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FUNCTION ALGEBRAS ON BIDISKS

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Abstract. We study sufficient conditions for function algebras generated by four smooth functions on a small closed bidisk near the origin in $\mathbb{C}^2$ to coincide with the space of continuous functions on the bidisk. This problem in one dimension has been studied by De Paepe and the second name author.

1. Introduction

Let K be a compact subset of $\mathbb{C}^n$. We denote by $C(K)$ the algebra of continuous complex valued functions on K, provided by the topology induced by the supremum norm $\|\cdot\|$. Let $P(K)$ be the uniform closure in $C(K)$ of all (restrictions to K) polynomials. We recall that for a given compact K in $\mathbb{C}^n$, by $\hat{K}$ we denote the polynomial convex hull of K, i.e.,

$$\hat{K} = \{z \in \mathbb{C}^n : |p(z)| \leq \|p\|_K \text{ for every polynomial } p \text{ in } \mathbb{C}^n\}.$$

We say that K is polynomially convex if $\hat{K} = K$. A compact $K \subset \mathbb{C}$ is polynomially convex if its complement is connected. In $\mathbb{C}^n$ $(n > 1)$ there no general condition that a compact subset is polynomially convex. It is well-known that $K, \hat{K}$ respectively can be identified with the space of maximal ideal of $C(K), P(K)$. The theory of polynomial convexity is curial in function theory of several complex variables. For instance, by the classical Oka-Weil theorem ensures that holomorphic function on neighborhoods of a compact polynomially convex set can be approximated uniformly by polynomials. Later on, Hörmander and Wermer proved that continuous function on a compact polynomially convex subset of smooth *totally real* manifold M in $\mathbb{C}^n$ can be approximated uniformly by polynomials (see Theorem 1.1, [3]). Recall that a manifold M is totally real at $p \in M$ if the real tangent space $T_p(M)$ of M at p contains no complex line. A manifolds M is totally real if it is totally real at any point of M. An example of totally real manifold is the real Euclidean space $\mathbb{R}^n$. In this case, the mentioned above theorem of Hörmander and Wermer

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reduces to classical Stone-Weierstrass theorem. Note that, if $f_1, \ldots, f_n$ are $\mathcal{C}^1$ functions on an open subset U of $\mathbb{C}^n$ and $\det(\frac{\partial f_i}{\partial \overline{z}_j}(a))_{ij} \neq 0$ with $a = (a_1, \ldots, a_n) \in U$, then $M = \{(z_1, \ldots, z_n, f_1, \ldots, f_n) : z \in U\}$ is totally real at $(a_1, \ldots, a_n, f_1(a), \ldots, f_n(a))$.

We now describe in details our problem. Let F, G be $\mathcal{C}^1$ functions defined on a neighborhood of the origin in $\mathbb{C}^2$. Assume that

$$\begin{aligned} F(0,0) = G(0,0) = \frac{\partial F}{\partial z}(0,0) = \frac{\partial G}{\partial z}(0,0) \\ = \frac{\partial F}{\partial w}(0,0) = \frac{\partial F}{\partial w}(0,0) = \frac{\partial F}{\partial \overline{w}}(0,0) \\ = \frac{\partial G}{\partial \overline{z}}(0,0) = 0, \end{aligned}$$

and

$$\frac{\partial F}{\partial \overline{z}}(0,0) = \frac{\partial G}{\partial \overline{w}}(0,0) = 1.$$

In other word, F (resp. G) looks like $\overline{z}$ (resp. $\overline{w}$) near the origin. The aim of this paper is to determine *explicitly* $[z^2, F^2, w^2, G^2]$, the closed function subalgebra of $C(D)$ generated by z^2, F^2, w^2, G^2 on a small bidisk $D = \tilde{D} \times \tilde{D}$, where $\tilde{D}$ is a closed disk around the origin in $\mathbb{C}$. In general, it is a difficult problem. We have partial answers in the case where the lowest order terms of $f := F - \overline{z}$ and $g := G - \overline{w}$ are polynomials whose coefficients satisfy certain conditions.

