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MAXIMALITY PRESERVING CONSTRUCTIONS OF MAXIMAL COMMUTATIVE SUBALGEBRAS OF MATRIX ALGEBRA

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Abstract. Let (R, m_R, k) be a local maximal commutative subalgebra of $M_n(k)$ with nilpotent maximal ideal m_R. In this paper, we will construct a maximal commutative subalgebra R^{ST} which is isomorphic to R and study some interesting properties related to R^{ST}. Moreover, we will introduce a method to construct an algebra in $MC_n(k)$ with $i(m_R) = n$ and $\dim(R) = n$.

1. Introduction

Throughout this paper, (R, m_R, k) is a local maximal commutative subalgebra of $M_n(k)$ with nilpotent maximal ideal m_R and residue class field k. The set of all local maximal commutative subalgebras (R, m_R, k) of $M_n(k)$ will be denoted by $MC_n(k)$. The socle of the algebra R is denoted by $soc(R)$ and $i(m_R)$ is the index of nilpotency of the maximal ideal m_R. Also, we will let E_{ij} be the (i, j)-th matrix unit.

The next theorem is known as the Kravchuk's theorem [10].

Theorem 1.1 ([6], [10]). *Let (R, m_R, k) be an algebra in $MC_n(k)$ with $i(m_R) \geq 3$. Then, the matrix r in m_R can be assumed to be of the following form*:

$$r = \begin{pmatrix} O_{\ell\times\ell} & O_{\ell\times p} & O_{\ell\times q} \\ A(r)_{p\times\ell} & B(r)_{p\times p} & O_{p\times q} \\ C(r)_{q\times\ell} & D(r)_{q\times p} & O_{q\times q} \end{pmatrix}.$$

Here, $n = \ell + p + q$, $\ell \neq 0, p \neq 0, q \neq 0$. Moreover, $soc(R)$ consists of all matrices of the form:

$$r = \begin{pmatrix} O_{(n-q)\times\ell} & O_{(n-q)\times(n-\ell)} \\ C(r)_{q\times\ell} & O_{q\times(n-\ell)} \end{pmatrix}.$$

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Remark 1.2 ([6], [10]). Let (R, m_R, k) be an algebra in $MC_n(k)$. If $i(m_R) = 3$, then the matrix r in m_R can be assumed to be of the following form:

$$r = \begin{pmatrix} O_{\ell\times\ell} & O_{\ell\times p} & O_{\ell\times q} \\ A(r)_{p\times\ell} & O_{p\times p} & O_{p\times q} \\ C(r)_{q\times\ell} & D(r)_{q\times p} & O_{q\times q} \end{pmatrix},$$

where $n = \ell + p + q$, $\ell \neq 0, p \neq 0, q \neq 0$.

Theorem 1.3 ([10]). *Let (R, m_R, k) be an algebra in $MC_n(k)$. Suppose the matrices r_i in m_R of the form*

$$r_i = \begin{pmatrix} O_{\ell\times\ell} & O_{\ell\times p} & O_{\ell\times q} \\ A(r_i)_{p\times\ell} & B(r_i)_{p\times p} & O_{p\times q} \\ C(r_i)_{q\times\ell} & D(r_i)_{q\times p} & O_{q\times q} \end{pmatrix}, \qquad i = 1, 2, \ldots, t$$

constitute a basis for m_R. Then, the rank of the following $p \times \ell t$ matrix H is p :

$$H = \begin{pmatrix} A(r_1) & A(r_2) & \cdots & A(r_t) \end{pmatrix}.$$

In Section 2, we will construct a maximal commutative subalgebra R^{ST} which is isomorphic to R and study some interesting properties related to R^{ST}.

In Section 3, we will find some conditions that the algebra $S = \{r \in R \mid r^{ST} = r\}$ can be an algebra in $MC_n(k)$.

In Section 4, we will introduce a method to construct an algebra in $MC_n(k)$ with $i(m_R) = n$ and $\dim(R) = n$ such that $r = r^{ST}$ for all $r \in R$.

2. ST-isomorphism

In this section, we will first define the skew transpose matrix r^{ST} of a matrix r in $M_{m\times n}(k)$ and construct some interesting isomorphic algebras R^{ST} in $MC_n(k)$.

