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LINEAR WEINGARTEN SPACELIKE HYPERSURFACES IN LOCALLY SYMMETRIC LORENTZ SPACE

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Abstract. Let M be a linear Weingarten spacelike hypersurface in a locally symmetric Lorentz space with $R = aH + b$, where R and H are the normalized scalar curvature and the mean curvature, respectively. In this paper, we give some conditions for the complete hypersurface M to be totally umbilical.

1. Introduction

Let N_1^{n+1} be an $(n+1)$-dimensional pseudo-Riemannian manifold of index 1, which is called Lorentz space. When the Lorentz space N_1^{n+1} is of constant curvature c, we call it Lorentz space form, denoted by $N_1^{n+1}(c)$. A hypersurface M of a Lorentz space N_1^{n+1} is said to be spacelike if the induced metric on M from that of the Lorentz space is positive definite.

It is well know that spacelike hypersurfaces in a Lorentz space form have been investigated by many differential geometers from both the physical and the mathematical points of view. Goddard [9] conjectured that a complete spacelike hypersurface in de Sitter space $N_1^{n+1}(1)$ with constant mean curvature H must be totally umbilical. Akutagawa [2] and Ramanathan [17] proved independently that the conjecture is true if $H^2 \leq 1$ when $n = 2$ and $n^2H^2 \leq 4(n-1)$ when $n \geq 3$. In [14], Montiel proved that Goddard's conjecture is true provided that M^n is compact. Montiel [15] proved that complete spacelike hypersurface M^n with $H^2 = 4(n-1)/n^2$ is isometric to a hyperbolic cylinder if M^n has at least two ends. Another natural Goddard-like problem is to study hypersurfaces of Lorentze space with constant scalar curvature. An interesting result of Cheng and Ishikawa [8] states that the totally umbilical round spheres are the only compact spacelike hypersurfaces in de Sitter space $N_1^{n+1}(1)$ with constant normalized scalar curvature $R < 1$. Some other authors, such as Brasil-Colares-Palmas [4], Camargo-Chaves-Sousa Jr [5], Caminha [6] and Li [10] have also worked on related problems.

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When M^n is a complete spacelike hypersurface in de Sitter space $N_1^{n+1}(1)$ with $R = kH$, Cheng [7] proved that if the sectional curvature is non-negative and H can obtain its maximum on M^n, then M^n is totally umbilical. Shu [18] obtained a characteristic theorem concerning such hypersurfaces in terms of the mean curvature.

All of the above results were obtained under the assumption that the ambient manifolds possess very nice symmetry properties. Many researchers have recently begun to study ambient manifolds which do not have symmetry in general, such as locally symmetric Lorentz space, see [3], [13], [12], [19], [20] and [21]. First of all, we recall that, for constants c_1 and c_2, Jin Ok Baek et al. [3] introduced the class of $(n+1)$-dimensional Lorentz spaces N_1^{n+1} of index 1 which satisfy the following two conditions (here and in the sequel, $\bar{K}$ denotes the sectional curvature on N_1^{n+1}):

(1) for any spacelike vector μ and timelike vector ν,

$$\bar{K}(\mu, \nu) = -c_1/n; \tag{1.1}$$

(2) for any spacelike vector μ and timelike vector ν,

$$\bar{K}(\mu, \nu) \geq c_2. \tag{1.2}$$

There are several examples of Lorentz spaces satisfying (1.1) and (1.2), for instance,

Example 1.1. The Lorentz space form $N_1^{n+1}(c)$, where $-c_1/n = c_2 = c$.

Example 1.2. Semi-Riemannian product manifold $H_1^k(-c_1/n) \times N^{n+1-k}(c_2)$, $c_1 > 0$, and $R_1^k \times S^{n+1-k}(1)$. In particular, $R_1^1 \times S^n(1)$ is so-called *Einstein Static Universe.*

Example 1.3. *Robertson-Walker spacetime* $N(c, f) = I \times_f N^3(c)$, where I denotes an open interval of R_1^1 and $f > 0$ a smooth function defined on the interval I, $N^3(c)$ a 3-dimensional Riemannian manifold of constant curvature c.

We denote by $\bar{K}_{CD}$ the components of the Ricci tensor of N_1^{n+1} satisfying (1.1) and (1.2), then the scalar curvature $\bar{R}$ of N_1^{n+1} is given by

$$\bar{R} = \sum_{A=1}^{n+1} \varepsilon_A \bar{K}_{AA} = -2 \sum_{i=1}^{n} \bar{K}_{n+1iin+1} + \sum_{i,j=1}^{n} \bar{K}_{ijji} = 2c_1 + \sum_{i,j=1}^{n} \bar{K}_{ijji}.$$

It is well known $\bar{R}$ is constant when Lorentz space N_1^{n+1} is locally symmetric, so $\sum_{i,j=1}^{n} \bar{K}_{ijji}$ is constant.

