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HYPERSURFACES WITH CONSTANT k-TH MEAN CURVATURE AND TWO DISTINCT PRINCIPAL CURVATURES IN SPHERES

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Abstract. In this paper, we investigate the hypersurface M in a unit sphere with constant k-th mean curvature and two distinct principal curvatures, and characterize such a hypersurface.

1. Introduction and main result

Let M be an n-dimensional hypersurface in an $(n+1)$-dimensional unit sphere $S^{n+1}(1)$. It is well known that a compact minimal hypersurface M with $S = n$ in $S^{n+1}(1)$ is isometric to a Clifford torus $S^1(\sqrt{\frac{1}{n}}) \times S^{n-1}(\sqrt{\frac{n-1}{n}})$, where S is the squared norm of the second fundamental form of the hypersurface (cf. [5], [7], [8]). In 1970, Otsuki [10] investigated the converse problem by using differential equation and proved that Riemannian product $S^k(\sqrt{\frac{k}{n}}) \times S^{n-k}(\sqrt{\frac{n-k}{n}})$ is the only compact minimal hypersurface in $S^{n+1}(1)$ with two distinct principal curvatures whose multiplicities are greater than 1. Furthermore, for compact minimal hypersurfaces with two distinct principal curvatures, one of which is simple, Otsuki also constructed infinitely many immersed minimal hypersurfaces other than the Clifford torus $S^1(\sqrt{\frac{1}{n}}) \times S^{n-1}(\sqrt{\frac{n-1}{n}})$ which are not congruent to each other.

In order to characterize the geometric structure of M, it is natural to add some additional geometrical or topological conditions on M. Based on this consideration, hypersurfaces immersed in $S^{n+1}(1)$ with constant mean curvature or constant scalar curvature have been into our insight and have its own interest. This class of hypersurfaces has been studied by many authors and obtained a series of rigidity or classification results, see [1], [2], [4], [6], [9], [10],

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[11], [12] and the references therein. For example, Wei [11] studied the hypersurfaces in $S^{n+1}(1)$ with constant mean curvature and two distinct principal curvatures and proved that:

Theorem A (Wei [11]). *Let M be an n-dimensional $(n \geq 3)$ connected complete hypersurface in $S^{n+1}(1)$ with constant mean curvature H and two distinct principal curvatures, one of which is simple. If the squared norm of the second fundamental form S of M satisfies*

$$S \leq n + \frac{n^3H^2}{2(n-1)} - \frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4 + 4(n-1)H^2}$$

or

$$S \geq n + \frac{n^3H^2}{2(n-1)} + \frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4 + 4(n-1)H^2},$$

then M is isometric to the Riemannian product $S^1(a) \times S^{n-1}(\sqrt{1-a^2})$, where $a^2 = \frac{2+nH^2 \pm \sqrt{n^2H^4+4(n-1)H^2}}{2n(1+H^2)}$.

Concerning the hypersurfaces in $S^{n+1}(1)$ with constant scalar curvature and two distinct principal curvatures, Wei [12] and Cheng [4] proved that:

Theorem B (Wei [12], Cheng [4]). *Let M be an n-dimensional $(n \geq 3)$ connected complete hypersurface in $S^{n+1}(1)$ with constant scalar curvature $n(n-1)r$ ($r \neq \frac{n-2}{n-1}$ is the normalized scalar curvature of M) and two distinct principal curvatures, one of which is simple. If*

$$S \leq (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2}$$

or

$$S \geq (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2},$$

then M is isometric to the Riemannian product $S^1(\sqrt{1-a^2}) \times S^{n-1}(a)$, where $a^2 = \frac{n-2}{nr}$.

