \perp \ker F_* \quad \text{for } R \in \{I, J, K\},
$$

given a local orthonormal frame $\{e_1, \ldots, e_n\}$ of $\ker F_*$, the set $\{e_1, \ldots, e_n, I(e_1), \ldots, I(e_n), J(e_1), \ldots, J(e_n), K(e_1), \ldots, K(e_n)\}$ is a local orthonormal frame of TM so that $\{I(e_1), \ldots, I(e_n), J(e_1), \ldots, J(e_n), K(e_1), \ldots, K(e_n)\}$ is a local orthonormal frame of $(\ker F_*)^\perp$.

Let

$$
e_{n+i} := I(e_i), \ e_{2n+i} := J(e_i), \ e_{3n+i} := K(e_i) \quad \text{for } i \in \{1, \ldots, n\}.
$$

Let

$$
\widetilde{e}_j := F_* e_j \quad \text{for } n+1 \leq j \leq 4n.
$$

Since F is a Riemannian submersion, we know

$$
F_*([X, Y]) = [F_* X, F_* Y] \quad \text{and} \quad F_*(\nabla_X Y) = \nabla_{F_* X} F_* Y
$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$ [11], where we denote conveniently by ∇ the Levi-Civita connections of the metrics g_M and g_N.

Assume that

$$
\nabla_{e_j} e_i = \sum_{k=1}^{4n} \Gamma_{ji}^k e_k
$$

for some Christoffel symbols $\Gamma_{ji}^k, \ 1 \leq i, j, k \leq 4n$.

Clearly,

$$
\Gamma_{ji}^k = -\Gamma_{jk}^i \quad \text{for } 1 \leq i, j, k \leq 4n.
$$

Since $\nabla_{e_j} Re_k = R\nabla_{e_j} e_k$ for $R \in \{I, J, K\}$ and $j, k \in \{1, \dots, 4n\}$, we have

$$\Gamma^l_{jn+k} = -\Gamma^{n+l}_{jk},\ \Gamma^{n+l}_{jn+k} = \Gamma^l_{jk},\ \Gamma^{2n+l}_{jn+k} = -\Gamma^{3n+l}_{jk},\ \Gamma^{3n+l}_{jn+k} = \Gamma^{2n+l}_{jk},$$
$$\Gamma^l_{j2n+k} = -\Gamma^{2n+l}_{jk},\ \Gamma^{n+l}_{j2n+k} = \Gamma^{3n+l}_{jk},\ \Gamma^{2n+l}_{j2n+k} = \Gamma^l_{jk},\ \Gamma^{3n+l}_{j2n+k} = -\Gamma^{n+l}_{jk},$$
$$\Gamma^l_{j3n+k} = -\Gamma^{3n+l}_{jk},\ \Gamma^{n+l}_{j3n+k} = -\Gamma^{2n+l}_{jk},\ \Gamma^{2n+l}_{j3n+k} = \Gamma^{n+l}_{jk},\ \Gamma^{3n+l}_{j3n+k} = \Gamma^l_{jk}$$

for $1 \leq k, l \leq n$ and $1 \leq j \leq 4n$.

Hence,

$$\begin{aligned}
\nabla_{\widetilde{e}_j} \widetilde{e}_i &= F_*(\nabla_{e_j} e_i) \\
&= \sum_{l=n+1}^{4n} \Gamma^l_{ji} \widetilde{e}_l \\
&= \begin{cases} \sum_{l=1}^{n} (\Gamma^l_{jk} \widetilde{e}_{n+l} - \Gamma^{3n+l}_{jk} \widetilde{e}_{2n+l} + \Gamma^{2n+l}_{jk} \widetilde{e}_{3n+l}), & i = n+k \\ \sum_{l=1}^{n} (\Gamma^{3n+l}_{jk} \widetilde{e}_{n+l} + \Gamma^l_{jk} \widetilde{e}_{2n+l} - \Gamma^{n+l}_{jk} \widetilde{e}_{3n+l}), & i = 2n+k \\ \sum_{l=1}^{n} (-\Gamma^{2n+l}_{jk} \widetilde{e}_{n+l} + \Gamma^{n+l}_{jk} \widetilde{e}_{2n+l} + \Gamma^l_{jk} \widetilde{e}_{3n+l}), & i = 3n+k \end{cases}
\end{aligned}$$

for $1 \leq k \leq n$.

4. Examples

Example 4.1. Define a map $F : \mathbb{R}^4 \mapsto \mathbb{R}^3$ by

$$F(x_1, \dots, x_4) = (x_1 \sin\alpha - x_3 \cos\alpha, x_2, x_4).$$

Then the map F is a h-slant submersion with the h-slant angle $\theta = \frac{\pi}{2}$.

Example 4.2. Let $F : \mathbb{R}^4 \mapsto \mathbb{R}^3$ be a Riemannian submersion. Then the map F is a h-slant submersion with the h-slant angle $\theta = \frac{\pi}{2}$.