A hypersurface in Lorentz space N_1^{n+1} is called linear Weingarten hypersurface if $cR + dH + e = 0$, where c, d, e are constants such that $c^2 + d^2 \neq 0$. When the constant d vanishes, linear Weingarten hypersurfaces reduce to hypersurfaces with constant scalar curvature; When the constant c vanishes, linear Weingarten hypersurfaces reduce to hypersurfaces with constant mean curvature; When the constant e vanishes, linear Weingarten hypersurfaces reduce to hypersurfaces with $R = kH$. In [11], Li-Suh-Wei studied compact linear Weingarten hypersurfaces with $R = aH + b$ in unit sphere $S^{n+1}(1)$. In this paper, we will consider spacelike linear Weingarten hypersurfaces with $R = aH + b$ in locally symmetric Lorentz space N_1^{n+1}. Our results as following generalizes some known ones.

Theorem 1.4. *Let N_1^{n+1} be a locally symmetric Lorentz space satisfying* (1.1) *and* (1.2), *M^n is a complete spacelike linear Weingarten hypersurface immersed in N_1^{n+1} with $R = aH + b$ satisfying $a \neq 0, b < c_2$. Suppose that the maximum of H can be attained on M^n. If the sectional curvature of M^n is not less than $-c_1/n - c_2$, then M^n is totally umbilical or an isometric hypersurface with two distinct principle curvature, one of which is simple.*

In particular, when N_1^{n+1} is de Sitter space $N_1^{n+1}(c)$ $(c = -c_1/n = c_2 > 0)$, from Theorem 1.4 we have the following corollary by means of the congruence Theorem of Abe-Koike-Yamaguchi [1].

Corollary 1.5. *Let M^n be a complete spacelike linear Weingarten hypersurface immersed in $N_1^{n+1}(c)$ with $R = aH + b$ satisfying $a \neq 0, b < c$. Suppose that the maximum of H can be attained on M^n. If the sectional curvature of M^n is non-negative, then M^n is totally umbilical or $H^1(c_1) \times S^{n-1}(c_2)$, where $1/c_1 + 1/c_2 = 1/c$.*

Theorem 1.6. *Let N_1^{n+1} be a locally symmetric Lorentz space satisfying* (1.1) *and* (1.2). *M^n is a complete spacelike linear Weingarten hypersurface immersed in N_1^{n+1} with $R = aH + b$ satisfying $b < c_2$. If $S \leq 2\sqrt{n-1}(2c_2 + c_1/n)$, then either*

(1) *M^n is totally umbilical, or*
(2) $\sup S = 2\sqrt{n-1}(2c_2 + c_1/n)$. *If $\sup S$ is attained at some point in M^n, then M^n is isometric to an isometric hypersurface with two distinct principle curvature, one of which is simple.*

2. Preliminaries

Let M^n be an n-dimensional spacelike hypersurface immersed in the Lorentz space N_1^{n+1}. We choose a local field of pseudo-Riemannian orthonormal frames $\{e_1, \ldots, e_{n+1}\}$ in N_1^{n+1} such that, restricted to M^n, $e_1, \ldots, e_n$ are tangent to M^n, and the vector e_{n+1} is normal to M^n. Let $\{\omega_1, \ldots, \omega_{n+1}\}$ be the dual frame field. In this paper, we make the following convention on the range of indices:

$$1 \leq A,\ B,\ C \leq n+1; \qquad 1 \leq i,\ j,\ k \leq n.$$

Then the structure equations of N_1^{n+1} are given by

$$d\omega_A = -\sum_B \varepsilon_B \omega_{AB} \wedge \omega_B, \qquad \omega_{AB} + \omega_{BA} = 0,$$

$$d\omega_{AB} = -\sum_C \varepsilon_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2} \sum_{C,D} \varepsilon_C \varepsilon_D \bar{K}_{ABCD} \omega_C \wedge \omega_D,$$

where $\varepsilon_i = 1$, $\varepsilon_{n+1} = -1$ and $\bar{K}_{ABCD}$ denotes the components of the Riemannian curvature tensor of N_1^{n+1}. Then

$$\bar{K}_{CD} = \sum_B \varepsilon_B \bar{K}_{BCDB}, \qquad \bar{K} = \sum_A \varepsilon_A \bar{K}_{AA}.$$

