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SENSITIVITY ANALYSIS FOR A CLASS OF IMPLICIT MULTIFUNCTIONS WITH APPLICATIONS

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Abstract. In this paper, under some suitable conditions and in virtue of a selection which depends on a vector-valued function and a feasible set map, the sensitivity analysis of a class of implicit multifunctions is investigated. Moreover, by using the results established, the solution sets of parametric vector optimization problems are studied.

1. Introduction

Sensitivity analysis is not only theoretically interesting but also practically important in optimization theory. It means the quantitative analysis, that is, the study of derivatives of perturbation functions. A number of useful results have been obtained in scalar optimization problems (see [4, 7, 24]), in vector optimization problems (see [9, 13, 29, 30]), in scalar variational inequalities (see [18, 19, 28, 31]) and in vector variational inequalities (see [15, 12, 16]).

Implicit multifunctions (or set-valued maps) can be obtained from generalized equations which were first introduced by Robinson (see [20, 21, 22]). The sensitivity analysis of implicit multifunctions is important in variational analysis for its role where it provides a means for quantifying the sensitivity to data perturbation of solutions to parameterized optimization problems. In this paper, we consider the following implicit multifunction which was introduced in [11]:

(1) $$S(u) = \{x \in X(u) \mid 0_{\mathbb{R}^p} \in f(x,u) + Q(x,u)\},\ \forall u \in \mathbb{R}^m,$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$ is a vector-valued map, $0_{\mathbb{R}^p}$ denotes the origin of $\mathbb{R}^p$, $X : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ and $Q : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^p$ are two multifunctions. It is well known that the stationary points for the parametric optimization problem

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whose objective function and feasible set depend on the parameter (see Section 5 of [10]) may be expressed as the implicit multifunction of form (1). Levy [10] investigated the derivative properties of the implicit multifunction of form (1) and obtained the following inclusion relation:

$$DS(\hat{u},\hat{x})(u) \subset \{x \in DX(\hat{u},\hat{x})(u) \mid 0_{\mathbb{R}^p} \in f'(\hat{x},\hat{u})(x,u) + DQ(\hat{x},\hat{u},-\hat{y})(x,u)\}, \tag{2}$$

where $DS(\hat{u},\hat{x})$ and $f'(\hat{x},\hat{u})$ denote the contingent derivative and the Hadamard directional derivative of S and f, respectively. However, the converse inclusion of (2), in general, does not hold without strong assumptions.

The implicit multifunction of form (1) contains the following particular type of implicit multifunctions:

$$S_1(u) := \{x \in X(u) \mid 0_{\mathbb{R}^p} \in f(x,u) + H(x)\}, \tag{3}$$

where $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ is a multifunction. The implicit multifunction (3) was first introduced by Robinson [20, 21, 22]. The stationary points associated with parametric optimization problems, where the objective functions only depend on the parameter, and the solution sets of parametric variational inequalities can be expressed as (3). In [10], Levy also discussed the contingent derivative of S_1 and obtained an inclusion relation which is similar to (2). King and Rockafellar [8] got an explicit expression of the contingent derivative of S_1 when it is a singleton. Dontchev [6] obtained a formula for the directionally differentiability of a selection solution map (a single-valued map) of S_1.

However, there is no paper to discuss explicit expressions of the contingent derivatives of S and S_1, respectively when they are general sets rather than singletons. In this paper, we introduce a selection map Y of Q which is defined by

$$Y(u) = \bigcup_{x \in X(u)} \{-f(x,u) \cap Q(x,u)\},\ \forall u \in \mathbb{R}^m. \tag{4}$$

Then, S can be rewritten as $S(u) = \{x \in X(u) \mid 0_{\mathbb{R}^p} \in f(x,u) + Y(u)\}$, $\forall u \in \mathbb{R}^m$. Under the above selection and other suitable conditions, we can obtain the following explicit expression:

$$DS(\hat{u},\hat{x})(u) = \{x \in DX(\hat{u},\hat{x})(u) \mid 0_{\mathbb{R}^p} \in f'(\hat{x},\hat{u})(x,u) + DY(\hat{u},-\hat{y})(u)\}. \tag{5}$$

The above selection means that the relevant information with S is reserved and others is ignored. In virtue of this selection, the implicit multifunction S can been interpreted as follows: If $-f$ is viewed as an objective function and X is considered as a feasible solution map, then Y and S are the optimal value map and the optimal solution map, respectively; and then (5) denotes the relationship between contingent derivative of the optimal value map and contingent derivative of the optimal solution map. Thus, it is clear that the selection multifunction Y is more accurate than Q to depict the contingent derivative of S and this selection plays the role of a bridge between S and Q.

Similarly, by using the selection Y_1 which is defined by

$$Y_1(u) = \bigcup_{x \in X(u)} \{-f(x,u) \cap H(x)\},$$

we obtain an explicit expression of the contingent derivative of S_1 which is similar to (5).

Furthermore, we also discuss the other particular type of implicit multifunctions:

$$S_2(u) := \{x \in X(u) \mid 0_{\mathbb{R}^p} \in f(x,u) + G(u)\}, \tag{6}$$

where $G : \mathbb{R}^m \rightrightarrows \mathbb{R}^p$ is a multifunction. We know that the solution sets of parametric vector optimization problems may be expressed as the implicit multifunctions (6). For parametric vector optimization problems, so far, people have only investigated the derivative properties of optimal value (weak optimal value) maps and feasible value maps (see [9, 13, 29, 30]). There is no paper to study the explicit expressions of various derivatives of solutions to parametric vector optimization problems. In this paper, by using a similar selection, we obtain an explicit expression of the contingent derivative of S_2 and apply this result to obtain an explicit expression of the contingent derivative of solution sets of parametric vector optimization problems.