Furthermore, the higher order mean curvature extends naturally the mean curvature and the scalar curvature as its special cases. So, studying the structures of hypersurfaces in $S^{n+1}(1)$ with constant k-th mean curvature H_k (see the Section 2 for definition) is another important research interest. For instance, in [13], Wei characterized the hypersurfaces with $H_k = 0$ and obtained:

Theorem C (Wei [13]). *Let M be an n-dimensional $(n \geq 3)$ connected complete hypersurface in $S^{n+1}(1)$ with constant k-th mean curvature $H_k = 0$ and two distinct principal curvatures, one of which is simple.*

(i) *If $S \geq \frac{n(k^2-2k+n)}{k(n-k)}$, then $S = \frac{n(k^2-2k+n)}{k(n-k)}$, and M is isometric to $S^1\big(\sqrt{\frac{k}{n}}\big) \times S^{n-1}\big(\sqrt{\frac{n-k}{n}}\big)$.*

(ii) *If* $S \leq \frac{n(k^2-2k+n)}{k(n-k)}$, *then* $S = \frac{n(k^2-2k+n)}{k(n-k)}$, *and* M *is isometric to* $S^1\big(\sqrt{\frac{k}{n}}\big) \times S^{n-1}\big(\sqrt{\frac{n-k}{n}}\big)$.

In this paper, we consider n-dimensional hypersurfaces with k-th mean curvature $H_k = \text{const.} > 0$ in a sphere $S^{n+1}(c)$ with constant curvature c. In fact, we prove the following result.

Theorem 1. *Let* M *be an* n*-dimensional* $(n \geq 3)$ *connected complete hypersurface in* $S^{n+1}(c)$ *with constant* k*-th* $(1 \leq k < n)$ *mean curvature* $H_k(> 0)$ *and two distinct principal curvatures, one of which is simple. If the squared norm of the second fundamental form* S *of* M *satisfies*

$$S \geq (n-1)t_0^{\frac{2}{k}} + c^2 t_0^{-\frac{2}{k}}, \tag{1}$$

or

$$S \leq (n-1)t_0^{\frac{2}{k}} + c^2 t_0^{-\frac{2}{k}}, \tag{2}$$

then M *is isometric to the Riemannian product* $S^1(c_1)\times S^{n-1}(c_2)$, *where* $c_1 > 0$, $c_2 > 0$, $\frac{1}{c_1} + \frac{1}{c_2} = \frac{1}{c}$, *and* t_0 *is the positive real root of the equation* $P_{H_k}(t) \equiv ckt^{\frac{k-2}{k}} - (n-k)t + nH_k = 0$ $(t > 0)$.

Remark 1. In Section 3, we will prove in Lemma 4 that the equation $P_{H_k}(t) = 0$ has actually a unique positive real root.

Remark 2. When $k = 1$, H_1 is exactly the mean curvature H. Let $c = 1$, then $P_{H_1}(t) \equiv \frac{1}{t} - (n-1)t + nH = 0$ $(t > 0)$ has one positive real root $t_0 = \frac{nH+\sqrt{n^2H^2+4(n-1)}}{2(n-1)}$ and Eq.(2) reduces to

$$S \leq n + \frac{n^3H^2}{2(n-1)} - \frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4 + 4(n-1)H^2}.$$

Therefore, Theorem 1 contains partially Theorem A as its special case. Meanwhile, Eq.(1) reduces to

$$S \geq n + \frac{n^3H^2}{2(n-1)} - \frac{n(n-2)}{2(n-1)}\sqrt{n^2H^4 + 4(n-1)H^2}.$$

It is obvious that the lower bound of S in Theorem 1 is less than that in Theorem A, this implies that Theorem 1 improves Theorem A partially.

Remark 3. When $k = 2$, we have $H_2 = r - c > 0$. Let $c = 1$, then M has constant positive second order mean curvature H_2 if and only if M has constant scalar curvature r and $r > 1$, this implies that $r > 1 - \frac{2}{n}$, and $r \neq \frac{n-2}{n-1}$. In this case, the equation $P_{H_2}(t) \equiv -(n-2)t + nH_2 + 2 = 0$ $(t > 0)$ has a unique positive real root $t_0 = \frac{n(r-1)+2}{n-2}$. Then, (1), (2) reduce to, respectively,

$$S \geq (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2}$$

and

$$S \leq (n-1)\frac{n(r-1)+2}{n-2} + \frac{n-2}{n(r-1)+2}.$$

We infer that Theorem 1 also contains Theorem B ([12, Theorem 1.3] and [4, Theorem 3.1]) as its special cases.