Next we define the covariant derivative of K_{ABCD} by

$$\sum_E \varepsilon_E \bar{K}_{ABCD;E}\omega_E = d\bar{K}_{ABCD} - \sum_E \varepsilon_E (\bar{K}_{EBCD}\omega_{EA} + \bar{K}_{AECD}\omega_{EB} + \bar{K}_{ABED}\omega_{EC} + \bar{K}_{ABCE}\omega_{ED}).$$

We restrict these forms to the spacelike hypersurface M^n in N_1^{n+1} and have $\omega_{n+1} = 0$. The induced metric ds^2 of M is written as $ds^2 = \sum_i \omega_i^2$. We may put

$$\omega_{n+1i} = \sum_j h_{ij}\omega_j, \qquad h_{ij} = h_{ji}. \tag{2.1}$$

The quadratic form $B = \sum_{i,j} h_{ij}\omega_i \otimes \omega_j \otimes e_{n+1}$ is the second fundamental form of M^n. We denote $L = (h_{ij})_{n\times n}$ and $S = \sum h_{ij}^2$. The mean curvature vector ξ of M^n is defined by

$$\xi = \frac{1}{n}\sum_i h_{ii}e_{n+1}.$$

The length of the mean curvature vector is called the mean curvature of M^n, denote by H. When $\xi \neq 0$, we choose e_{n+1} to assure $H = \frac{1}{n}\sum_i h_{ii}^{n+1} > 0$.

We can obtain the structure equations of M^n

$$d\omega_i = -\sum_j \omega_{ij} \wedge \omega_j, \qquad \omega_{ij} + \omega_{ji} = 0,$$

$$d\omega_{ij} = -\sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l,$$

and the Gauss equation

$$R_{ijkl} = \bar{K}_{ijkl} - (h_{il}h_{jk} - h_{ik}h_{jl}), \tag{2.2}$$

where $\{R_{ijkl}\}$ is the component of the curvature tensor of M^n. Let R_{ij} and R denote the components of the Ricci curvature and the normalized scalar curvature of M^n respectively. From (2.2) we have

$$R_{ik} = \sum_j \bar{K}_{jikj} - nHh_{ik} + \sum_j h_{ij}h_{jk}, \tag{2.3}$$

$$n(n-1)R = \sum_{i,j} \bar{K}_{jiij} - n^2H^2 + S. \tag{2.4}$$

Let h_{ijk} denote the covariant derivative of h_{ij} so that

$$\sum_k h_{ijk}\omega_k = dh_{ij} - \sum_k h_{kj}\omega_{ki} - \sum_k h_{ik}\omega_{kj}.$$

Then by exterior differentiation of (2.1), we obtain the Codazzi equation

$$h_{ijk} = h_{ikj} + \bar{K}_{n+1ijk}. \tag{2.5}$$

Next, we define the second covariant derivative of h_{ij} by

$$\sum_l h_{ijkl}\omega_l = dh_{ijk} - \sum_m h_{mjk}\omega_{mi} - \sum_m h_{imk}\omega_{mj} - \sum_m h_{ijm}\omega_{mk}.$$

By exterior differentiation of (2.5), we can get the following Ricci identity

$$h_{ijkl} - h_{ijlk} = -\sum_m (h_{mj}R_{mikl} + h_{im}R_{mjkl}). \tag{2.6}$$

Restricting the covariant derivative $\bar{K}_{ABCD;E}$ of $\bar{K}_{ABCD}$ on M^n, then $\bar{K}_{n+1ijk;l}$ is given by

$$\bar{K}_{(n+1)ijk;l} = \bar{K}_{(n+1)ijkl} + \bar{K}_{(n+1)i(n+1)k}h_{jl} + \bar{K}_{(n+1)ij(n+1)}h_{kl} + \bar{K}_{mijk}h_{ml}, \tag{2.7}$$

where $\bar{K}_{(n+1)ijkl}$ denotes the covariant derivative of $\bar{K}_{(n+1)ijk}$ as a tensor on M^n.

The Laplacian of h_{ij} is defined by $\triangle h_{ij} = \sum_k h_{ijkk}$. From (2.5) and (2.6) we obtain

$$\triangle h_{ij} = nH_{ij} + \sum_{i,j,k}(\bar{K}_{n+1kikj} + \bar{K}_{n+1ijkk}) - \sum_{i,j,k,m}(h_{mk}R_{mijk} + h_{im}R_{mkjk}). \tag{2.8}$$