The paper is organized as follows. In Section 2, we recall preliminary material from sensitivity analysis needed for the subsequent sections. In Section 3, by using the preceding selection, we first investigate the differentiability properties of S. Then, we study the differentiability properties of S_1 and S_2, respectively. In Section 4, by using the results of Section 3, we study the differentiability properties of optimal (weak optimal) solution maps to parametric vector optimization problems.

2. Mathematical preliminaries

In this section, suppose that $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ is a multifunction and $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$ is a vector-valued map. The effective domain and graph of F are defined by $\text{dom}F := \{x \in \mathbb{R}^n \mid F(x) \neq \emptyset\}$ and $\text{gph}F := \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^p \mid y \in F(x)\}$, respectively. Let S be a nonempty set of $\mathbb{R}^n$. Let $\mathbb{B}_{\mathbb{R}^n}, \mathbb{B}_{\mathbb{R}^m}$ and $\mathbb{B}_{\mathbb{R}^p}$ denote the closed unit balls of $\mathbb{R}^n, \mathbb{R}^m$ and $\mathbb{R}^p$, respectively. Let $C \subset \mathbb{R}^p$ be a pointed closed convex cone.

Definition 1 ([3])**.** Let $\hat{x} \in \text{cl}\, S$, where $\text{cl}\, S$ denotes the closure of S.

The contingent cone $T(S,\hat{x})$ of S at $\hat{x}$ is the set of all $x \in \mathbb{R}^n$ such that there exist sequences $h_n \downarrow 0$ and $\{x_n\} \subset \mathbb{R}^n$ with $x_n \to x$ and $\hat{x} + h_n x_n \in S$, $\forall n$.

The adjacent cone $T^\flat(S,\hat{x})$ of S at $\hat{x}$ is the set of all $x \in X$ such that for any sequence $h_n \downarrow 0$, there exists a sequence $\{x_n\} \subset \mathbb{R}^n$ with $x_n \to x$ and $\hat{x} + h_n x_n \in S$, $\forall\, n$.

S is said to be derivable at $\hat{x}$ if and only if $T(S,\hat{x}) = T^\flat(S,\hat{x})$.

Definition 2 ([3, 23])**.** Let $(\hat{x}, \hat{y}) \in \text{gph}F$.

The contingent derivative of F at $(\hat{x}, \hat{y})$ is the set-valued map $DF(\hat{x}, \hat{y}) : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ whose graph is $T(\text{gph}F, (\hat{x}, \hat{y}))$.

The adjacent derivative of F at $(\hat{x}, \hat{y})$ is the set-valued map $D^\flat F(\hat{x}, \hat{y}) : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ whose graph is $T^\flat(\text{gph}F, (\hat{x}, \hat{y}))$.

F is said to be proto-differentiable at $(\hat{x}, \hat{y})$ whose proto-derivative is denoted by $F'_{(\hat{x},\hat{y})}$ if and only if $\text{gph}F'_{(\hat{x},\hat{y})} = T(\text{gph}F, (\hat{x}, \hat{y})) = T^\flat(\text{gph}F, (\hat{x}, \hat{y}))$, i.e., $\text{gph}F$ is derivable at $(\hat{x}, \hat{y})$.

Definition 3 ([17, 23])**.** Let $(\hat{x}, \hat{y}) \in \text{gph}F$. F is said to be semi-differentiable at $(\hat{x}, \hat{y})$ if and only if for any $y \in DF(\hat{x}, \hat{y})(x)$, any $t_n \downarrow 0$ and any $x_n \to x$, there exists a sequence $y_n \to y$ such that $\hat{y} + t_n y_n \in F(\hat{x} + t_n x_n)$.

Remark 2.1. Semi-differentiability is a more exacting property than proto-differentiability. The relationship between them has been obtained by Rockafellar (See [23]). About their equivalent definitions can also be found in [23].

Definition 4 ([2])**.** Let $\hat{x} \in \mathbb{R}^n$ and $\mathbb{B}_{\mathbb{R}^p}$ be the closed unit ball in $\mathbb{R}^p$. F is said to be upper locally Lipschitz at $\hat{x}$ if and only if there exist a constant $L > 0$ and a neighborhood $N_{\hat{x}}$ of $\hat{x}$ such that for each $x \in N_{\hat{x}}$, $F(x) \subset F(\hat{x}) + L||x - \hat{x}||\mathbb{B}_{\mathbb{R}^p}$.

F is said to be Lipschitz around $\hat{x}$ if and only if there exist a constant $L > 0$ and a neighborhood $N_{\hat{x}}$ of $\hat{x}$ such that for each $x_1, x_2 \in N_{\hat{x}}$, $F(x_1) \subset F(x_2) + L||x_1 - x_2||\mathbb{B}_{\mathbb{R}^p}$.