Definition 2.1. Let $A = (a_{ij})_{m\times n} \in M_{m\times n}(k)$ be a matrix with a_{ij} as its (i, j)-th entry. Define $A^{ST} = (b_{ij})_{n\times m} \in M_{n\times m}(k)$ be the matrix as follows:

$$b_{ij} = a_{(m-j+1)(n-i+1)} \qquad (1 \leq i \leq n,\ 1 \leq j \leq m).$$

We will call the matrix A^{ST} the skew transpose of A.

Thus, the skew transpose A^{ST} of A is the following $n \times m$ matrix:

$$A^{ST} = \begin{pmatrix} a_{mn} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{11} \end{pmatrix}.$$

Let $A = (a_{ij})_{n\times n} \in M_n(k)$ be a square matrix. If we call the line from $(1, n)$-th entry to $(n, 1)$-th entry as the skew-diagonal line, then the skew transpose A^{ST} can be obtained by symmetric moving the entries of A with respect to the skew-diagonal line of A.

For the skew transpose A^{ST} of the matrix A, the following properties can be easily proved.

Theorem 2.2. *Let $A, B \in M_{m\times n}(k)$ and $\alpha \in k$. Then the following properties hold*:

(1) $(A+B)^{ST} = A^{ST} + B^{ST}$,
(2) $(\alpha A)^{ST} = \alpha A^{ST}$,
(3) $(A^{ST})^{ST} = A$,
(4) $(A^T)^{ST} = (A^{ST})^T$, *where* A^T *is the transpose of* A.

Theorem 2.3. *Let* $A \in M_{m\times n}(k)$ *and* $B \in M_{n\times \ell}(k)$. *Then* $(AB)^{ST} = B^{ST}A^{ST}$.

Theorem 2.4. *Suppose* R *is an algebra in* $MC_n(k)$. *Then the algebra* $R^{ST} = \{r^{ST} \mid r \in R\}$ *is also in* $MC_n(k)$.

Proof. Define a map $f : R \to R^{ST}$ by $f(r) = r^{ST}$ for all $r \in R$. Then, straightforward computations show that f is an isomorphism as k-algebras by the properties in Theorem 2.2 and Theorem 2.3. Thus, the algebra R^{ST} is also in $MC_n(k)$. □

Definition 2.5. The isomorphism f in the proof of Theorem 2.4 will be called an ST-isomorphism as k-algebras.

By Theorem 2.4, the ST-isomorphism as k-algebras is a maximality preserving map.

Example 2.6. (1) Let R be the following algebra in $MC_4(k)$:

$$R = \left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ b & a & 0 & 0 \\ c & b & a & 0 \\ d & 0 & 0 & a \end{pmatrix} \mid a, b, c, d \in k \right\}.$$

Then,

$$R^{ST} = \left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & b & a & 0 \\ d & c & b & a \end{pmatrix} \mid a, b, c, d \in k \right\}$$

is also an algebra in $MC_4(k)$.

(2) Let R be the following algebra in $MC_4(k)$:

$$R = \left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ b & a & 0 & 0 \\ c & b & a & 0 \\ d & c & b & a \end{pmatrix} \mid a, b, c, d \in k \right\}.$$

Then, $R^{ST} = R$ and is also an algebra in $MC_4(k)$.

More generally, if we let $R = k[E_{21} + \cdots + E_{nn-1},\ E_{31} + \cdots + E_{nn-2}, \ldots, E_{n1}]$ be an algebra in $MC_n(k)$. Then, $R^{ST} = R$ and is also an algebra in $MC_n(k)$.

Using the properties of skew transpose matrix, the following properties can be proved:

Theorem 2.7. *Suppose R is an algebra in $MC_n(k)$. If $f : R \to R^{ST}$ is the ST-isomorphism as k-algebras, then the following properties hold*:

(1) $\alpha f : R \longrightarrow R^{ST}$ *defined by* $(\alpha f)(r) = \alpha f(r)$ *for all* $r \in R$ *is a maximality preserving isomorphism as* k*-vector spaces for all nonzero* α *in* k.