2. Preliminaries and lemmas

Let M be an n-dimensional hypersurface in an $(n+1)$-dimensional Euclidean sphere $S^{n+1}(c)$ with constant curvature c. We choose a local orthonormal frame field $\{e_1, \ldots, e_{n+1}\}$ in $S^{n+1}(c)$, such that $e_1, \ldots, e_n$ are tangent to M, e_{n+1} is the unit normal vector field. Let $\{\omega_1, \ldots, \omega_{n+1}\}$ denote the corresponding dual coframe field. Using the same symbols as in [10], then the structure equations and the Gauss equations of M can be written as

$$d\omega_i = \sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0,$$

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l,$$

$$R_{ijkl} = c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + (h_{ik}h_{jl} - h_{il}h_{jk}),$$

where h_{ij} denotes the components of the second fundamental form of M. The covariant derivative h_{ijk} of h_{ij} is defined by

$$\sum_k h_{ijk}\omega_k = dh_{ij} + \sum_k h_{kj}\omega_{ki} + \sum_k h_{ik}\omega_{kj}, \tag{3}$$

then, we obtain the Codazzi equation

$$h_{ijk} = h_{ikj}. \tag{4}$$

For $1 \leq k \leq n$, the k-th mean curvature H_k of M is defined by

$$\binom{n}{k} H_k = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k}, \tag{5}$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, λ_i $(1 \leq i \leq n)$ are the principal curvatures of M. In particular, when $k = 1$, $H_1 = H$ is nothing but the mean curvature of M; while $k = 2$, a simple calculation by using Gauss equations of M yields $H_2 = r - c$, where r is the normalized scalar curvature of M. So we know that the k-th mean curvature H_k generalizes the mean curvature and the scalar curvature naturally.

Now, we assume that M is a hypersurface in $S^{n+1}(c)$ with constant k-th mean curvature $H_k(> 0)$ and two distinct principal curvatures λ (multiplicity $n-1$) and μ (multiplicity 1). Choosing a proper frame field $\{e_1, \ldots, e_{n+1}\}$ in $S^{n+1}(c)$ such that $h_{ij} = \lambda_i \delta_{ij}$, and taking the convention on the range of indices that $1 \leq i, j, k, \ldots \leq n$, $1 \leq a, b, c, \ldots \leq n-1$, then

$$h_{ab} = \lambda\delta_{ab}, \ h_{nn} = \mu, \ h_{an} = 0. \tag{6}$$

On the other hand, by virtue of the definition of H_k, we have from (5) that $\binom{n}{k}H_k = \binom{n-1}{k}\lambda^k + \binom{n-1}{k-1}\lambda^{k-1}\mu$, equivalently,

$$\lambda^{k-1}((n-k)\lambda + k\mu) = nH_k. \tag{7}$$

Notice our assumption $H_k > 0$, (7) implies $\lambda \neq 0$ and

$$\mu = \frac{nH_k - (n-k)\lambda^k}{k\lambda^{k-1}}, \tag{8}$$

$$\lambda - \mu = n\frac{\lambda^k - H_k}{k\lambda^{k-1}} \neq 0. \tag{9}$$

By means of the following Lemma 3, together with (3), (4), (6) and (9), making use of the similar methods to [10], we get

$$\omega_{an} = \frac{\lambda_{,n}}{\lambda - \mu}\omega_a = \frac{k\lambda^{k-1}\lambda_{,n}}{n(\lambda^k - H_k)}\omega_a = \frac{\mathrm{d}\{\log|\lambda^k - H_k|^{\frac{1}{n}}\}}{\mathrm{d}s}\omega_a. \tag{10}$$