Definition 5 ([4, 25])**.** g is called directionally differentiable at $(\hat{x}, \hat{u})$ in the Hadamard sense with Hadamard directional derivative $g'(\hat{x}, \hat{u}) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$, if and only if for any sequences $x_n \to x$, $u_n \to u$ and $t_n \downarrow 0$, it holds that

$$g'(\hat{x}, \hat{u})(x, u) = \lim_{n\to\infty} \frac{g(\hat{x} + t_n x_n, \hat{u} + t_n u_n) - g(\hat{x}, \hat{u})}{t_n}, \forall (x, u) \in \mathbb{R}^n \times \mathbb{R}^m.$$

Remark 2.2. It is shown in [27] that the directionally differentiability in the Hadamard sense is weaker than the Fréchet differentiability. It follows from Proposition 2.46 of [4] that $g'(\hat{x}, \hat{u})(\cdot, \cdot)$ is continuous on $\mathbb{R}^n \times \mathbb{R}^m$. And it follows from the directionally differentiability at $(\hat{x}, \hat{u})$ in the Hadamard sense that $g(\cdot, \cdot)$ is continuous at $(\hat{x}, \hat{u})$ and $g(\hat{x} + t_n x_n, \hat{u} + t_n u_n) = g(\hat{x}, \hat{u}) + t_n g'(\hat{x}, \hat{u})(x_n, u_n) + o(t_n)$, where $o(t_n)$ denotes the remainder term with $\frac{o(t_n)}{t_n} \to 0_{\mathbb{R}^p}$ as $n \to \infty$. It is also clear that $g'(\hat{x}, \hat{u})$ is positively homogeneous, i.e., $g'(\hat{x}, \hat{u})(tx, tu) = tg'(\hat{x}, \hat{u})(x, u), \forall t \geq 0$.

Definition 6 ([26, 30])**.** F is called C-minicomplete by W near $\hat{x}$ if and only if $F(x) \subseteq W(x) + C$, $\forall x \in U_{\hat{x}}$, where $W(x) := \min_C F(x)$ and $U_{\hat{x}}$ is a neighborhood of $\hat{x}$.

Remark 2.3. Since $W(x) \subseteq F(x)$, if F is C-minicomplete by W near $\hat{x}$, then, $F(x) + C = W(x) + C$, $\forall x \in U_{\hat{x}}$.

3. Sensitivity analysis to a class of implicit multifunctions

In this section, we investigate the differentiability properties of S to the form (1). And then we compare our results to much of the existing results.

Theorem 3.1. *Let $S : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ and $Y : \mathbb{R}^m \rightrightarrows \mathbb{R}^p$ be defined by* (1) *and* (4)*, respectively, $\hat{x} \in S(\hat{u})$ and $\hat{y} = f(\hat{x}, \hat{u})$. Suppose that f is directionally differentiable at $(\hat{x}, \hat{u})$ in the Hadamard sense, X is semi-differentiable at $(\hat{u}, \hat{x})$ and there exist a constant $L > 0$ and a neighborhood $N_{\hat{u}}$ of $\hat{u}$ such that*

$$||f(x_1, u) - f(x_2, u)|| \geq L||x_1 - x_2||, \ \forall u \in N_{\hat{u}} \text{ and } x_1, x_2 \in X(u). \tag{7}$$

Then, for each $u \in \mathrm{dom}DS(\hat{u}, \hat{x})$ we have

$$DS(\hat{u}, \hat{x})(u) = \{x \in DX(\hat{u}, \hat{x})(u) \mid 0_{\mathbb{R}^p} \in f'(\hat{x}, \hat{u})(x, u) + DY(\hat{u}, -\hat{y})(u)\}. \tag{8}$$

Proof. Let $x \in DS(\hat{u}, \hat{x})(u)$. Then, there exist sequences $t_n \downarrow 0$ and $(u_n, x_n) \to (u, x)$ such that

$$\hat{x} + t_n x_n \in S(\hat{u} + t_n u_n).$$

Therefore, we get

$$\hat{x} + t_n x_n \in X(\hat{u} + t_n u_n) \tag{9}$$

and

$$-f(\hat{x} + t_n x_n, \hat{u} + t_n u_n) \in Y(\hat{u} + t_n u_n). \tag{10}$$

It follows from the directionally differentiability of f at $(\hat{x}, \hat{u})$ that

$$f(\hat{x} + t_n x_n, \hat{u} + t_n u_n) = f(\hat{x}, \hat{u}) + t_n f'(\hat{x}, \hat{u})(x_n, u_n) + o(t_n). \tag{11}$$

Thus, by (9), (10) and (11), we have $x \in DX(\hat{u}, \hat{x})(u)$ and $-f'(\hat{x}, \hat{u})(x, u) \in DY(\hat{u}, -\hat{y})(u)$.

Suppose that x belongs to the right part of (8). Then, there exist sequences $t_n \downarrow 0$ and $(u_n, y_n) \to (u, f'(\hat{x}, \hat{u})(x, u))$ such that

$$-\hat{y} - t_n y_n \in Y(\hat{u} + t_n u_n).$$

Thus, there exists $x_n \in X(\hat{u} + t_n u_n)$ such that

$$-\hat{y} - t_n y_n = -f(x_n, \hat{u} + t_n u_n) \text{ and } x_n \in S(\hat{u} + t_n u_n).$$

Since X is semi-differentiable at $(\hat{u}, \hat{x})$, for the above sequences t_n and u_n there exists a sequence $x'_n \to x$ such that $\hat{x} + t_n x'_n \in X(\hat{u} + t_n u_n)$. Because $\hat{u} + t_n u_n \in N_{\hat{u}}$ for sufficiently large n, it follows from (7) that

$$||f(\hat{x} + t_n x'_n, \hat{u} + t_n u_n) - f(x_n, \hat{u} + t_n u_n)|| \geq L||\hat{x} + t_n x'_n - x_n||.$$