Our study is motivated by a similar problem in one complex variable. More precisely, let F be a smooth function which looks likes $\overline{z}$ near the origin in $\mathbb{C}$. Under what conditions the closed function subalgebras $[z^2, F^2; \tilde{D}]$ on a small closed $\tilde{D}$ coincides with $C(\tilde{D})$? A complete answer to this question is still unknown. However, using the theory of polynomial convexity, sufficient conditions on F are obtained in the case where $F - \overline{z}$ are perturbations of homogeneous polynomials in z and $\overline{z}$ whose coefficients satisfy certain easy checked conditions. For more details see [1], [2], [5] and the references listed therein.

In our work, we adopt the same approach like the one variable case. Namely, we consider the image of D under the differentiable mapping

$$S : (z, w) \mapsto \left(z^2, (\overline{z} + f)^2, w^2, (\overline{w} + g)^2\right).$$

Under some technical conditions on f and g, we first show that $P\big(S(D)\big) = C\big(S(D)\big)$. To prove this fact, we look at the *inverse* of $S(D)$ under the map

$$\pi : (z, w, u, v) \mapsto (z^2, w^2, u^2, v^2).$$

Then $\pi^{-1}\big(S(D)\big)$ is union of totally real graphs over D. By using a key lemma of Kallin giving a sufficient condition for polynomial convexity of union of two compact polynomially convex, we can show that $\pi^{-1}\big(S(D)\big)$ is polynomially convex for D small enough. Thus, by some variant of above cited theorem of Hömander and Wermer, it follows that $P(\pi^{-1}(S(D))) = P(\pi^{-1}(S(D)))$ for

D sufficiently small. By the technical lemma that is an extension in [5], we can conclude that $P(S(D)) = C(S(D))$ for such D. We finish the proof by analyzing the nature of the map S. Roughly speaking, if S is a diffeomorphism onto $S(D)$, then the function algebra $[z^2, (\overline{z}+f)^2, w^2, (\overline{w}+g)^2; D]$ equals $C(D)$. Otherwise, this function algebra coincides with a proper closed subalgebra of $C(D)$ depending on the locus where S is not injective.

Finally, we note that the most difficult step in the above approach is to show the polynomially convexity of a union of totally graphs. Unlike the one dimension case, in the high dimension, we have difficulties while applying Kallin's lemma, since the pairwise intersections of these graphs may be of real dimension two.

2. Preliminaries

We will frequently invoke the following useful result (see [5]).

Theorem 2.1. (Kallin's lemma) *Suppose that*:
(i) X_1, X_2 *are polynomially convex subsets of* $\mathbb{C}^n$;
(ii) Y_1, Y_2 *are polynomially convex subsets of* $\mathbb{C}$ *such that* 0 *is a common boundary point for* Y_1 *and* Y_2, *and* $Y_1 \cap Y_2 = \{0\}$;
(iii) $p : \mathbb{C}^n \to \mathbb{C}$ *is a polynomial map such that* $p(X_1) \subset Y_1, p(X_2) \subset Y_2$;
(iv) $p^{-1}(0) \cap (X_1 \cup X_2)$ *is polynomially convex.*
Then $X_1 \cup X_2$ *is polynomially convex.*

In practice, we will try to find p such that $p(X_1)$ (resp. $p(X_2)$) is contained in upper (resp. lower) half plane and $p(X_1) \cap p(X_2) = \{0\}$. It should be noticed that Kallin's lemma is applicable mostly in the cases where $X_1 \cap X_2$ is "small".

The next result in [4] is a generalization of the Hörmander-Wermer theorem mentioned at the beginning of our paper.

Theorem 2.2 (O'Farrell, Preskenis and Walsh)**.** *Let* X *be a compact polynomially convex set in* $\mathbb{C}^n$ *and* E *be a closed subset of* X. *Assume that* $X \setminus E$ *is totally real* (*that is, locally contained in a totally real manifold*). *Then*

$$P(X) = \{f \in C(X) : f\big|_E \in P(E)\}.$$

We say that a general homogeneous polynomial p of degree $k \geq 2$ in variables z and $\overline{z}$

$$p(z) = \sum_{l=-\infty}^{+\infty} a_l \overline{z}^l z^{k-l}$$

satisfies the *coefficient condition* if there exists $l_0 \leq \frac{k}{2}$ such that

$$|a_{l_0}| > \sum_{l \neq l_0} |a_l|.$$

The interest for introducing the above notion lies in the following fact.