Since $\frac{1}{2}\Delta S = \sum_{i,j,k}(h_{ijk})^2 + \sum_{i,j} h_{ij}\Delta h_{ij}$, then it follows from (2.7) and (2.8) that

$$\begin{aligned} &\frac{1}{2}\triangle S \\ &= \sum_{i,j,k} h_{ijk}^2 + \sum_{i,j,k} nh_{ij}H_{ij} + \sum_{i,j} h_{ij}(\bar{K}_{n+1ijk;k} + \bar{K}_{n+1kik;j}) \\ &\quad - (S\sum_k \bar{K}_{n+1kn+1k} + nH\sum_{i,j} h_{ij}\bar{K}_{n+1ijn+1}) \\ &\quad - \sum_{i,j,k,m}(h_{ij}h_{mk}\bar{K}_{mijk} + h_{ij}h_{mj}\bar{K}_{mkik} + h_{ij}h_{mk}R_{mijk} + h_{ij}h_{im}R_{mkjk}). \end{aligned} \tag{2.9}$$

Let $T = \sum_{i,j} T_{ij}\omega_i\omega_j$ be a symmetric tensor on M^n defined by

$$T_{ij} = nH\delta_{ij} - h_{ij}.$$

We introduce an operator $\square$ associated to T acting on $f \in C^2(M^n)$ by

$$\square f = \sum_{i,j} T_{ij}f_{ij} = \sum_{i,j}(nH\delta_{ij} - h_{ij})f_{ij}. \tag{2.10}$$

Setting $f = nH$ in (2.10) and from (2.4) we obtain

$$
\begin{aligned}
\Box(nH) &= \sum_{i,j}(nH\delta_{ij} - h_{ij})(nH)_{ij} \\
&= \sum_{i}(nH)(nH)_{ii} - \sum_{i,j} nh_{ij}H_{ij} \qquad (2.11)\\
&= \frac{1}{2}\Delta(nH)^2 - \sum_{i}(nH_i)^2 - \sum_{i,j} nh_{ij}H_{ij} \\
&= \frac{1}{2}\Delta S - \frac{1}{2}n(n-1)\Delta R - n^2|\nabla H|^2 - \sum_{i,j} nh_{ij}H_{ij}.
\end{aligned}
$$

We introduce another operator

$$L = \Box + \frac{n-1}{2}a\Delta.$$

Then it follows from $R = aH + b$ and (2.11) that

$$
\begin{aligned}
L(nH) &= \Box(nH) + \frac{n-1}{2}a\Delta(nH) \\
&= \Box(nH) + \frac{1}{2}n(n-1)\triangle R \qquad (2.12)\\
&= \frac{1}{2}\Delta S - n^2|\nabla H|^2 - \sum_{i,j} nh_{ij}H_{ij}.
\end{aligned}
$$

Substituting (2.9) into (2.12) we have

(2.13)

$$
\begin{aligned}
&L(nH) \\
&= \sum_{i,j,k} h_{ijk}^2 - n^2|\nabla H|^2 + \sum_{i,j,k} h_{ij}(\bar{K}_{n+1ijk;k} + \bar{K}_{n+1kik;j}) \\
&\quad - (S\sum_{k}\bar{K}_{n+1kn+1k} + nH\sum_{i,j} h_{ij}\bar{K}_{n+1ijn+1}) \\
&\quad - \sum_{i,j,k,m}(h_{ij}h_{mk}\bar{K}_{mijk} + h_{ij}h_{mj}\bar{K}_{mkik} + h_{ij}h_{mk}R_{mijk} + h_{ij}h_{im}R_{mkjk}).
\end{aligned}
$$

Lemma 2.1 ([16]). *Let μ_i $(1 \le i \le n)$ be real numbers such that $\sum_i \mu_i = 0$ and $\sum_i \mu_i^2 = \beta^2$, where $\beta = constant \ge 0$. Then*

$$-\frac{n-2}{\sqrt{n(n-1)}}\beta^3 \le \sum_i \mu_i^3 \le \frac{n-2}{\sqrt{n(n-1)}}\beta^3 \qquad (2.14)$$

and the equality holds if and only if at least $(n-1)$ of the μ_i are equal.

Proposition 2.2. *Let M^n be an n-dimensional spacelike linear Weingarten hypersurface immersed in a locally symmetric Lorentz space L^{n+1} satisfying (1.1) and (1.2) with $R = aH + b$. If $a \ne 0, b < c_2$, then L is elliptic.*

Proof. If $H=0$, we have $R=b<c_2$. It follows from (2.4) that $S=n(n-1)b-\sum_{ij}\bar{K}_{ijij}\le n(n-1)(b-c_2)<0$. This is impossible. Therefore we have $H>0$. It follows from (2.4) and $R=aH+b$ that

$$S=n^2H^2+n(n-1)(aH+b)-\sum_{i,j}\bar{K}_{jiij}, \tag{2.15}$$

then

$$a=\frac{1}{n(n-1)H}\left(S-n^2H^2-n(n-1)b+\sum_{i,j}\bar{K}_{jiij}\right). \tag{2.16}$$