Since f is directionally differentiable at $(\hat{x}, \hat{u})$ in the Hadamard sense,

$$f(\hat{x} + t_n x'_n, \hat{u} + t_n u_n) = f(\hat{x}, \hat{u}) + t_n f'(\hat{x}, \hat{u})(x'_n, u_n) + o(t_n).$$

Thus, there exists $b_n \in \mathbb{B}_{\mathbb{R}^n}$ such that

$$x_n = \hat{x} + t_n \left[x'_n - \frac{1}{L}||f'(\hat{x}, \hat{u})(x'_n, u_n) + \frac{o(t_n)}{t_n} - y_n||b_n\right] \in S(\hat{u} + t_n u_n).$$

Since $||f'(\hat{x},\hat{u})(x_n,u_n)+\frac{o(t_n)}{t_n}-y_n|| \to 0$ and $\{b_n\}$ is bounded, $x \in DS(\hat{u},\hat{x})(u)$. □

The following two examples show that the condition (7) is essential to Theorem 3.1.

Example 3.1. Let $n=m=p=1$ and f, X, Q be given by

$$f(x,u) := x^2+u^2,\ X(u) := [u+1,+\infty),$$
$$Q(x,u) := \begin{cases} [-x^2-u^2-1,0] & \text{if } u+1 \le x \le u+2 \\ (x^2+u^2,+\infty) & \text{otherwise.} \end{cases}$$

Then, we easily get

$$Y(u) = \{-x^2-u^2 \mid u+1 \le x \le u+2\} \text{ and } S(u) = [u+1,u+2].$$

Let $\hat{u}=0, \hat{x}=2$. Then, $\hat{y}=4$. We can easily verify that there exist $N_{\hat{u}} = [-0.1,0.1]$ and $L=1.8$ satisfying (7) and X is semi-differentiable at $(0,2)$ with $DX(0,2)(u)=\mathbb{R}$. Consequently, the conclusions of Theorem 3.1 hold. And we can easily get that

$$DS(0,2)(u) = \{x \in DX(0,2)(u) \mid 0 \in f'(2,0)(x,u)+DY(0,-4)(u)\} = (-\infty,u],$$

where $f'(2,0)(x,u)=4x$ and $DY(0,-4)(u)=[-4u,+\infty)$.

Example 3.2. Let $n=m=p=1$ and f, X, Q be given by

$$f(x,u) := u(x^2-x),\ X(u) := [u,+\infty),$$
$$Q(x,u) := \begin{cases} (-\infty,|u(x^2-x)|] & \text{if } u \le x \le u+1 \\ \emptyset & \text{otherwise.} \end{cases}$$

Then, we easily get

$$Y(u) = \{-u(x^2-x) \mid u \le x \le u+1\} \text{ and } S(u) = [u,u+1].$$

Let $\hat{u}=0$, $\hat{x}=1$. Then, $\hat{y}=0$. We can easily verify that X is semi-differentiable at $(0,1)$ with $DX(0,1)(u)=\mathbb{R}$ and for each neighborhoods of 0, (7) does not hold when $u=0$. And we can easily get that (8) does not hold when $u>0$, since

$$DS(0,1)(u) = (-\infty,u] \text{ and}$$
$$\{x \in DX(0,1)(u) \mid 0 \in f'(1,0)(x,u)+DY(0,0)(u)\} = \mathbb{R}.$$

In fact, it is obvious that $f'(1,0)(x,u)=0,\ \forall x,u \in \mathbb{R}$. Let $y \in DY(0,0)(u)$ and $u>0$. Then, there exist sequences $t_n \downarrow 0$ and $(u_n,y_n) \to (u,y)$ with $t_ny_n \in Y(t_nu_n)$. Because $u>0$, $t_nu_n \ge 0$ and $1+t_nu_n \ge 1$ for sufficiently large n. Thus,

$$-t_n^2u_n^2(1+t_nu_n) \le t_ny_n \le \frac{t_nu_n}{4}.$$

And then we get $DY(0,0)(u)=[0,\frac{u}{4}]$. Hence,

$$\{x \in DX(0,1)(u) \mid 0 \in f'(1,0)(x,u)+DY(0,0)(u)\} = \mathbb{R}.$$

Let a set-valued map $\tilde{X} : \mathbb{R}^m \times \mathbb{R}^p \rightrightarrows \mathbb{R}^n$ be defined by

$$\tilde{X}(u, y) := \{x \in X(u) \mid f(x, u) = y\}.$$

We consider the differentiability properties to Y of the form (4).

Theorem 3.2. *Let $\hat{x} \in X(\hat{u})$ and $\hat{y} = f(\hat{x}, \hat{u})$. Suppose that $\tilde{X}$ is upper locally Lipschitz at $(\hat{u}, \hat{y})$, $\tilde{X}(\hat{u}, \hat{y})$ contains the finite number of points and f is directionally differentiable at $(\hat{x}, \hat{u})$ in the Hadamard sense. Then, for each $u \in \mathrm{dom} DY(\hat{u}, -\hat{y})$, one has*

(12)
$$DY(\hat{u}, -\hat{y})(u) \subset \bigcup_{x_0 \in \tilde{X}(\hat{u}, \hat{y})} \bigcup_{x \in DX(\hat{u}, x_0)(u)} \{-f'(x_0, \hat{u})(x, u) \cap DQ(x_0, \hat{u}, -\hat{y})(x, u)\}.$$