(2) $f^{-1} : R^{ST} \longrightarrow R$ *is a maximality preserving isomorphism as* k*-algebras.*

(3) $f^T : R \longrightarrow (R^{ST})^T$ *defined by* $f^T(r) = f(r)^T = (r^{ST})^T$ *for all* $r \in R$ *is a maximality preserving isomorphism as* k*-algebras.*

Furthermore, we can consider some maximality preserving maps on $MC_n(k)$ as following theorem:

Theorem 2.8. *Suppose R is an algebra in $MC_n(k)$. Then the following properties hold*:

(1) *For an invertible matrix* $P \in M_n(k)$, *the map* $\psi : R \longrightarrow (PRP^{-1})^{ST}$ *defined by* $\psi(r) = (PrP^{-1})^{ST}$ *for all* $r \in R$ *is a maximality preserving isomorphism as* k*-algebras.*

(2) *For an invertible matrix* $P \in M_n(k)$, *the map* $\phi : R \longrightarrow PR^{ST}P^{-1}$ *defined by* $\phi(r) = Pr^{ST}P^{-1}$ *for all* $r \in R$ *is a maximality preserving isomorphism as* k*-algebras.*

Proof. (1) Since PRP^{-1} is an algebra in $MC_n(k)$, $(PRP^{-1})^{ST}$ is an algebra in $MC_n(k)$ by Theorem 2.4. Thus, the map ψ is a well defined maximality preserving map. Moreover, we can show that ψ is an isomorphism as k-algebras by using Theorem 2.2 and Theorem 2.3. Also, (2) can be proved by similar way. □

We can easily prove the following properties.

Lemma 2.9. *Suppose* (R, m_R, k) *is an algebra in* $MC_n(k)$. *Then, the set* $\Delta = \{r_1, r_2, \ldots, r_t\}$ *is a basis of* m_R *if and only if the set* $\Delta^{ST} = \{r_1^{ST}, r_2^{ST}, \ldots, r_t^{ST}\}$ *is a basis of* m_R^{ST}.

By the straightforward calculations using Lemma 2.9, the next properties hold. We will let $m_{R^{ST}}$ be the maximal ideal of R^{ST} and $i(m_{R^{ST}})$ be the index of nilpotency of maximal ideal $m_{R^{ST}}$.

Theorem 2.10. *Suppose* (R, m_R, k) *is an algebra in* $MC_n(k)$. *Then, we have the following properties*:

(1) $\dim(m_R) = \dim(m_{R^{ST}})$,
(2) $\dim(soc(R)) = \dim(soc(R^{ST}))$,
(3) $i(m_R) = i(m_{R^{ST}})$.

3. Algebras in $MC_n(k)$ with $r^{ST} = r$

Let R be an algebra as in Theorem 1.1 and define a set S as follows:

$$S = \{r \in R \mid r^{ST} = r\}.$$

Then, obviously the maximal ideal of S consists of the following form of matrices:

$$r = \begin{pmatrix} O_{\ell\times\ell} & O_{\ell\times p} & O_{\ell\times\ell} \\ A(r)_{p\times\ell} & B(r)_{p\times p} & O_{p\times\ell} \\ C(r)_{\ell\times\ell} & A(r)^{ST}_{\ell\times p} & O_{\ell\times\ell} \end{pmatrix},$$

where $C(r)^{ST}_{\ell\times\ell} = C(r)_{\ell\times\ell}$ and $B(r)^{ST}_{p\times p} = B(r)_{p\times p}$.

Moreover, we have the following Remark 3.1:

Remark 3.1. Suppose R is an algebra in $MC_n(k)$. Then, the algebra $S = \{r \in R \mid r^{ST} = r\}$ is a commutative subalgebra of R but we can not guarantee that S is an algebra in $MC_n(k)$ since S doesn't contain all the elements of the following form

$$r = \begin{pmatrix} O_{\ell\times\ell} & O & O \\ O & O_{p\times p} & O \\ C(r) & O & O_{\ell\times\ell} \end{pmatrix}$$

which should be in $soc(S)$. Thus, we can not say S is an algebra in $MC_n(k)$.

Example 3.2. Let

$$R = \left\{ \begin{pmatrix} a & 0 & 0 \\ b & a & 0 \\ c & 0 & a \end{pmatrix} \mid a, b, c \in k \right\}.$$

Then, $S = \{r \in R \mid r^{ST} = r\} = k[E_{31}] \notin MC_3(k)$.

But, if $\dim_k(soc(R)) = 1$, then the algebra $S = \{r \in R \mid r^{ST} = r\}$ can be in $MC_n(k)$ as the following Theorem 3.3.