Lemma 2.3. *Let p be a homogeneous polynomial in variables z and $\overline{z}$ of degree k. Assume that p satisfies the coefficient condition. Let f, g be functions of class $\mathcal{C}^1$ near the origin in $\mathbb{C}$ satisfying $f(z) = o(|z|^k)$, $g(z) = o(|z|^k)$. Then for every closed disk $\tilde{D}$ around $0 \in \mathbb{C}$ small enough, we have that $M_1 \cup M_2$ is polynomial convex, where*

$$M_1 = \{\big(z, \overline{z} + p(z) + f(z)\big) : z \in \tilde{D}\},$$
$$M_2 = \{\big(z, \overline{z} - p(z) + g(z)\big) : z \in \tilde{D}\}.$$

Proof. We write

$$p(z) = \sum_{l=-\infty}^{+\infty} a_l \overline{z}^l z^{k-l}.$$

Since p satisfies the coefficient condition, we can find l_0 such that $2l_0 \leq k$ and

$$|a_{l_0}| - \sum_{l \neq l_0} |a_l| > 0. \tag{1}$$

Consider the polynomial $\varphi : \mathbb{C}^2 \to \mathbb{C}$ defined by

$$\varphi(z, w) = a_{l_0} z^{k-2l_0+1} + \overline{a_{l_0}} w^{k-2k_0+1},\ (z, w) \in \mathbb{C}^2.$$

We claim that for $\tilde{D}$ small enough $\varphi(M_1)$ is contained in the upper half plane. Indeed,

$$\begin{aligned}\varphi\big(z, \overline{z} + p(z) + f(z)\big) &= a_{l_0} z^{k-2l_0+1} + \overline{a_{l_0}}\Big(\overline{z} + p(z) + f(z)\Big)^{k-2l_0+1} \\ &= a_{l_0} z^{k-2l_0+1} + \overline{a_{l_0} z^{k-2l_0+1}} \\ &\quad + (k - 2l_0 + 1)\overline{a}_{l_0} z^{k-2l_0} p(z) + o(|z|^{2k-2l_0}).\end{aligned}$$

This follows that

$$\mathrm{Im}\varphi\big(z, \overline{z}+p(z)+f(z)\big) \geq (k-2l_0+1)|a_{l_0}|\Big(|a_{l_0}| - \sum_{l \neq l_0} |a_l|\Big)|z|^{2k-2l_0} + o(|z|^{2k-2l_0}).$$

Combining this with (1), we infer that for $\tilde{D}$ small enough $\varphi(M_1)$ is contained in the upper half plane and $\varphi(M_1) \cap \mathbb{R} = \{0\}$. By the same computation we also deduce that $\varphi(M_2)$ is contained in the lower half plane and $\varphi(M_2) \cap \mathbb{R} = \{0\}$. Thus $\varphi^{-1}(0) \cap (M_1 \cup M_2) = \{(0, 0)\}$ for $\tilde{D}$ small enough. The desired conclusion now follows from Kallin's lemma. $\square$

We need the following lemma which is a generalization of Lemma of [5].

Lemma 2.4. *Let X be a compact subset of $\mathbb{C}^m$, and a polynomial mapping $\pi : \mathbb{C}^m \to \mathbb{C}^m$ defined by*

$$\pi(z_1, \ldots, z_m) = \big(z_1^{k_1}, \ldots, z_m^{k_m}\big).$$

Let $\pi^{-1}(X) = X_{1,\ldots,1} \cup \cdots \cup X_{i_1,\ldots,i_m} \cup \cdots \cup X_{k_1,\ldots,k_m}$, with $X_{1,\ldots,1}$ compact, and

$$X_{i_1,\ldots,i_m} = \Big\{\big(\rho_{k_1}^{k_1-1} z_1, \ldots, \rho_{k_m}^{k_m-1} z_m\big) : (z_1, \ldots, z_m) \in X_{1,\ldots,1}\Big\}$$

for $1 \le i_1 \le k_1, \dots, 1 \le i_m \le k_m$ *where* $\rho_{k_i} = \exp\left(\frac{2\pi i}{k_j}\right)$ *with* $j = 1, \dots, m$. *If*

$$P\Big(\pi^{-1}(X)\Big) = C\Big(\pi^{-1}(X)\Big),$$

then $P(X) = C(X)$.