Proof. Suppose that $-y \in DY(\hat{u}, -\hat{y})(u)$. Then, there exist sequences $t_n \downarrow 0$ and $(u_n, y_n) \to (u, y)$ with

$$-\hat{y} - t_n y_n \in Y(\hat{u} + t_n u_n).$$

Thus, there exists $x_n \in X(\hat{u} + t_n u_n)$ with

$$-\hat{y} - t_n y_n = -f(x_n, \hat{u} + t_n u_n) \tag{13}$$

and

$$-\hat{y} - t_n y_n \in Q(x_n, \hat{u} + t_n u_n). \tag{14}$$

Since $\tilde{X}$ is upper locally Lipschitz at $(\hat{u}, \hat{y})$, there exists a constant $M > 0$ with $\tilde{X}(\hat{u}+t_n u_n, \hat{y}+t_n y_n) \subset \tilde{X}(\hat{u}, \hat{y}) + M t_n(||u_n||+||y_n||)\mathbb{B}_{\mathbb{R}^n}$ for sufficiently large n. Therefore, there exists $b_n \in \mathbb{B}_{\mathbb{R}^n}$ such that

$$x_n - M t_n(||u_n|| + ||y_n||) b_n \in \tilde{X}(\hat{u}, \hat{y}).$$

Because $\tilde{X}(\hat{u}, \hat{y})$ contains the finite number of points, we assume, without loss of generality, that

$$x_n - M t_n(||u_n|| + ||y_n||) b_n \to x_0 \in \tilde{X}(\hat{u}, \hat{y}).$$

It is clear that

$$x_n - M t_n(||u_n|| + ||y_n||) b_n = x_0 \text{ for sufficiently large } n.$$

It follows from the boundness of b_n that $x_n \to x_0$ and we assume, without loss of generality, that $b_n \to b \in \mathbb{B}_{\mathbb{R}^n}$. Consequently, for sufficiently large n, we obtain

$$\frac{x_n - x_0}{t_n} = M(||u_n|| + ||y_n||) b_n \to M(||u|| + ||y||) b. \tag{15}$$

Thus, it follows from

$$x_0 + t_n \frac{x_n - x_0}{t_n} = x_n \in X(\hat{u} + t_n u_n)$$

that $M(||u||+||y||)b \in DX(\hat{u}, x_0)(u)$. Since f is directionally differentiable at $(\hat{x}, \hat{u})$ in the Hadamard sense,

$$f(x_n, \hat{u}+t_n u_n) = f(x_0, \hat{u}) + t_n f'(x_0, \hat{u})(\frac{x_n - x_0}{t_n}, u_n) + o(t_n).$$

Then, by (13) and (15) we have $\hat{y} = f(x_0, \hat{u})$ and

$$y = f'(x_0, \hat{u})(M(||u||+||y||)b, u).$$

Moreover, by (14) we get

$$-y \in DQ(x_0, \hat{u}, -\hat{y})(M(||u||+||y||)b, u).$$

Hence, (12) is obtained. □

The following example illustrates that Theorem 3.2 is applicable.

Example 3.3. Let $n = m = p = 1$ and f, X, Q be given by

$$f(x,u) := x+u,\ X(u) := [u-1, u+2], \text{ and}$$

$$Q(x,u) := \begin{cases} [-|x|-|u|, |x|+|u|] & \text{if } u \le x \le u+1 \\ (-\infty, -|x|-|u|) & \text{otherwise.} \end{cases}$$

Then, we easily get

$$Y(u) = \{-x-u \mid u \le x \le u+1\}.$$

Let $\hat{u} = 0, \hat{x} = 1$. Then, $\hat{y} = 1$. We can verify that $\tilde{X}(0,1) = \{1\}$ and $\tilde{X}$ is upper locally Lipschitz at $(0,1)$. Thus, the conclusions of Theorem 3.2 hold. Moreover, the left part of (12) is the proper subset of the right part of (12). Indeed, for $u = 0$ we can easily get that $DY(0,-1)(0) = [0, +\infty)$ and

$$\bigcup_{x \in DX(0,1)(0)} \{-f'(1,0)(x,0) \cap DQ(1,0,-1)(x,0)\} = \mathbb{R},$$

where $DX(0,1)(0) = \mathbb{R}$, $f'(1,0)(x,0) = x$ and

$$DQ(1,0,-1)(x,0) = \begin{cases} [-x, +\infty) & \text{if } x < 0 \\ \mathbb{R} & \text{if } x = 0 \\ (-\infty, -x] & \text{if } x > 0. \end{cases}$$

Remark 3.1. It follows from the above example that the converse inclusion of (12) may not hold. In fact, let $I : \mathbb{R}^m \to \mathbb{R}^m$ be an identical map, i.e., $I(u) = u$, $\forall u \in \mathbb{R}^m$ and $G : \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m$ be defined by $G(u) := X(u) \times I(u)$. For the special case, Y can be rewritten as

$$Y(u) = -(f \circ G)(u) \cap (Q \circ G)(u). \tag{16}$$

Although Li and Meng [13] obtained a formula for the contingent derivative of $f \circ G$ and Li et al. [14] discussed the contingent derivative of $Q \circ G$ and the contingent derivative of intersection of two set-valued maps, too many strict conditions are needed to get an explicit expression of the contingent derivatives of (16). So, it is very difficult to make (12) become an equality.

Next, we give a necessary condition of (7), which is very useful to investigate the sensitivity analysis of implicit multifunctions.