Theorem 3.3. *Suppose (R, m_R, k) is an algebra in $MC_n(k)$ with $i(m_R) = 3$ as in Remark 1.2 and $\dim(soc(R)) = 1$. Let the maximal ideal of S be $m_S = \{r_i \mid i = 1, 2, \ldots, t\}$. If the rank of the matrix $H = (A(r_1), \ldots, A(r_t))$ is p, then S is an algebra in $MC_n(k)$.*

Proof. Obviously, the algebra S is a commutative algebra. Now, let $L \in M_n(k)$ be a matrix in the centralizer of S. Then, $Lr_i = r_iL$ for all $r_i \in m_S$. Let L and r_i be as following block matrices:

$$L = \begin{pmatrix} T_1 & T_2 & T_3 \\ T_4 & T_5 & T_6 \\ T_7 & T_8 & T_9 \end{pmatrix}, \quad r_i = \begin{pmatrix} O_{1\times1} & O & O \\ A(r_i) & O_{p\times p} & O \\ C(r_i) & A(r_i)^{ST} & O_{1\times1} \end{pmatrix},$$

where $T_1 \in k, T_5 \in M_{p\times p}(k), T_9 \in k,\ i = 1, \ldots, t$. Then, from the relation, $Lr_i = r_iL$, the following equations hold:

(1) $T_2A(r_i) + T_3C(r_i) = 0$,
(2) $T_3A(r_i)^{ST} = O_{1\times q}$,
(3) $T_5A(r_i) + T_6C(r_i) = A(r_i)T_1$,
(4) $T_6A(r_i)^{ST} = A(r_i)T_2$,
(5) $A(r_i)T_3 = O_{q\times1}$,

(6) $T_8A(r_i) + T_9C(r_i) = C(r_i)T_1 + A(r_i)^{ST}T_4$,
(7) $T_9A(r_i)^{ST} = C(r_i)T_2 + A(r_i)^{ST}T_5$,
(8) $C(r_i)T_3 + A(r_i)^{ST}T_6 = 0$.

In the equation (1), if we let $C(r_i) = 0 \in k$, then we have $T_2A(r_i) = 0$ for all i. Thus, we obtain $T_2 = O_{1\times p}$ since the rank of the matrix $H = (A(r_1), \ldots, A(r_t))$ is p.

Again, from the equation (1), $T_3C(r_i) = 0$ implies $T_3 = 0$ by letting $C(r_i) \neq 0 \in k$. Thus, $A(r_i)^{ST}T_6 = 0$ for all i in the equation (8) and so $T_6^{ST}A(r_i) = (A(r_i)^{ST}T_6)^{ST} = 0$. Thus, $T_6^{ST} = O_{1\times p}$ and so $T_6 = O_{p\times 1}$.

Now, by letting $A(r_i) = O_{p\times 1}$, and $C(r_i) \neq 0 \in k$ in the equation (6), we have $T_1 = T_9 = a \in k$.

Finally, in the equation (3), by letting $C(r_i) = 0$, we have $T_5A(r_i) = A(r_i)T_1 = aA(r_i)$. This implies, if we let I_p be the $p \times p$ identity matrix, we obtain $(T_5 - aI_p)A(r_i) = O_{p\times 1}$ for all i. Thus, $T_5 - aI_p = O_{p\times p}$ and so $T_5 = aI_p$. Therefore, the matrix L is of the form

$$L = \begin{pmatrix} a & O & O \\ T_4 & aI_p & O \\ T_7 & T_8 & a \end{pmatrix}.$$

In the equation (6), by letting $C(r_i) = 0$, we obtain

$$T_8A(r_i) = A(r_i)^{ST}T_4 = (A(r_i)^{ST}T_4)^{ST} = T_4^{ST}A(r_i)$$

since $A(r_i)^{ST}T_4 \in k$. Thus, we have $(T_8 - T_4^{ST})A(r_i) = 0$ and so $T_8 = T_4^{ST}$. Therefore, L is of the following form:

$$L = \begin{pmatrix} a & O & O \\ T_4 & aI_p & O \\ T_7 & T_4^{ST} & a \end{pmatrix} \in S$$

and we can conclude that the algebra S is in $MC_n(k)$. □

Theorem 3.4. *Suppose (R, m_R, k) is an algebra in $MC_n(k)$ with $i(m_R) = 3$. If S is an algebra in $MC_n(k)$, then* $\dim(soc(R)) = 1$.