Taking exterior differentiation of (10), we have

$$\begin{aligned}\mathrm{d}\omega_{an} = &\left\{-\frac{\mathrm{d}^2\{\log|\lambda^k - H_k|^{\frac{1}{n}}\}}{\mathrm{d}s^2} + \left[\frac{\mathrm{d}\{\log|\lambda^k - H_k|^{\frac{1}{n}}\}}{\mathrm{d}s}\right]^2\right\}\omega_a \wedge \mathrm{d}s \\ &+ \frac{\mathrm{d}\{\log|\lambda^k - H_k|^{\frac{1}{n}}\}}{\mathrm{d}s}\sum_{b=1}^{n-1}\omega_{ab}\wedge\omega_b.\end{aligned} \tag{11}$$

Alternatively, from (10), structure equations and Gauss equations of M, a direct calculation gives

$$\mathrm{d}\omega_{an} = \frac{\mathrm{d}\{\log|\lambda^k - H_k|^{\frac{1}{n}}\}}{\mathrm{d}s}\sum_{b=1}^{n-1}\omega_{ab}\wedge\omega_b - (\lambda\mu + c)\omega_a \wedge \mathrm{d}s. \tag{12}$$

Comparing (11) with (12), we get the following lemma.

Lemma 2. *If M is an n-dimensional connected complete hypersurface in $S^{n+1}(c)$ with constant k-th mean curvature $H_k(>0)$ and two distinct principal curvatures λ and μ with multiplicities $n-1$ and 1, respectively. Then M is the locus of a family of moving submanifolds $M_1^{n-1}(s)$ (where the parameter s is the arc length of the integral curves of μ), and λ^k, H_k satisfy the following differential equation of order 2 :*

$$-\frac{d^2\{\log|\lambda^k - H_k|^{\frac{1}{n}}\}}{ds^2} + \left\{\frac{d\{\log|\lambda^k - H_k|^{\frac{1}{n}}\}}{ds}\right\}^2 + (\lambda\mu + c) = 0.$$

Lemma 3 (Otsuki [10])**.** *Let M be a hypersurface in a sphere $S^{n+1}(c)$ such that the multiplicities of the principal curvatures are constants, then the distribution of the space of the principal vectors corresponding to each principal curvature is completely integrable. In particular, if the multiplicity of a principal curvature is greater than 1, then this principal curvature is constant on* *each integral submanifold of the corresponding distribution of the space of the principal vectors.*

3. Proof of the main theorem

Define a positive function $\overline{w}(s)$ over $s \in (-\infty, +\infty)$ by $\overline{w}(s) = |\lambda^k - H_k|^{-\frac{1}{n}}$, from (8) and Lemma 2, we get

$$\frac{\mathrm{d}^2\overline{w}}{\mathrm{d}s^2} + \overline{w}\frac{ck\lambda^{k-2} - (n-k)\lambda^k + nH_k}{k\lambda^{k-2}} = 0. \tag{13}$$

In order to prove our main theorem, we will prove, at first, that $\frac{\mathrm{d}^2\overline{w}}{\mathrm{d}s^2} \geq 0$ or $\frac{\mathrm{d}^2\overline{w}}{\mathrm{d}s^2} \leq 0$ by using the equation (13), then to analysis the monotonicity of the functions $\frac{\mathrm{d}\overline{w}}{\mathrm{d}s}$ and $\overline{w}(s)$. As a result, we will know that $\overline{w}(s)$ is a constant. According to the results due to Cartan [3], taking similar arguments as in [10], we will complete the proof of the main theorem. Whatever, we prove firstly the following lemmas.