We choose a local frame of orthonormal vector fields $\{e_i\}$ such that $h_{ij}=\lambda_i\delta_{ij}$. For any i, from (2.16) we have

$$\begin{aligned} & nH-\lambda_i+\frac{n-1}{2}a \\ (2.17)\quad = \;& nH-\lambda_i+\frac{1}{2nH}\left(S-n^2H^2-n(n-1)b+\sum_{i,j}\bar{K}_{jiij}\right) \\ = \;& \left(\frac{1}{2}(nH)^2-nH\lambda_i+\frac{1}{2}S+\frac{1}{2}\sum_{i,j}\bar{K}_{jiij}-\frac{1}{2}n(n-1)b\right)(nH)^{-1}. \end{aligned}$$

Since $\bar{K}_{jiij}\ge c_2$, we have

$$\begin{aligned} & nH-\lambda_i+\frac{n-1}{2}a \\ \ge\; & \left\{\frac{1}{2}(\sum_j\lambda_j)^2-\lambda_i\sum_j\lambda_j+\frac{1}{2}\sum_j\lambda_j^2+\frac{1}{2}n(n-1)(c_2-b)\right\}(nH)^{-1} \\ =\; & \left\{\sum_j\lambda_j^2+\frac{1}{2}\sum_{l\ne j}\lambda_l\lambda_j-\lambda_i\sum_j\lambda_j+\frac{1}{2}n(n-1)(c_2-b)\right\}(nH)^{-1} \\ =\; & \left\{\sum_{i\ne j}\lambda_j^2+\frac{1}{2}\sum_{l\ne j,l,j\ne i}\lambda_l\lambda_j+\frac{1}{2}n(n-1)(c_2-b)\right\}(nH)^{-1} \\ =\; & \frac{1}{2}\left\{\sum_{i\ne j}\lambda_j^2+(\sum_{j\ne i}\lambda_j)^2+n(n-1)(c_2-b)\right\}(nH)^{-1}. \end{aligned}$$

It follows from $b<c_2$ that

$$nH-\lambda_i+\frac{n-1}{2}a>0. \tag{2.18}$$

Thus L is an elliptic operator. $\square$

Proposition 2.3. *Let M^n be an n-dimensional spacelike linear Weingarten hypersurface immersed in a locally symmetric Lorentz space L^{n+1} satisfying* (1.1) *and* (1.2) *with $R = aH + b$. If $(n-1)a^2 + 4n(c_2 - b) \geq 0$, then we have*

$$\sum_{i,j,k} h_{ijk}^2 \geq n^2 |\nabla H|^2. \tag{2.19}$$

Moreover, suppose that the equality holds on M^n in (2.19)*, then H is constant.*

Proof. From (2.4) and $R = aH + b$, we have

$$S = n^2H^2 + n(n-1)(aH + b) - \sum_{i,j} \bar{K}_{jiij}. \tag{2.20}$$

Since $K_{ABCD;E} = 0$, taking the covariant derivative of (2.20), we have

$$2\sum_{i,j} h_{ij}h_{ijk} = S_k = \left(2n^2H + n(n-1)a\right) H_k \tag{2.21}$$

for every k. Hence, by Cauchy-Schwartz's inequality, we have

$$\sum_{i,j} h_{ij}^2 \sum_{i,j,k} h_{ijk}^2 \geq (n^2H + \frac{1}{2}n(n-1)a)^2 |\nabla H|^2, \tag{2.22}$$

that is

$$S\sum_{i,j,k} h_{ijk}^2 \geq (n^2H + \frac{1}{2}n(n-1)a)^2 |\nabla H|^2. \tag{2.23}$$

On the other hand, it follows from (2.20) that

$$\begin{aligned}
& \left(n^2H + \frac{1}{2}n(n-1)a\right)^2 - n^2S \\
= \;& n^2\left(n^2H^2 + n(n-1)Ha - S\right) + \frac{1}{4}n^2(n-1)^2a^2 \\
= \;& n^2\sum_{i,j} K_{jiij} - n^3(n-1)b + \frac{1}{4}n^2(n-1)^2a^2 \\
\geq \;& \frac{1}{4}n^2(n-1)\left((n-1)a^2 + 4n(c_2 - b)\right). \qquad (2.24)
\end{aligned}$$

Since $(n-1)a^2 + 4n(c_2 - b) \geq 0$, we have

$$\left(n^2H + \frac{1}{2}n(n-1)a\right)^2 \geq n^2S. \tag{2.25}$$

It follows from (2.23) and (2.25) that

$$S\sum_{i,j,k} h_{ijk}^2 \geq (n^2H + \frac{1}{2}n(n-1)a)^2 |\nabla H|^2 \geq n^2S|\nabla H|^2. \tag{2.26}$$

Hence either $S = 0$ and $\sum_{i,j,k} h_{ijk}^2 = n^2|\nabla H|^2$ or $\sum_{i,j,k} h_{ijk}^2 \geq n^2|\nabla H|^2$.