Proof. Suppose $\dim_k(soc(R)) \neq 1$. Then, $\dim_k(soc(R)) = q^2$ for some positive integer q. From the condition $r = r^{ST}$, all the matrices in $soc(R)$ can't be contained in $soc(S)$, which contradicts to the fact in Theorem 1.1. Thus we have the result. □

Corollary 3.5. *Suppose (R, m_R, k) is an algebra in $MC_n(k)$ with $i(m_R) = 3$. Furthermore, assume $r^{ST} = r$ for all $r \in R$. Then $R = k[E_{21} + E_{n(n-1)}, \ldots, E_{(n-1)1} + E_{n2}, E_{n1}]$.*

Proof. Since $\dim_k(soc(R)) = 1$ by Theorem 3.4, $\ell = q = 1$ in Theorem 1.1. Also, we may assume R is the algebra of the form in Theorem 1.3. Since the rank of the matrix $H = (A(r_1), \ldots, A(r_t))$ is p and $A(r_1), \ldots, A(r_t)$ are linearly independent, we should have $p = t = n - 2$ and $R = k[E_{21} + E_{n(n-1)}, \ldots, E_{(n-1)1} + E_{n2}, E_{n1}]$. □

Example 3.6. Suppose (R, m_R, k) is an algebra in $MC_4(k)$ with $i(m_R) = 3$ and $r^{ST} = r$ for all $r \in R$. Then, by Theorem 3.4 and Corollary 3.5, we obtain $\dim(soc(R)) = 1$ and $R = k[E_{21} + E_{43}, E_{31} + E_{42}, E_{41}]$.

Corollary 3.7. *Suppose (R, m_R, k) is an algebra in $MC_n(k)$ with $i(m_R) = 3$ and $r^{ST} = r$ for all $r \in R$. Then $\dim_k(R) = n$.*

Proof. Since $R = k[E_{21} + E_{n(n-1)}, \dots, E_{(n-1)1} + E_{n2}, E_{n1}]$ by Corollary 3.5, we obtain $\dim_k(R) = 1 + (n-2) + 1 = n$. □

By Corollary 3.7, we can always construct an algebra $R \in MC_n(k)$ with $i(m_R) = 3$ and $\dim(R) = n$. Furthermore, the next corollary holds.

Corollary 3.8. *Suppose (R, m_R, k) is an algebra in $MC_n(k)$ such that $r^{ST} = r$ for all $r \in R$. Then, $i(m_R) \geq 3$ for all $n \geq 3$.*

Proof. Suppose $i(m_R) = 2$. Then, $m_R = soc(R)$. Since $r^{ST} = r$ for all $r \in R$, $C(r)_{q\times\ell}$ should be square matrix in Theorem 1.1. Hence if $r \in soc(R)$, then $C(r)_{q\times q} = C(r)^{ST}_{q\times q}$ and so $soc(R)$ can't contain all the matrices of the form in Theorem 1.1 which is impossible. Therefore, $i(m_R) \geq 3$ for all $n \geq 3$. □

4. Algebras in $MC_n(k)$ with $i(m_R) = n = \dim(R)$

In this section, we will provide a method to construct algebras (R, m_R, k) in $MC_n(k)$ that $i(m_R) = n = \dim(R)$. Specially, $r = r^{ST}$ for all $r \in m_R$.

Let (B, m_B, k) be a finite dimensional commutative local k-algebra with identity and N a finitely generated faithful B-module. Suppose

$$B \cong \mathrm{Hom}_B(N, N)$$

via the regular representation. Define an algebra R as follows:

$$R = B[X_1, X_2, \dots, X_{n-2}]/I,$$

where I is the following ideal:

$$I = (m_B X_1, \dots, m_B X_{n-2}, X_1^2 - X_2, X_1^3 - X_3, \dots, X_1^{n-2} - X_{n-2}, X_1^{n-1} - z, X_1^n).$$

Here, z is a nonzero element in $soc(B)$ with $\dim_k(Nz) = 1$.