Lemma 4. *Let $P_{H_k}(t) = ckt^{\frac{k-2}{k}} - (n-k)t + nH_k$, where $c > 0$, $t > 0$, $1 \leq k < n$, $n \geq 3$, and $H_k = const. > 0$. Then $P_{H_k}(t)$ has a unique positive real root t_0. Furthermore,*

(1) *When $0 < t \leq t_0$, we have $P_{H_k}(t) \geq 0$;*
(2) *When $t \geq t_0$, we have $P_{H_k}(t) \leq 0$.*

Proof. (i) When $k = 1$, we have $\frac{\mathrm{d}P_{H_1}(t)}{\mathrm{d}t} = -ct^{-2} - (n-1) < 0$, which implies that $P_{H_1}(t)$ is a strictly monotone decreasing function. The unique positive solution of $P_{H_1}(t) = 0$ is $t_0 = \frac{nH_1 + \sqrt{n^2H_1^2 + 4(n-1)c}}{2(n-1)}$, thus $P_{H_1}(t) \geq 0$ for $0 < t \leq t_0$ and $P_{H_1}(t) \leq 0$ for $t \geq t_0$.

(ii) When $k = 2$, by making use of the similar methods to (i), we reach the conclusion.

(iii) When $k \geq 3$, a direct calculation then gives $\frac{\mathrm{d}P_{H_k}(t)}{\mathrm{d}t} = c(k-2)t^{-\frac{2}{k}} - (n-k)$ and $\frac{\mathrm{d}^2P_{H_k}(t)}{\mathrm{d}t^2} = -\frac{2(k-2)}{k}ct^{-\frac{2+k}{k}} < 0$, which implies that $\frac{\mathrm{d}P_{H_k}(t)}{\mathrm{d}t}$ is a strictly monotone decreasing function of t. Put $\frac{\mathrm{d}P_{H_k}(t)}{\mathrm{d}t} = 0$, we get $t_1 = (\frac{n-k}{c(k-2)})^{-\frac{k}{2}} > 0$. Thus, if $0 < t < t_1$, then $\frac{\mathrm{d}P_{H_k}(t)}{\mathrm{d}t} > 0$ and $P_{H_k}(t)$ is strictly monotone increasing. If $t > t_1$, $\frac{\mathrm{d}P_{H_k}(t)}{\mathrm{d}t} < 0$ and $P_{H_k}(t)$ is strictly monotone decreasing. Furthermore, since $\lim_{t\to 0^+} P_{H_k}(t) = nH_k > 0$, $\lim_{t\to+\infty} P_{H_k}(t) = -\infty$, from the continuous property of $P_{H_k}(t)$, we infer that $P_{H_k}(t)$ has a unique positive real root, denoted by t_0. Finally, we conclude that $P_{H_k}(t) \geq 0$ for $0 < t \leq t_0$ and $P_{H_k}(t) \leq 0$ for $t \geq t_0$, which completes the proof of Lemma 4. □

Lemma 5. *Let $f(t) = \frac{1}{k^2t^{\frac{2k-2}{k}}}\{(n-1)k^2t^2 + (nH_k - (n-k)t)^2\}$ for $t > 0$, $H_k = \text{const.} > 0$, $1 \leq k < n$ and $n \geq 3$, then $f(t_0) = (n-1)t_0^{\frac{2}{k}} + c^2t_0^{-\frac{2}{k}}$, where t_0 is the positive real root of $P_{H_k}(t) = 0$. Furthermore, if $t \geq H_k$, $f(t)$ is a* *monotone increasing function*; *if* $0 < t \leq H_k$, $f(t)$ *is a monotone decreasing function.*

Proof. Notice that $P_{H_k}(t_0) = ckt_0^{\frac{k-2}{k}} - (n-k)t_0 + nH_k = 0$, so

$$\begin{aligned} f(t_0) &= \frac{1}{k^2 t_0^{\frac{2k-2}{k}}}\Big\{(n-1)k^2t_0^2 + \big((ckt_0^{\frac{k-2}{k}} - (n-k)t_0 + nH_k) - ckt_0^{\frac{k-2}{k}}\big)^2\Big\} \\ &= \frac{1}{k^2 t_0^{\frac{2k-2}{k}}}\Big\{(n-1)k^2t_0^2 + (-ckt_0^{\frac{k-2}{k}})^2\Big\} \\ &= (n-1)t_0^{\frac{2}{k}} + c^2 t_0^{-\frac{2}{k}}. \end{aligned}$$