Proposition 3.3. *If* (7) *holds, then, we get*

$$d(x, S(u)) \leq \frac{1}{L} d(-f(x,u), Y(u)), \forall u \in N_{\hat{u}}, x \in X(u). \tag{17}$$

Proof. Assume that (17) does not hold. Then, there exist $u \in N_{\hat{u}}$, $x \in X(u)$ and $\bar{x} \in X(u)$ with $-f(\bar{x}, u) \in Y(u)$ such that

$$d(x, S(u)) > \frac{1}{L} d(-f(x,u), Y(u)) = \frac{1}{L} ||f(x,u) - f(\bar{x}, u)||.$$

Since $\bar{x} \in S(u)$, $d(x, S(u)) \leq ||x - \bar{x}||$. Thus,

$$\frac{1}{L} ||f(x,u) - f(\bar{x}, u)|| < ||x - \bar{x}||,$$

which contradicts (7). Hence, (17) holds. □

From the above proposition, we know that (17) is generally weaker than (7). However, (7) is more easier than (17) to be verified. Next, we use a condition which is similar to (17) to prove (8).

Proposition 3.4. *If* (7) *is replaced by*

$$d(x, S(u)) \leq \frac{1}{L} d(-f(x,u), Y(u)), \forall u \in N_{\hat{u}}, x \in X(u) \cap N_{\hat{x}}. \tag{18}$$

and other conditions of Theorem 3.1 hold, then (8) *still holds.*

Proof. From Theorem 3.1, we only need to prove that $DS(\hat{u}, \hat{x})(u)$ contains the right part of (8). Suppose that x belongs to the right part of (8). Then, there exist sequences $t_n \downarrow 0$ and $(u_n, y_n) \to (u, f'(\hat{x}, \hat{u})(x, u))$ such that

$$-f(\hat{x}, \hat{u}) - t_n y_n \in Y(\hat{u} + t_n u_n).$$

Thus, there exists $x_n \in X(\hat{u} + t_n u_n)$ such that

$$-f(\hat{x}, \hat{u}) - t_n y_n = -f(x_n, \hat{u} + t_n u_n).$$

Since X is semi-differentiable at $(\hat{u}, \hat{x})$, for the above sequences t_n and u_n there exists a sequence $x'_n \to x$ such that $\hat{x} + t_n x'_n \in X(\hat{u} + t_n u_n)$. Because $\hat{u} + t_n u_n \in N_{\hat{u}}$ for sufficiently large n and $x_n, \hat{x} + t_n x'_n \in X(\hat{u} + t_n u_n)$, from (18) we have

$$\begin{aligned} d(\hat{x} + t_n x'_n, S(\hat{u} + t_n u_n)) &\leq \frac{1}{L} d(-f(\hat{x} + t_n x'_n, \hat{u} + t_n u_n), Y(\hat{u} + t_n u_n)) \\ &\leq \frac{1}{L} ||f(\hat{x} + t_n x'_n, \hat{u} + t_n u_n) - f(x_n, \hat{u} + t_n u_n)||. \end{aligned}$$

Thus, there exists $b_n \in \mathbb{B}_{\mathbb{R}^n}$ such that

$$\hat{x} + t_n \left[x'_n - \frac{1}{L} ||f'(\hat{x}, \hat{u})(x'_n, u_n) + \frac{o(t_n)}{t_n} - y_n|| b_n \right] \in S(\hat{u} + t_n u_n).$$

Since $||f'(\hat{x},\hat{u})(x_n,u_n)+\frac{o(t_n)}{t_n}-y_n|| \to 0$ and $\{b_n\}$ is bounded, $x \in DS(\hat{u},\hat{x})(u)$. □

Remark 3.2. If (18) is replaced by

$$d(x,S(u)) \leq \frac{1}{L} d(-f(x,u),Y(u)), \forall u \in N_{\hat{u}}, x \in N_{\hat{x}},$$

then, it follows from the above process of proof that the semi-differentiability of X is no longer necessary and (8) still holds.

The remainder of this section concerns applications of Theorems 3.1 and 3.2 to three particular types of solution multifunctions encountered in parametric optimization problems.

Corollary 3.5. *Suppose that $S_1 : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is defined by* (3) *and $Y_1 : \mathbb{R}^m \rightrightarrows \mathbb{R}^p$ is defined by $Y_1(u) := \bigcup_{x \in X(u)} \{-f(x,u) \cap H(x)\}$.*

(i) *Under the conditions of Theorem 3.1 or Proposition 3.4, we get*

$$DS_1(\hat{u},\hat{x})(u) = \{x \in DX(\hat{u},\hat{x})(u) \mid 0_{\mathbb{R}^p} \in f'(\hat{x},\hat{u})(x,u) + DY_1(\hat{u},-\hat{y})(u)\}.$$

(ii) *Assume that all the conditions of Theorem 3.2 hold. Then, we have*

$$DY_1(\hat{u},-\hat{y})(u) \subset \bigcup_{x_0 \in \tilde{X}(\hat{u},\hat{y})} \bigcup_{x \in DX(\hat{u},x_0)(u)} \{-f'(x_0,\hat{u})(x,u) \cap DH(x_0,-\hat{y})(x)\}.$$

Remark 3.3. In Theorem 4.1 of [8], King and Rockafellar obtained the semi-differentiability of S_1 at some point when $DS_1(\hat{u},\hat{x})$ is a single-valued map. In Theorem 2.4 of [6], Dontchev investigated the directionally differentiability of a selection solution map (a single-valued map) of S_1. However, it follows from Example 3.1 that S_1 and $DS_1(\hat{u},\hat{x})$, in general, are set-valued maps. So, Corollary 3.5 is different from results of [6] and [8].