Theorem 4.1. *Suppose R is an algebra as in the above statements. If we let $M = N \oplus (\oplus_{i=1}^{n-2} Nz)$, then the k-algebra R is isomorphic to $\mathrm{Hom}_R(M, M)$ via the regular representation. In other words, R is in $MC_n(k)$, where $n = \dim_k(M)$.*

Proof. Obviously $M = N \oplus (\oplus_{i=1}^{n-2} Nz)$ is a $B[X_1, X_2, \ldots, X_{n-2}]$-module via the following operations:

$$\begin{array}{ccc}(u, u_1z, \ldots, u_{n-2}z)b & = & (bu, u_1zb, \ldots, u_{n-2}zb) \\ (u, u_1z, \ldots, u_{n-2}z)X_1 & = & (u_1z, u_2z, \ldots, u_{n-2}z, uz) \\ (u, u_1z, \ldots, u_{n-2}z)X_2 & = & (u_2z, u_3z, \ldots, uz, u_1z^2) \\ (u, u_1z, \ldots, u_{n-2}z)X_3 & = & (u_3z, u_4z, \ldots, u_1z^2, u_2z^2) \\ \vdots & \vdots & \vdots \\ (u, u_1z, \ldots, u_{n-2}z)X_{n-2} & = & (u_{n-2}z, uz, \ldots, u_{n-3}z^2),\end{array}$$

where $b \in B$ and $u, u_j \in N$ for $j = 1, 2, \ldots, n-2$.

Moreover, if we let x_j is the image of X_j in R for each $j = 1, 2, \ldots, n-2$, then M is an R-module via the following operations:

$$\begin{array}{ccc}(u, u_1z, \ldots, u_{n-2}z)b & = & (bu, u_1zb, \ldots, u_{n-2}zb) \\ (u, u_1z, \ldots, u_{n-2}z)x_1 & = & (u_1z, u_2z, \ldots, u_{n-2}z, uz) \\ (u, u_1z, \ldots, u_{n-2}z)x_2 & = & (u_2z, u_3z, \ldots, uz, u_1z^2) \\ (u, u_1z, \ldots, u_{n-2}z)x_3 & = & (u_3z, u_4z, \ldots, u_1z^2, u_2z^2) \\ \vdots & \vdots & \vdots \\ (u, u_1z, \ldots, u_{n-2}z)x_{n-2} & = & (u_{n-2}z, uz, \ldots, u_{n-3}z^2),\end{array}$$

where $b \in B$ and $u, u_j \in N$ for $j = 1, 2, \ldots, n-2$.

Furthermore, by straightforward calculations, we obtain

$$I \subseteq Ann_{B[X_1, X_2, \ldots, X_{n-2}]}(M).$$

To show the faithfulness of M, let

$$(u, 0, \ldots, 0)(b + \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_{n-2} x_{n-2}) = (0, 0, \ldots, 0)$$

for $u \in N$ and $\alpha_j \in k$ for all j. Then we have

$$(ub, u\alpha_{n-2}z, \ldots, u\alpha_1 z) = (0, 0, \ldots, 0).$$

Since N is a faithful B-module by assumption, we have

$$b = 0, \quad \alpha_j = 0, \qquad j = 1, 2, \ldots, n-2$$

which implies M is a finitely generated faithful R-module.

Now, let $f \in \mathrm{Hom}_R(M, M)$. Define $\phi_1 : N \to M$ and $\phi_2 : M \to N$ by

$$\phi_1(u) = (u, 0, \ldots, 0), \qquad \phi_2(u, u_1z, u_2z, \ldots, u_{n-2}z) = u.$$

Then, obviously ϕ_1 and ϕ_2 are B-module homomorphisms. Moreover, the map $\phi : N \to N$ defined by

$$\phi = \phi_2 f \phi_1$$

is a B-module homomorphism. Since $B \cong \mathrm{Hom}_B(N, N)$ via the regular representation, $\phi = \mu_a$ for some $a \in B$, where $\mu_a : N \to N$ is the natural homomorphism defined by $\mu_a(u) = ua$ for all $u \in N$. Thus, we have

$$\phi_2(f(u, 0, \ldots, 0)) = \phi_2 f \phi_1(u) = \phi(u) = \mu_a(u) = ua.$$

By the definition of ϕ_2, there exist $n-2$ number of B-module homomorphisms $\psi_j; N \to Nz$ for $j = 1, 2, \ldots, n-2$ such that

$$f(u, 0, \ldots, 0) = (ua, \psi_1(u), \psi_2(u), \ldots, \psi_{n-2}(u)).$$

Since $\dim_k(Nz) = 1$, there exists an element $v \in N$ such that $\{vz\}$ is a k-vector space basis of Nz. Thus, there exist $c_1, c_2, \ldots, c_{n-2} \in k$ such that