Proof. Let $Q(z_1, \dots, z_m) = \sum a_{j_1,\dots,j_m} z_1^{j_1} \cdots z_m^{j_m}$ be a polynomial in m variables. For each $(i_1, \dots, i_m)$ with $1 \le i_1 \le k_1, \dots, 1 \le i_m \le k_m$, we put

$$Q_{i_1,\dots,i_m}(z_1, \dots, z_m) := Q\big(\rho_{k_1}^{i_1-1} z_1, \dots, \rho_{k_m}^{i_m-1} z_m\big),\ (z_1, \dots, z_m) \in X_{1,\dots,1}.$$

We claim that

$$\frac{1}{k_1 \cdots k_m} \sum Q_{i_1,\dots,i_m}(z_1, \dots, z_m) = \sum a_{p_1 k_1,\dots,p_m k_m} z_1^{k_1 p_1} \cdots z_m^{k_m p_m}.$$

Indeed, since every polynomial Q can be written by as a finite sum of monomials of the forms $a z_1^{s_1} \cdots z_m^{s_m}$ we only need check for $Q(z_1, \dots, z_n) = a z_1^{s_1} \cdots z_m^{s_m}$. We have

$$\begin{aligned} &\frac{1}{k_1 \cdots k_m} \sum Q_{i_1,\dots,i_m}(z_1, \dots, z_m) \\ &= \frac{a}{k_1 \cdots k_m} z_1^{s_1} \cdots z_m^{s_m} \sum \rho_{k_1}^{(i_1-1)s_1} \cdots \rho_{k_m}^{(i_m-1)s_m} \\ &= \frac{a}{k_1 \cdots k_m} z_1^{s_1} \cdots z_m^{s_m} \prod_{j=1}^{m} \Big(\sum_{1 \le i_j \le k_j} \big(\rho_{k_j}^{s_j}\big)^{i_j-1} \Big). \end{aligned}$$

If there exists $1 \le j \le m$ such that $s_j \ne p_j k_j$, then

$$\sum_{1 \le i_j \le k_j} \big(\rho_{k_j}^{s_j}\big)^{i_j-1} = \frac{\big(\rho_{k_j}^{s_j}\big)^{k_j} - 1}{\rho_{k_j}^{s_j} - 1} = 0,$$

where $\rho_{k_j}^{k_j} = (\exp \frac{2\pi i}{k_j})^{k_j} = 1$. It follows that

$$\frac{1}{k_1 \cdots k_m} \sum Q_{i_1,\dots,i_m}(z_1, \dots, z_m) = 0.$$

In the case $s_j = p_j k_j$ for all $j = 1, \dots, m$ we have

$$\sum_{1 \le i_j \le k_j} \big(\rho_{k_j}^{s_j}\big)^{i_j-1} = \sum_{1 \le i_j \le k_j} \big(\rho_{k_j}^{k_j}\big)^{p_j(i_j-1)} = k_j.$$

This implies that

$$\begin{aligned} \frac{1}{k_1 \cdots k_m} \sum Q_{i_1,\dots,i_m}(z_1, \dots, z_m) &= \frac{a}{k_1 \cdots k_m} z_1^{s_1} \cdots z_m^{s_m} \prod_{j=1}^{m} k_j \\ &= a z_1^{p_1 k_1} \cdots z_n^{k_m p_m}. \end{aligned}$$

The claim is proved. Now, suppose that $P(\pi^{-1}(X)) = C(\pi^{-1}(X))$. Let $f \in C(X)$. Then $f \circ \pi \in C(\pi^{-1}(X))$, so there is a polynomial Q in m variables with $f \circ \pi \approx Q$ in $\pi^{-1}(X)$. In particular, this is true for $X_{i_1,..,i_m}$, so

$$f(z_1^{k_1}, \dots, z_m^{k_m}) \approx Q(\rho_{k_1}^{i_1-1} z_1, \dots, \rho_{k_m}^{i_m-1} z_m) = Q_{i_1,\dots,i_m}(z_1, \dots, z_m)$$

on $X_{1,\dots,1}$. It follows that
(2)
$$f(z_1^{k_1},\dots,z_m^{k_m}) \approx \frac{1}{k_1\cdots k_m}\sum_{1\le i_1\le k_1,\dots,1\le i_m\le k_m} Q_{i_1,\dots,i_m}(z_1,\dots,z_m) \text{ on } X_{1,\dots,1}.$$