Corollary 3.6. *Suppose that $S_2 : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is defined by* (6) *and $Y_2 : \mathbb{R}^m \rightrightarrows \mathbb{R}^p$ is defined by $Y_2(u) := \bigcup_{x \in X(u)} \{-f(x,u)\} \cap G(u)$.*

(i) *Under the conditions of Theorem 3.1 or Proposition 3.4, we get*

$$DS_2(\hat{u},\hat{x})(u) = \{x \in DX(\hat{u},\hat{x})(u) \mid 0_{\mathbb{R}^p} \in f'(\hat{x},\hat{u})(x,u) + DY_2(\hat{u},-\hat{y})(u)\}.$$

(ii) *Assume that all the conditions of Theorem 3.2 hold. Then, we have*

$$DY_2(\hat{u},-\hat{y})(u) \subset \bigcup_{x_0 \in \tilde{X}(\hat{u},\hat{y})} \bigcup_{x \in DX(\hat{u},x_0)(u)} \{-f'(x_0,\hat{u})(x,u)\} \cap DG(\hat{u},-\hat{y})(u).$$

Corollary 3.7. *Suppose that $S_3 : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ and $Y_3 : \mathbb{R}^m \rightrightarrows \mathbb{R}^p$ are defined by $S_3(u) := \{x \in X \mid 0_{\mathbb{R}^p} \in f(x,u) + M\}$ and $Y_3(u) := \bigcup_{x \in X} \{-f(x,u)\} \cap M$, respectively, where X and M are subsets of $\mathbb{R}^n$ and $\mathbb{R}^p$, respectively.*

(i) *If X is derivable at $\hat{x}$ and* (18) *holds, then we get*

$$DS_3(\hat{u},\hat{x})(u) = \{x \in T(X,\hat{x}) \mid 0_{\mathbb{R}^p} \in f'(\hat{x},\hat{u})(x,u) + DY_3(\hat{u},-f(\hat{x},\hat{u}))(u)\}.$$

(ii) *Assume that all the conditions of Theorem 3.2 hold. Then, we have*

$$DY_3(\hat{u}, -\hat{y})(u) \subset \bigcup_{x_0 \in \tilde{X}(\hat{u},\hat{y})} \bigcup_{x \in T(X,\hat{u})} \{-f'(x_0, \hat{u})(x, u)\} \cap T(M, -\hat{y}),$$

where $\tilde{X}(u, y) := \{x \in X \mid f(x, u) = y\}$.

Remark 3.4. In Corollary 5.7 of [1], Ahmaroq and Thibault got

$$DS_3(\hat{u}, \hat{x})(u) = \{x \in T(X, \hat{x}) \mid 0_{\mathbb{R}^p} \in f'(\hat{x}, \hat{u})(x, u) + T(M, -\hat{y})\},$$

under the following conditions: X is derivable at $\hat{x}$ and

$$d(x, S_3(u)) \leq \frac{1}{L} d(-f(x, u), M), \forall u \in N_{\hat{u}}, x \in X \cap N_{\hat{x}}. \tag{19}$$

Since the condition (18) can be obtained by replacing M with Y_3 in the condition (19), the above corollary is a generalization of Corollary 5.7 of [1]. In addition, under another condition Rockafellar [23] also got the same result as Corollary 5.7 in [1]. But by Theorem 3.2 of [5], we know that (19) is weaker than Rockafellar's condition.

4. Applications to parametric vector optimization problems

In this section, we apply Corollary 3.6 to the following parametric vector optimization problem:

$$VOP(u) \qquad \min_K f(x, u) \quad \text{s.t. } x \in X(u),$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$ is a vector-valued map, $X : \mathbb{R}^m \to \mathbb{R}^n$ is a multifunction which is the feasible solution map of $VOP(u)$ and $K \subset \mathbb{R}^p$ is a pointed closed convex cone with nonempty interior int$K$. Let $F : \mathbb{R}^m \to \mathbb{R}^p$ be a multifunction defined by

$$F(u) := \{f(x, u) \mid x \in X(u)\}, \ \forall u \in \mathbb{R}^m,$$

which is the feasible value map of $VOP(u)$. Let $0_{\mathbb{R}^n}, 0_{\mathbb{R}^m}$ and $0_{\mathbb{R}^p}$ denote the origins of $\mathbb{R}^n, \mathbb{R}^m$ and $\mathbb{R}^p$, respectively. Two set-valued maps E and W are defined respectively by

$$E(u) := \{y \in F(u) \mid (F(u) - y) \cap (-K) = \{0_{\mathbb{R}^p}\}\}, \ \forall u \in \mathbb{R}^m$$

and

$$W(u) := \min_{\text{int}K} F(u) = \{y \in F(u) \mid (F(u) - y) \cap (-\text{int}K) = \emptyset\}, \ \forall u \in \mathbb{R}^m.$$

By $S_4(u)$ and $S_5(u)$ we denote the optimal solutions and the weakly optimal solutions of $VOP(u)$, respectively, where

$$S_4(u) = \{x \mid f(x, u) \in E(u)\} \text{ and } S_5(u) = \{x \mid f(x, u) \in W(u)\}.$$

Naturally, for $S_4(u)$ and $S_5(u)$, we have

$$Y_4(u) = \bigcup_{x \in X(u)} f(x, u) \cap E(u) = E(u) \text{ and } Y_5(u) = \bigcup_{x \in X(u)} f(x, u) \cap W(u) = W(u).$$

It follows from Proposition 3.4 that the following result holds.