$$\psi_j(v) = c_j vz, \qquad j = 1, 2, \ldots, n-2.$$

Then, $a + c_{n-2}x_1 + c_{n-2}x_2 + \cdots + c_1 x_{n-2} \in R$ and we want to show

$$f = \mu_{a+c_{n-2}x_1+c_{n-3}x_2+\cdots+c_1x_{n-2}}.$$

Since vz generates Nz, we can write

$$u_j z = s_j vz, \qquad j = 1, 2, \ldots, n-2$$

for some $s_j \in k$, $j = 1, 2, \ldots, n-2$. Thus,

$$(u, u_1 z, u_2 z, \ldots, u_{n-2} z) = (u, s_1 vz, s_2 vz, \ldots, s_{n-2} vz)$$

and so we want to show that

$$\begin{aligned} &f(u, s_1 vz, \ldots, s_{n-2} vz) \\ &= \mu_{a+c_{n-2}x_1+c_{n-3}x_2+\cdots+c_1x_{n-2}}(u, s_1 vz, s_2 vz, \ldots, s_{n-2} vz). \end{aligned}$$

Briefly, let

$$\begin{aligned} r &= a + c_{n-2}x_1 + c_{n-3}x_2 + \cdots + c_1 x_{n-2}, \\ w &= (u, s_1 vz, \ldots, s_{n-2} vz). \end{aligned}$$

Then, $\mu_r(w)$ is as follows:

$$\begin{aligned} \mu_r(w) &= (u, s_1 vz, \ldots, s_{n-2} vz)(a + c_{n-2}x_1 + c_{n-3}x_2 + \cdots + c_1 x_{n-2}) \\ &= (ua, s_1 vza, \ldots, s_{n-2} vza) + (c_{n-2} s_1 vz, c_{n-2} s_2 vz, \ldots, c_{n-2} uz) \\ &\quad + \cdots + (c_1 s_{n-2} vz, c_1 uz, \ldots, c_1 s_{n-3} vz^2). \end{aligned}$$

But, for each i and j, we have

$$\psi_i(s_j v) = s_j \psi_i(v) = s_j c_i vz.$$

Thus, we have the following identities:

$$\begin{aligned} f(0, s_1 vz, \ldots, s_{n-2} vz) &= f((s_{n-2} v, 0, \ldots, 0)x_1 + (s_{n-3} v, 0, \ldots, 0)x_2 \\ &\quad + \cdots + (s_1 v, 0, \ldots, 0)x_{n-2}) \\ &= (s_{n-2} va, \psi_1(s_{n-2} v), \ldots, \psi_{n-2}(s_{n-2} v))x_1 \\ &\quad + (s_{n-3} va, \psi_1(s_{n-3} v), \ldots, \psi_{n-2}(s_{n-3} v))x_2 \\ &\quad + \cdots + (s_1 va, \psi_1(s_1 v), \ldots, \psi_{n-2}(s_1 v))x_{n-2} \\ &= (\psi_1(s_{n-2} v), \psi_2(s_{n-2} v), \ldots, \psi_{n-2}(s_{n-2} v), s_{n-2} vaz) \\ &\quad + (\psi_2(s_{n-3} v), \psi_3(s_{n-3} v), \ldots, s_{n-3} vaz, \psi_1(s_{n-3} v)z) \\ &\quad + \cdots + (\psi_{n-2}(s_1 v), s_1 vaz, \ldots, \psi_{n-3}(s_1 v)z) \end{aligned}$$

$$\begin{aligned}
&= (c_1 s_{n-2} vz, c_2 s_{n-2} vz, \ldots, c_{n-2} s_{n-2} vz, s_{n-2} vaz) \\
&\quad + (c_2 s_{n-3} vz, c_3 s_{n-3} vz, \ldots, s_{n-3} vaz, c_1 s_{n-3} vz^2) \\
&\quad + \cdots + (c_{n-2} s_1 vz, s_1 vaz, \ldots, c_{n-3} s_1 vz^2) \\
&= (c_1 s_{n-2} vz + \cdots + c_{n-2} s_1 vz, \ldots, s_{n-2} vaz).
\end{aligned}$$