Furthermore, we have

$$\frac{\mathrm{d}f(t)}{\mathrm{d}t} = \frac{2t^{\frac{2-3k}{k}}}{k^3}\Big\{(n^2 - 2nk + nk^2)t^2 + n(k-2)(n-k)H_k t + (1-k)n^2H_k^2\Big\}.$$

Putting $g(t) \equiv (n^2 - 2nk + nk^2)t^2 + n(k-2)(n-k)H_k t + (1-k)n^2H_k^2$, a direct calculation gives that $g(H_k) = 0$. We will discuss the monotone property of $f(t)$ for $k = 1$ and $k \geq 2$ separately.

(1) When $k = 1$, $g(t) = n(n-1)t(t - H_1)$. Henceforth, if $0 < t \leq H_1$, then $g(t) \leq 0$ and $\frac{\mathrm{d}f(t)}{\mathrm{d}t} \leq 0$, it follows that $f(t)$ is a decreasing function; if $t \geq H_1$, then $g(t) \geq 0$ and $\frac{\mathrm{d}f(t)}{\mathrm{d}t} \geq 0$, this leads to $f(t)$ be an increasing function.

(2) When $k \geq 2$, we infer that $\frac{\mathrm{d}g(t)}{\mathrm{d}t} = 2n(k^2-2k+n)t + n(k-2)(n-k)H_k > 0$, so $g(t)$ is strictly monotone increasing and H_k is the only zero point of $g(t)$. Hence, if $0 < t \leq H_k$, then $g(t) \leq 0$, $\frac{\mathrm{d}f(t)}{\mathrm{d}t} \leq 0$ and $f(t)$ is a decreasing function; if $t \geq H_k$, then $g(t) \geq 0$, $\frac{\mathrm{d}f(t)}{\mathrm{d}t} \geq 0$ and $f(t)$ is an increasing function. This completes the proof of Lemma 5. □

Proof of Theorem 1. Put $t = H_k$, then $P_{H_k}(H_k) = ckH_k^{\frac{k-2}{k}} + kH_k > 0$, we know $H_k < t_0$ from the monotone property of $P_{H_k}(t)$ (Lemma 4(1)). We also assert that $\lambda^k > H_k$. In fact, if on the contrary $\lambda^k < H_k$ (because of $\lambda^k \neq H_k$ from (9)), then $\lambda^k < t_0$. Review the process of the proof of Lemma 4, it is not difficult to find that $P_{H_k}(\lambda^k) > 0$. Recall the definition of $P_{H_k}(t)$, (13) can be rewritten as

$$\frac{\mathrm{d}^2\overline{w}}{\mathrm{d}s^2} + \overline{w}\frac{P_{H_k}(\lambda^k)}{k\lambda^{k-2}} = 0, \tag{14}$$

therefore $\frac{\mathrm{d}^2\overline{w}}{\mathrm{d}s^2} < 0$, this implies $\frac{\mathrm{d}\overline{w}}{\mathrm{d}s}$ is a strictly monotone decreasing function of t, and it has at most one zero point for $s \in (-\infty, +\infty)$. If $\frac{\mathrm{d}\overline{w}}{\mathrm{d}s}$ has no zero point in $(-\infty, +\infty)$, then $\overline{w}(s)$ is a monotone function in $s \in (-\infty, +\infty)$; if $\frac{\mathrm{d}\overline{w}}{\mathrm{d}s}$ has one zero point s_0 in $(-\infty, +\infty)$, then $\overline{w}(s)$ is a monotone function both in $(-\infty, s_0]$ and $[s_0, +\infty)$. Since $\overline{w}(s)$ is bounded ([13]), we know that both