Proposition 4.1. *Let $\hat{x} \in S_4(\hat{u})$ and $\hat{y} = f(\hat{x}, \hat{u})$. Suppose that f is directionally differentiable at $(\hat{x}, \hat{u})$ in the Hadamard sense and X is semi-differentiable at $(\hat{u}, \hat{x})$.*

(i) *If there exist a constant $L > 0$ and neighborhoods $N_{\hat{u}}$ and $N_{\hat{x}}$ of $\hat{u}$ and $\hat{x}$, respectively, such that*

$$d(x, S_4(u)) \leq Ld(f(x, u), E(u)), \forall u \in N_{\hat{u}}, x \in X(u) \cap N_{\hat{x}}, \tag{20}$$

then, for each $u \in \mathrm{dom} DS_4(\hat{u}, \hat{x})$ we have

$$DS_4(\hat{u}, \hat{x})(u) = \{x \in DX(\hat{u}, \hat{x})(u) \mid f'(\hat{x}, \hat{u})(x, u) \in DE(\hat{u}, \hat{y})(u)\}.$$

(ii) *If there exist a constant $L > 0$ and neighborhoods $N_{\hat{u}}$ and $N_{\hat{x}}$ of $\hat{u}$ and $\hat{x}$, respectively, such that*

$$d(x, S_5(u)) \leq Ld(f(x, u), W(u)), \forall u \in N_{\hat{u}}, x \in X(u) \cap N_{\hat{x}}, \tag{21}$$

then, for each $u \in \mathrm{dom} DS_5(\hat{u}, \hat{x})$ we get

$$DS_5(\hat{u}, \hat{x})(u) = \{x \in DX(\hat{u}, \hat{x})(u) \mid f'(\hat{x}, \hat{u})(x, u) \in DW(\hat{u}, \hat{y})(u)\}.$$

Remark 4.1. If we assume that (7) holds, then, from Propositions 3.3 and 3.4 we have the above results still hold.

Theorem 4.2. *Let $\hat{x} \in S_4(\hat{u})$, $\hat{y} = f(\hat{x}, \hat{u})$ and $\tilde{X} : \mathbb{R}^m \times \mathbb{R}^p \rightrightarrows \mathbb{R}^n$ be defined by $\tilde{X}(u, y) := \{x \in X(u) \mid f(x, u) = y\}$. Suppose that*

(i) *there exists a pointed closed convex cone $\tilde{K}$ such that $\tilde{K} \subset \mathrm{int} K \cup \{0_{\mathbb{R}^p}\}$;*
(ii) *X is Lipschitz around $\hat{u}$ and $X(\hat{u})$ is compact;*
(iii) *F is proto-differentiable at $(\hat{u}, \hat{y})$;*
(iv) *F is $\tilde{K}$-minicomplete by E near $\hat{u}$;*
(v) *$\tilde{X}$ is upper locally Lipschitz at $(\hat{u}, \hat{y})$ and $\tilde{X}(\hat{u}, \hat{y})$ contains the finite number of points;*
(vi) *if there exist a constant $L > 0$ and neighborhoods $N_{\hat{u}}$ of $\hat{u}$ and $N_{\hat{x}}$ of $\hat{x}$ such that $d(x, S_4(u)) \leq Ld(f(x, u), W(u)), \forall u \in N_{\hat{u}}, x \in X(u) \cap N_{\hat{x}}$.*

Then, for each $u \in \mathrm{dom} DS_4(\hat{u}, \hat{x})$, one has

$$\begin{aligned} & DS_4(\hat{u}, \hat{x})(u) \\ = {} & DS_5(\hat{u}, \hat{x})(u) \\ = {} & \{x \in DX(\hat{u}, \hat{x})(u) \mid f'(\hat{x}, \hat{u})(x, u) \in \min\nolimits_{\mathrm{int} K} DF(\hat{u}, \hat{y})(u)\} \\ = {} & \left\{ x \mid f'(\hat{x}, \hat{u})(x, u) \in \min\nolimits_{\mathrm{int} K} \bigcup_{x_0 \in \tilde{X}(\hat{u}, \hat{y})} \bigcup_{x \in DX(\hat{u}, x_0)(u)} f'(x_0, \hat{u})(x, u) \right\}. \end{aligned}$$

Proof. By (v) and Theorem 4.1 of [13], we obtain that

$$DF(\hat{u}, \hat{y})(u) = \bigcup_{x_0 \in \tilde{X}(\hat{u}, \hat{y})} \bigcup_{x \in DX(\hat{u}, x_0)(u)} f'(x_0, \hat{u})(x, u).$$

By (i)-(iv) and Theorem 5.1 of [13], we get

$$DE(\hat{u}, \hat{y})(u) = DW(\hat{u}, \hat{y})(u) = \min_{\text{int}K} DF(\hat{u}, \hat{y})(u).$$

Since $S_4(u) \subset S_5(u)$ and $E(u) \subset W(u)$,

$$d(x, S_5(u)) \leq d(x, S_4(u)) \text{ and } d(f(x,u), W(u)) \leq d(f(x,u), E(u)).$$

Thus, (vi) deduces that (20) and (21) both hold. Hence, it follows from Proposition 4.1 that our results hold. □

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