If Q has form of
$$Q(z_1,\dots,z_m)=\sum a_{r_1,\dots,r_m} z_1^{r_1}\cdots z_m^{r_m},$$
then the right hand of (2) to equal
$$\sum a_{p_1k_1,\dots,p_mk_m} z_1^{k_1p_1}\cdots z_m^{k_mp_m},$$
so equals to $P(z_1^{k_1},\dots,z_m^{k_m})$, where P is a polynomial in m variables. We conclude that
$$f(z_1^{k_1},\dots,z_m^{k_m}) \approx P(z_1^{k_1},\dots,z_m^{k_m}) \ \text{ on } \ X_{1,\dots,1}.$$
That is $f\approx P$ on X. So $P(X)=C(X)$. The lemma is proved. □

3. The main results

The aim of this work is to prove the following results.

Theorem 3.1. *Suppose that p (resp. q) is a general homogenous polynomial in $z,\overline{z}$ of even degree k, (resp. in $w,\overline{w}$ of even degree k') satisfying the coefficient condition. Let $m\ge k'$ and $m'\ge k$ be nonnegative integers and $\hat{f}$ (resp. $\hat{g}$) be $\mathcal{C}^1$ function near the origin in $\mathbb{C}^2$ such that*
$$\lim_{(z,w)\to(0,0)}\frac{\hat{f}(z,w)}{|z|^k|w|^m}=\lim_{(z,w)\to(0,0)}\frac{\hat{g}(z,w)}{|z|^{k'}|w|^{m'}}=0.$$
Define
$$f(z,w)=p(z)|w|^m+\hat{f}(z,w),\ \ g(z,w)=q(w)|z|^{m'}+\hat{g}(z,w).$$
Then
$$\begin{aligned}&[z^2,(\overline{z}+f)^2,w^2,(\overline{w}+g)^2;D]\\ &=\{h\in C(D): h(z,0)=h(-z,0)=h(0,-w)=h(0,w)\}\end{aligned}$$
for every bidisk D sufficiently small.

We have not been able to describe the algebra $[z^2,(\overline{z}+f)^2,w^2,(\overline{w}+g)^2;D]$ in the case of $0<m<k'$ or $0<m'<k$. The following theorem deals to $m=m'=0$.

Theorem 3.2. *Suppose that p (resp. q) is a general homogenous polynomial in $z,\overline{z}$ of even degree k, (resp. in $w,\overline{w}$ of even degree k') satisfying the coefficient condition. Let $\hat{f}$ and $\hat{g}$ be $\mathcal{C}^1$ functions near the origin in $\mathbb{C}^2$ such that*
$$\lim_{(z,w)\to(0,0)}\frac{\hat{f}(z,w)}{|z|^k|w|^k}=\lim_{(z,w)\to(0,0)}\frac{\hat{g}(z,w)}{|z|^{k'}|w|^{k'}}=0.$$

Define

$$f(z,w) = p(z) + \hat{f}(z,w), \quad g(z,w) = q(w) + \hat{g}(z,w).$$

Then

$$[z^2, (\overline{z}+f)^2, w^2, (\overline{w}+g)^2; D] = C(D)$$

for every bidisk D sufficiently small.

Proof of Theorem 3.1. Let S be the map defined near the origin in $\mathbb{C}^2$ by

$$S(z,w) := \Big(z^2, \big(\overline{z}+f(z,w)\big)^2, w^2, \big(\overline{w}+g(z,w)\big)^2\Big).$$