We suppose $\sum_{i,j,k} h_{ijk}^2 = n^2|\nabla H|^2$ on M^n. Then equalities in (2.22), (2.23), (2.24), (2.25) and (2.26) hold.

If $(n-1)a^2 + 4n(c_2 - b) > 0$, then $(n^2H + \frac{1}{2}n(n-1)a)^2 > n^2S$ from (2.24). Since the second equality in (2.26) holds, we have $|\nabla H| = 0$ and hence H is constant on M^n.

If $(n-1)a^2 + 4n(c_2 - b) = 0$, since the equality holds in (2.24), we have $(n^2H + \frac{1}{2}n(n-1)a)^2 = n^2S$. This together with (2.21) forces that

$$S_k^2 = 4n^2SH_k^2, \qquad k = 1, \dots, n. \tag{2.27}$$

Since the equality holds in (2.22), there exists a real function c_k on M^n such that

$$h_{ijk} = c_k h_{ij}, \qquad i,\ j = 1, \dots, n, \tag{2.28}$$

for every k. Taking the sum on both sides of equation (2.28) with respect to $i = j$, we get

$$H_k = c_k H, \qquad k = 1, \dots, n. \tag{2.29}$$

From (2.28), we have

$$S_k = 2\sum_{i,j} h_{ij}h_{ijk} = 2c_kS, \qquad k = 1, \dots, n. \tag{2.30}$$

Multiplying both sides of equations in (2.30) by H and by using (2.29), we have

$$HS_k = 2H_kS, \qquad k = 1, \dots, n. \tag{2.31}$$

It follows from (2.27) and (2.31) that

$$H_k^2S = H_k^2n^2H^2, \qquad k = 1, \dots, n. \tag{2.32}$$

Hence we have

$$|\nabla H|^2(S - n^2H^2) = 0. \tag{2.33}$$

Suppose that H is not constant on M^n, we assert that $S = n^2H^2$. In fact, since H is not constant, we have that $|\nabla H|$ is not vanishing identically on M^n. We denote $M_0 = \{x \in M \mid |\nabla H| \neq 0\}$, then M_0 is open in M. Let $T = S - n^2H^2$, it follows form (2.33) that $T = 0$ in M_0. From the continuity of T, we have that $T = 0$ on the closure $\mathrm{cl}(M_0)$ of M_0. If $M/\mathrm{cl}(\mathrm{M}_0) \neq \emptyset$, then H is constant in $M/\mathrm{cl}(\mathrm{M}_0)$. It follows from (2.20) that S is constant and hence T is constant in $M/\mathrm{cl}(\mathrm{M}_0)$. From the continuity of T, we have that $T = 0$ on M^n and hence $S = n^2H^2$. It follows from (2.4) that $n(n-1)R = \sum_{i,j} \bar{K}_{jiij}$. Since $\sum_{i,j} \bar{K}_{jiij}$ is constant, we have that R is constant and hence H is constant. This is contradict to the assumption. Hence the mean curvature H is constant. □

Lemma 2.4 ([7])**.** *Let M be a complete Riemannian manifold whose Ricci curvature is bounded from below. Let f be a C^2-function which is bounded from above. Then there exists a sequence $\{q_k\}$ such that*

$$\lim_{k\to\infty} f(q_k) = \sup f, \quad \lim_{k\to\infty} \|\nabla f(q_k)\| = 0, \quad \limsup_{k\to\infty} Lf(q_k) \leq 0,$$

where $Lf = \sum b_j f_{jj}$, $b_j \geq 0$ *is bounded.*

3. Proof of theorems

Proof of Theorem 1.4. We choose $e_1, \ldots, e_n$ such that $h_{ij} = \lambda_i \delta_{ij}$, then (2.13) becomes

$$L(nH) = \sum_{i,j,k} h_{ijk}^2 - n^2|\nabla H|^2 - S\sum_k \bar{K}_{n+1kn+1k} - nH\sum_i \lambda_i \bar{K}_{n+1iin+1}$$
$$- \frac{1}{2}\sum_{i,k}(\lambda_i - \lambda_k)^2(\bar{K}_{ikik} + R_{ikik}). \tag{3.1}$$