$\lim_{s\to-\infty}\overline{w}(s)$ and $\lim_{s\to+\infty}\overline{w}(s)$ exist and we have

$$\lim_{s\to-\infty}\frac{\mathrm{d}\overline{w}(s)}{\mathrm{d}s}=\lim_{s\to+\infty}\frac{\mathrm{d}\overline{w}(s)}{\mathrm{d}s}=0.$$

This is impossible because $\frac{\mathrm{d}\overline{w}(s)}{\mathrm{d}s}$ is a strictly monotone decreasing function. Therefore we prove the assertion that $\lambda^k > H_k$. By the way, keep in mind that $H_k < t_0$ as we have proved at the beginning of the proof of Theorem 1.

Evaluating the function $f(t)$ (defined in Lemma 5) at λ^k and using (7), we easily obtain

$$\begin{aligned} f(\lambda^k) &= \frac{1}{k^2\lambda^{2k-2}}\{(n-1)k^2\lambda^{2k}+[nH_k-(n-k)\lambda^k]^2\} \\ &= (n-1)\lambda^2+\frac{1}{k^2\lambda^{2k-2}}k^2\lambda^{2k-2}\mu^2 = S. \end{aligned}$$

Case 1. If the assumption (1) holds in Theorem 1, i.e., $S=f(\lambda^k)\ge f(t_0)$, we know from Lemma 5 that $\lambda^k \ge t_0$, thus Lemma 4 tells us $P_{H_k}(\lambda^k)\le P_{H_k}(t_0)=0$. So we have $\frac{\mathrm{d}^2\overline{w}}{\mathrm{d}s^2}\ge 0$ from (14), this means that $\frac{\mathrm{d}\overline{w}}{\mathrm{d}s}$ is a monotone increasing function of s. Therefore, $\overline{w}(s)$ must be monotonic when s tends to infinity. On the other hand, since $\overline{w}(s)$ is bounded (cf. [13]), we find that both $\lim_{s\to-\infty}\overline{w}(s)$ and $\lim_{s\to+\infty}\overline{w}(s)$ exist and we have

$$\lim_{s\to-\infty}\frac{\mathrm{d}\overline{w}(s)}{\mathrm{d}s}=\lim_{s\to+\infty}\frac{\mathrm{d}\overline{w}(s)}{\mathrm{d}s}=0.$$

By the monotonicity of $\frac{\mathrm{d}\overline{w}(s)}{\mathrm{d}s}$, we see that $\frac{\mathrm{d}\overline{w}}{\mathrm{d}s}\equiv 0$, thus $\overline{w}(s)$ is a constant. Then, according to $\overline{w}(s)=|\lambda^k-H_k|^{-\frac{1}{n}}$ and (8), we infer that λ, μ are constants on M. Therefore, we know from the results due to Cartan in [3] that M is an isoparametric hypersurface, it is isometric to the Riemannian product $S^1(c_1)\times S^{n-1}(c_2)$, where $c_1>0,\ c_2>0,\ \frac{1}{c_1}+\frac{1}{c_2}=\frac{1}{c}$.

Case 2. If the assumption (2) holds in Theorem 1, i.e., $S=f(\lambda^k)\le f(t_0)$, we obtain from Lemma 5 again that $\lambda^k\le t_0$, thus $P_{H_k}(\lambda^k)\ge 0$ by Lemma 4. So we have $\frac{\mathrm{d}^2\overline{w}}{\mathrm{d}s^2}\le 0$, this means that $\frac{\mathrm{d}\overline{w}}{\mathrm{d}s}$ is a monotone increasing function of s. By the similar discussion to the Case 1, we know that λ, μ are constants on M and M is an isoparametric hypersurface, it is isometric to the Riemannian product $S^1(c_1)\times S^{n-1}(c_2)$. We complete the proof of Theorem 1. □

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