We can check it as follows: Given coordinates (x_1, x_2, x_3, x_4) on $\mathbb{R}^4$, we can naturally choose the complex structures I, J, and K on $\mathbb{R}^4$ defined by

$$I(\frac{\partial}{\partial x_1}) = \frac{\partial}{\partial x_2}, I(\frac{\partial}{\partial x_2}) = -\frac{\partial}{\partial x_1}, I(\frac{\partial}{\partial x_3}) = \frac{\partial}{\partial x_4}, I(\frac{\partial}{\partial x_4}) = -\frac{\partial}{\partial x_3},$$
$$J(\frac{\partial}{\partial x_1}) = \frac{\partial}{\partial x_3}, J(\frac{\partial}{\partial x_2}) = -\frac{\partial}{\partial x_4}, J(\frac{\partial}{\partial x_3}) = -\frac{\partial}{\partial x_1}, J(\frac{\partial}{\partial x_4}) = \frac{\partial}{\partial x_2},$$
$$K(\frac{\partial}{\partial x_1}) = \frac{\partial}{\partial x_4}, K(\frac{\partial}{\partial x_2}) = \frac{\partial}{\partial x_3}, K(\frac{\partial}{\partial x_3}) = -\frac{\partial}{\partial x_2}, K(\frac{\partial}{\partial x_4}) = -\frac{\partial}{\partial x_1}.$$

Since F is a Riemannian submersion, the dimension of the space $\ker(F_*)_p$ is equal to 1 for any $p \in \mathbb{R}^4$. Using the properties $\langle RX, X \rangle = 0$ for $X \in T_p\mathbb{R}^4$ and $R \in \{I, J, K\}$, where $\langle , \rangle$ denotes the Euclidean metric on $\mathbb{R}^4$, we have the result.

Example 4.3. Let (M, I, J, K, g_M) be a $4n$-dimensional hyperkähler manifold and (N, g_N) a $(4n-1)$-dimensional Riemannian manifold. Let $F : (M, I, J, K, g_M) \mapsto (N, g_N)$ be a Riemannian submersion. Then the map F is a h-slant submersion with the h-slant angle $\theta = \frac{\pi}{2}$.

Example 4.4. Define a map $F : \mathbb{R}^8 \mapsto \mathbb{R}^6$ by

$$F(x_1, \ldots, x_8) = (x_1 \sin\alpha - x_3 \cos\alpha, x_2, x_4, x_5 \sin\beta - x_7 \cos\beta, x_6, x_8).$$

Then the map F is a h-slant submersion with the h-slant angle $\theta = \frac{\pi}{2}$.

Example 4.5. Let $(M_1, I_1, J_1, K_1, g_1)$ be a $4m$-dimensional hyperkähler manifold and $(M_2, I_2, J_2, K_2, g_2)$ $4n$-dimensional hyperkähler manifold. Let (N_1, g_1') be a $(4m-1)$-dimensional Riemannian manifold and (N_2, g_2') a $(4n-1)$-dimensional Riemannian manifold. Let $F_i : (M_i, I_i, J_i, K_i, g_i) \mapsto (N_i, g_i')$ be a Riemannian submersion for $i \in \{1, 2\}$. Consider the product map $F_1 \times F_2 : M_1 \times M_2 \mapsto N_1 \times N_2$ given by

$$(F_1 \times F_2)(x, y) = (F_1(x), F_2(y)) \quad \text{for } x \in M_1 \text{ and } y \in M_2.$$

Then the map $F_1 \times F_2$ is a h-slant submersion with the h-slant angle $\theta = \frac{\pi}{2}$.

Example 4.6. Let (M, E, g) be an almost quaternionic Hermitian manifold. Let $\pi : TM \mapsto M$ be the natural projection. Then the map π is a h-slant submersion with the h-slant angle $\theta = 0$ [6].

Example 4.7. Let (M, E_M, g_M) and (N, E_N, g_N) be almost quaternionic Hermitian manifolds. Let $F : M \mapsto N$ be a quaternionic submersion. Then the map F is a h-slant submersion with the h-slant angle $\theta = 0$ [6].

Example 4.8. Define a map $F : \mathbb{R}^4 \mapsto \mathbb{R}^2$ by

$$F(x_1, \ldots, x_4) = \left(\frac{x_1}{\sqrt{2}} - \frac{x_3}{\sqrt{2}}, \frac{x_1}{\sqrt{2}} - \frac{x_4}{\sqrt{2}}\right).$$

Then the differential F_* does not preserves the lengths of horizontal vectors so that F is not a Riemannian submersion. But the map F has the h-slant angle θ with $\cos\theta = \frac{\sqrt{3}}{3}$.

Example 4.9. Define a map $F : \mathbb{R}^8 \mapsto \mathbb{R}^4$ by

$$F(x_1, \ldots, x_8) = \left(\frac{x_1}{\sqrt{2}} - \frac{x_3}{\sqrt{2}}, x_4, \frac{x_5}{\sqrt{2}} - \frac{x_7}{\sqrt{2}}, x_6\right).$$

Then the map F is an almost h-slant submersion with the slant angles $\{\frac{\pi}{4}, \frac{\pi}{2}, \frac{\pi}{4}\}$.

Example 4.10. Define a map $F : \mathbb{R}^4 \mapsto \mathbb{R}^2$ by

$$F(x_1, \ldots, x_4) = (x_1 \cos\alpha - x_3 \sin\alpha, x_2 \sin\beta - x_4 \cos\beta).$$

Then the map F is an almost h-slant submersion with the slant angles $\{\theta_I, \frac{\pi}{2}, \theta_K\}$ such that $\cos\theta_I = |\sin(\alpha + \beta)|$ and $\cos\theta_K = |\cos(\alpha + \beta)|$.

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