Since we can rewrite $uz = svz$ for some $s \in k$, we have the following identities:

$$\begin{aligned}
(\psi_1(u), \psi_2(u), \ldots, \psi_{n-2}(u), uaz) &= (ua, \psi_1(u), \psi_2(u), \ldots, \psi_{n-2}(u))x_1 \\
&= f(u, 0, \ldots, 0)x_1 = f((u, 0, \ldots, 0)x_1) \\
&= f(0, 0, \ldots, 0, uz) = f(0, 0, \ldots, 0, svz) \\
&= (c_1 svz, c_2 svz, \ldots, c_{n-2} svz, svaz) \\
&= (c_1 uz, c_2 uz, \ldots, c_{n-2} uz, suz).
\end{aligned}$$

This implies that

$$\psi_j(u) = c_j uz$$

for all $j = 1, 2, \ldots, n-2$.
From the above results, we have the following identity:

$$f(u, 0, \ldots, 0) = (ua, \psi_1(u), \ldots, \psi_{n-2}(u)) = (ua, c_1 uz, \ldots, c_{n-2} uz).$$

Therefore, we have proved

$$f(u, s_1 vz, \ldots, s_{n-2} vz) = \mu_{a + c_{n-2} x_1 + \ldots + c_1 x_{n-2}}(u, s_1 vz, \ldots, s_{n-2} vz)$$

for all $(u, s_1 vz, \ldots, s_{n-2} vz) \in N \oplus (\oplus_{i=1}^{n-2} Nz)$, which implies

$$f = \mu_{a + c_{n-2} x_1 + \cdots + c_1 x_{n-2}}.$$

Since M is a faithful R-module, we can conclude R is isomorphic to

$$\mathrm{Hom}_R(M, M)$$

via the regular representation and so R is in $MC_n(k)$. □

Definition 4.2. We will call the algebra R of the form in Theorem 4.1 a C_3-construction.

Corollary 4.3. *Let (R, m_R, k) be an algebra as in Theorem 4.1. Then,*

(1) $\dim_k(R) = \dim_k(B) + (n-2)$.
(2) $x_1^j = x_j$ *for all* $j = 1, 2, \ldots, n-2$.
(3) $x_1^{n-1} = z$.
(4) m_R *is an ideal generated by* x_1.
(5) $i(m_R) = n$.

Remark 4.4. If we choose an algebra B with $\dim(B) = 2$ in Theorem 4.1, then $\dim(R) = n$ and so we can always construct algebras $(R, m_R, k) \in MC_n(k)$ with $i(m_R) = n = \dim(R)$.

The following is an example of a C_3-construction.

Example 4.5. Let (R, m_R, k) be a k-algebra in $MC_n(k)$ with $m_R = (E_{21} + E_{32} + \cdots + E_{n(n-1)})$, the ideal generated by $E_{21} + E_{32} + \cdots + E_{n(n-1)}$. Then, R is the algebra defined as following:

$$R = \left\{ \begin{pmatrix} a & 0 & 0 & \cdots\cdots\cdots & 0 & 0 \\ a_1 & a & 0 & \cdots\cdots\cdots & 0 & 0 \\ a_2 & a_1 & a & \cdots\cdots\cdots & 0 & 0 \\ a_3 & a_2 & a_1 & \cdots\cdots\cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n-3} & a_{n-4} & a_{n-5} & \cdots\cdots\cdots & 0 & 0 \\ a_{n-2} & a_{n-3} & a_{n-4} & \cdots\cdots\cdots & a & 0 \\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots\cdots\cdots & a_1 & a \end{pmatrix} \mid a, a_i \in k, i = 1, \ldots, n-1 \right\}$$

Note that $r^{ST} = r$ for all elements $r \in R$. Moreover, if we let

$$\begin{array}{rcl} x_1 & = & E_{21} + E_{32} + \cdots + E_{n(n-1)} \\ x_2 & = & E_{31} + E_{42} + \cdots + E_{n(n-2)} \\ \vdots & \vdots & \vdots \\ x_{n-2} & = & E_{(n-1)1} + E_{n2}, \end{array}$$

then

(1) $\dim_k(R) = n$.
(2) m_R is an ideal generated by x_1.
(3) $x_1^j = x_j$ for all $j = 1, 2, \ldots, n-2$.
(4) $x_1^{n-1} = E_{n1}$.
(5) $i(m_R) = n$.
(6) $r^{ST} = r$ for all $r \in R$.

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