Next we estimate the right hand of formula (3.1) one by one. Using (1.1) and (1.2), we have

$$- S\sum_k \bar{K}_{n+1kn+1k} - nH\sum_i \lambda_i \bar{K}_{n+1iin+1}$$
$$= \sum_k (S - nH\lambda_k)\frac{c_1}{n} = c_1(S - nH^2), \tag{3.2}$$

and

$$-\frac{1}{2}\sum_{i,k}(\lambda_i - \lambda_k)^2(\bar{K}_{ikik} + R_{ikik}) \geq \frac{1}{2}\sum_{i,k}(\lambda_i - \lambda_k)^2(c_2 + K_{\min})$$
$$= n(c_2 + K_{\min})(S - nH^2), \tag{3.3}$$

where $K_{\min}$ denotes the infimum of the sectional curvature of M^n. Substituting (3.2), (3.3) into (3.1) and from Proposition 2.3, we get

$$L(nH) \geq \sum_{i,j,k} h_{ijk}^2 - n^2|\nabla H|^2 + n(K_{\min} + \frac{c_1}{n} + c_2)(S - nH^2) \geq 0, \tag{3.4}$$

here we used the assumption $K_{\min} \geq -\frac{c_1}{n} - c_2$. Since L is elliptic and H can obtain its maximum on M, we deduce that H is constant and the equalities in (3.4) hold. Thus

$$\sum_{i,j,k} h_{ijk}^2 = n^2|\nabla H|^2 = 0, \tag{3.5}$$

and

$$(K_{\min} + \frac{c_1}{n} + c_2)(S - nH^2) = 0. \tag{3.6}$$

It follows from (3.6) that either $S = nH^2$ and M^n is totally umbilical or $K_{\min} + \frac{c_1}{n} + c_2 = 0$. In the latter case, since the equality in (3.3) holds, we have that $R_{jiij} = K_{\min} = -\frac{c_1}{n} - c_2$ and $\bar{K}_{jiij} = c_2$. It follows from Gauss equation (2.2) that, for any i, j

$$\lambda_i \lambda_j = \frac{c_1}{n} + 2c_2. \tag{3.7}$$

If $\frac{c_1}{n}+2c_2=0$, then all the λ_i are zero, and M^n is totally geodesic. Otherwise, if $\frac{c_1}{n}+2c_2\neq 0$, we conclude that M^n has at most two distinct principal curvature. In fact, without loss of generality, we assume that M^n has three distinct principle curvature $\lambda_{i_1},\lambda_{i_2},\lambda_{i_3}$. Then $\lambda_{i_1}\lambda_{i_2}=\lambda_{i_2}\lambda_{i_3}=\frac{c_1}{n}+2c_2$ and hence $\lambda_{i_1}=\lambda_{i_3}$. This is a contradiction. So M^n has at most two distinct principal curvature. If all the principle curvatures are equal, we have that M^n is totally umbilical. Otherwise, without loss of generality, we may suppose that

$$\lambda_1=\cdots=\lambda_k=\lambda,\quad \lambda_{k+1}=\cdots=\lambda_n=\mu$$

for some $k=1,\ldots,n-1$, and $\lambda\mu=\frac{c_1}{n}+2c_2$. We can prove $k=1$ or $n-1$. In fact, if $1<k<n-1$, it follows from (3.7) that $\lambda^2=\mu^2=\lambda\mu=\frac{c_1}{n}+2c_2$. This is contradict to $\lambda\neq\mu$. Hence we have $k=1$ or $n-1$.

On the other hand, it follows from (3.5) that λ_i is constant for every i. Hence M^n is an isometric hypersurface with two distinct principal curvatures, one of which is simple. This completes the proof of Theorem 1.4. □

Proof of Theorem 1.6. When the constant a vanishes, Theorem 1.3 of [21] implies that Theorem 1.6 holds. Next we assume that a is not zero. It follows from Gauss formula (2.2) that (2.13) becomes

$$\begin{aligned} L(nH) = &\sum_{i,j,k} h_{ijk}^2 - n^2|\nabla H|^2 - S\sum_k \bar{K}_{n+1kn+1k} - nH\sum_i \lambda_i\bar{K}_{n+1iin+1} \\ &-\sum_{i,k}(\lambda_i-\lambda_k)^2\bar{K}_{ikik} - nH\sum_j\lambda_j^3 + S^2. \end{aligned} \tag{3.8}$$

Let $\mu_i=\lambda_i-H$ and $|\Phi|^2=\sum_i\mu_i^2$, we get

$$\sum_i\mu_i=0,\quad |\Phi|^2=S-nH^2,\quad \sum_i\lambda_i^3=\sum_i\mu_i^3+3H|\Phi|^2+nH^3.$$