Define the following transformations from $\mathbb{C}^4$ to $\mathbb{C}^4$

$$\theta_1(z,w,u,v) \mapsto (-z,w,u,v); \quad \theta_2(z,w,u,v) \mapsto (z,w,-u,v);$$

$$\theta_3(z,w,u,v) = (-z,w,-u,v).$$

Let $\pi : \mathbb{C}^4 \to \mathbb{C}^4$ be the polynomial mapping

$$\pi(z,w,u,v) = (z^2, w^2, u^2, v^2).$$

Then we may express

$$\pi^{-1}\big(S(D)\big) = \big(M_1 \cup M_2 \cup M_3 \cup M_4\big) \cup \Big(\bigcup_{j=1}^{3} \theta_j(M_1 \cup M_2 \cup M_3 \cup M_4)\Big),$$

where

$$\begin{aligned}
M_1 &:= \{\big(z, \overline{z}+f(z,w), w, \overline{w}+g(z,w)\big) : (z,w) \in D\};\\
M_2 &:= \{\big(z, \overline{z}+f(z,w), -w, -\overline{w}-g(z,w)\big) : (z,w) \in D\};\\
M_3 &:= \{\big(-z, -\overline{z}-f(z,w), w, \overline{w}+g(z,w)\big) : (z,w) \in D\};\\
M_4 &:= \{\big(-z, -\overline{z}-f(z,w), -w, \overline{w}-g(z,w)\big) : (z,w) \in D\}.
\end{aligned}$$

We split the proof into several steps.

Step 1. We will show that for D sufficiently small, the union $M_1 \cup M_2 \cup M_3 \cup M_4$ is polynomially convex. To do this, first we write

$$q(w) = \sum_{l=-\infty}^{\infty} a_l \overline{w}^l w^{k'-l}.$$

Since q satisfies the coefficient condition, we can find l_0 such that $2l_0 \le k'$ and

$$b := |a_{l_0}| - \sum_{l \ne l_0} |a_l| > 0. \tag{3}$$

Consider the polynomial $\varphi : \mathbb{C}^4 \to \mathbb{C}$ defined by

$$\varphi(z,w,u,v) = zw\Big(a_{l_0} u^{k'-2l_0+1} + \overline{a_{l_0}} v^{k'-2l_0+1}\Big).$$

Now we claim that $\varphi(M_1)\cup\varphi(M_3)$ is contained in the upper half plane for D small enough. Indeed,

$$\begin{aligned}&\varphi\big(z,\overline{z}+f(z,w),w,\overline{w}+g(z,w)\big)\\ &=\Big[|z|^2+zp(z)|w|^m+z\hat{f}(z)\Big]\Big[a_{l_0}w^{k'-2l_0+1}\\ &\quad+\overline{a_{l_0}}\big(\overline{w}+p(w)|z|^{m'}+\hat{g}(z,w)\big)^{k'-2l_0+1}\Big]\\ &=\Big[|z|^2+|w|^m zp(z)+z\hat{f}(z,w))\Big]\Big[\big(a_{l_0}w^{k'-2l_0+1}+\overline{a}_{l_0}\overline{w}^{k'-2l_0+1}\big)\\ &\quad+(k'-2l_0+1)\overline{a_{l_0}}w^{k'-2l_0}q(w)|z|^{m'}+o\big(|w|^{2k'-2l_0}|z|^{m'}\big)\Big].\end{aligned}$$

Since $m\geq k'$, it follows form (3) that

$$\begin{aligned}&\mathrm{Im}\varphi\big(z,\overline{z}+f(z,w),w,\overline{w}+g(z,w)\big)\\ &\geq(k'-2l_0+1)b|a_{l_0}||z|^{m'+2}|w|^{2k'-2l_0}+o\big(|z|^{m'+2}|w|^{2k'-2l_0}\big).\end{aligned}\tag{4}$$

Hence $\varphi(M_1)$ is contained in the upper half plane for D small enough. The same conclusion for $\varphi(M_3)$ holds by the same computation. This proves the claim. Now we look at the image of $M_2\cup M_4$ under the map φ. Since k' is even, we obtain

$$\begin{aligned}&\varphi\big(z,\overline{z}+f(z,w),w,\overline{w}+g(z,w)\big)\\ &=\Big[|z|^2+zp(z)|w|^m+z\hat{f}(z)\Big]\Big[a_{l_0}w^{k'-2l_0+1}\\ &\quad+\overline{a_{l_0}}\big(-\overline{w}-p(w)|z|^{m'}-\hat{g}(z,w)\big)^{k'-2l_0+1}\Big]\\ &=\Big[|z|^2+|w|^m zp(z)+z\hat{f}(z,w))\Big]\Big[\big(-a_{l_0}w^{k'-2l_0+1}-\overline{a}_{l_0}\overline{w}^{k'-2l_0+1}\big)\\ &\quad-(k'-2l_0+1)\overline{a_{l_0}}w^{k'-2l_0}q(w)|z|^{m'}+o\big(|w|^{2k'-2l_0}|z|^{m'}\big)\Big].\end{aligned}$$