M^n is totally umbilical if and only if $|\Phi|^2=0$. It follows from Lemma 2.1 that

$$-nH\sum_i\lambda_i^3 \geq -n|H|\frac{n-2}{\sqrt{n(n-1)}}|\Phi|^3+3H|\Phi|^2+nH^3. \tag{3.9}$$

From (3.2), we have

$$-S\sum_k\bar{K}_{n+1kn+1k} - nH\sum_i\lambda_i\bar{K}_{n+1iin+1} = c_1|\Phi|^2. \tag{3.10}$$

It follows from condition (1.2) that

$$-\sum_{i,k}(\lambda_i-\lambda_k)^2\bar{K}_{ikik} \geq \sum_{i,k}(\lambda_i-\lambda_k)^2 c_2 = nc_2|\Phi|^2. \tag{3.11}$$

Substituting (3.9), (3.10) and (3.11) into (3.8) and from Proposition 2.3, we get

$$L(nH) \geq |\Phi|^2\Big(2nc_2+c_1-nH^2-\frac{n(n-2)}{\sqrt{n(n-1)}}|H||\Phi|+|\Phi|^2\Big). \tag{3.12}$$

Consider the quadratic form

$$Q(x,y) = -x^2 - \frac{n-2}{\sqrt{n-1}}xy + y^2. \tag{3.13}$$

By the orthogonal transformation

$$(*) \qquad \begin{cases} u = \frac{1}{\sqrt{2n}}\left\{(1+\sqrt{n-1})y + (1-\sqrt{n-1})x)\right\}, \\ v = \frac{1}{\sqrt{2n}}\left\{(1+\sqrt{n-1})y + (1+\sqrt{n-1})x)\right\}, \end{cases}$$

the equation (3.13) becomes

$$Q(x,y) = \frac{n}{2\sqrt{n-1}}(u^2 - v^2). \tag{3.14}$$

Let $x = \sqrt{nH^2}$, $y = |\Phi|$. Then $u^2 + v^2 = x^2 + y^2 = |\Phi|^2 + nH^2 = S$. Hence we have

$$\begin{aligned} 2nc_2 + c_1 + Q(x,y) &= 2nc_2 + c_1 - \frac{n}{2\sqrt{n-1}}(u^2+v^2) + \frac{n}{\sqrt{n-1}}u^2 \\ &\geq 2nc_2 + c_1 - \frac{n}{2\sqrt{n-1}}S. \end{aligned} \tag{3.15}$$

It follows from (3.12) and (3.15) that

$$L(nH) \geq |\Phi|^2(2nc_2 + c_1 - \frac{n}{2\sqrt{(n-1)}}S). \tag{3.16}$$

Since $S \leq 2\sqrt{n-1}(2c_2 + c_1/n)$, then λ_i are bounded. It follows from (2.20) that

$$2\sqrt{n-1}(2c_2 + c_1/n) \geq n^2H^2 + n(n-1)(aH+b) - \sum_{i,j}\bar{K}_{jiij}. \tag{3.17}$$

Since $\bar{K}_{jiij}$ is constant, from (3.17) we have that H is bounded. Hence the Ricci curvature of M^n is bounded from below and $nH - \lambda_i + \frac{n-1}{2}a$ is bounded. It follows from Lemma 2.4 that there exists a sequence $\{q_k\}$ such that

$$\lim_{k\to\infty}(nH)(q_k) = \sup(nH), \ \lim_{k\to\infty}\|\nabla(nH)(q_k)\| = 0, \ \limsup_{k\to\infty} L(nH)(q_k) \leq 0.$$

It follows from (2.20) and $\sum_{i,j=1}^n \bar{K}_{ijji} =$ constant that $\lim_{k\to\infty} S(q_k) = \sup S$. Evaluating (3.16) at points q_k, we have

$$0 \geq (\sup S - n\sup H^2)(2nc_2 + c_1 - \frac{n}{2\sqrt{(n-1)}}\sup S).$$

Since $S \leq 2\sqrt{n-1}(2c_2 + c_1/n)$, we have

$$\sup(S - nH^2)(2nc_2 + c_1 - \frac{n}{2\sqrt{(n-1)}}\sup S) = 0. \tag{3.18}$$

If $\sup(S - nH^2) = 0$, then $S = nH^2$ and M^n is totally umbilical.

If $\sup S = 2\sqrt{n-1}(2c_2 + c_1/n)$ and $\sup S$ is attained on M^n, then $\sup H$ can be attained on M^n. This together with $L(nH) \geq 0$ forces that H is constant. Then all the inequalities to obtain (3.16) become equalities. Since the equality in Lemma 2.1 holds, we have that M^n has two distinct principal curvature. This completes the proof of Theorem 1.6. □

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