Since $m\geq k'$, it follows from (3) that

$$\begin{aligned}&\mathrm{Im}\varphi\big(z,\overline{z}+f(z,w),w,\overline{w}+g(z,w)\big)\\ &\leq\;-(k'-2l_0+1)b|a_{l_0}||z|^{m'+2}|w|^{2k'-2l_0}+o\big(|z|^{m'+2}|w|^{2k'-2l_0}\big).\end{aligned}\tag{5}$$

Thus $\varphi(M_2)$ is contained in the lower half plane. This is also true for $\varphi(M_4)$ by the same argument.

Next we claim that $\varphi(M_1)\cap\varphi(M_2)=\{0\}$ and $\varphi^{-1}(0)\cap(M_1\cup M_2)$ is polynomially convex for every bidisk D small enough. Indeed, on one hand we have

$$\varphi(M_1)\cap\varphi(M_2)\subset\{x\in\mathbb{C}:\mathrm{Im}x=0\}.$$

On the other hand, from (4) and (5), we infer that for D small enough

$$\begin{aligned}&\varphi^{-1}\big(\varphi(M_1)\cap\varphi(M_2)\big)\cap\big(M_1\cup M_2\big)\\ &=\{(z,\overline{z},0,0):z\in\tilde{D}\}\cup\{(0,0,w,\overline{w}):z\in\tilde{D}\}:=\tilde{M}.\end{aligned}$$

Thus by the definition of φ we get $\varphi(M_1) \cap \varphi(M_2) = \{0\}$. Now we claim that $\tilde{M}$ is polynomially convex for any closed disk $\tilde{D}$ around $0 \in \mathbb{C}$. Indeed, by the Stone-Weierstrass theorem, we have $\tilde{M}_1 = \{(z, \overline{z}, 0, 0) : z \in \tilde{D}\}$ and $\tilde{M}_2 = \{(0, 0, w, \overline{w}) : w \in \tilde{D}\}$ are polynomially convex. Consider the polynomial $\lambda : \mathbb{C}^4 \to \mathbb{C}$ defined by $\lambda(z, w, u, v) = zw$. The claim is implied by applying Kallin's lemma to λ, $\tilde{M}_1$ and $\tilde{M}_2$. The same reasoning also implies that $\varphi(M_3) \cap \varphi(M_4) = \{0\}$ and $\varphi^{-1}(0) \cap (M_3 \cup M_4)$ is polynomially convex for every bidisk D sufficiently small. At this point, we may apply Kallin's lemma to get $M_1 \cup M_2$ and $M_3 \cup M_4$ are polynomially convex for every bidisk D small enough.

Next we show that $M_1 \cup M_2 \cup M_3 \cup M_4$ is polynomially convex for D small enough. To this end, we write

$$p(z) = \sum_{l=-\infty}^{+\infty} b_l \overline{z}^l z^{k-l}.$$

Since p satisfies the coefficient condition, we can find l_1 such that $2l_1 \leq k$ and

$$|b_{l_1}| > \sum_{l \neq l_1} |b_l|.$$

Consider the polynomial $\psi : \mathbb{C}^4 \to \mathbb{C}$ defined by

$$\psi(z, w, u, v) = uv\Big(b_{l_1} z^{k-2l_1+1} + \overline{b_{l_1}} w^{k-2l_1+1}\Big).$$

Following exactly the above reasoning, we obtain that for D small enough the set $\psi(M_1 \cup M_2)$ (resp. $\psi(M_3 \cup M_4)$) is contained in the upper (resp. lower) half plane. Furthermore

$$\psi(M_1 \cup M_2) \cap \psi(M_3 \cup M_4) = \{0\}.$$

So in view of Kallin's lemma we have $M_1 \cup M_2 \cup M_3 \cup M_4$ is polynomially convex for D small enough.

Step 2. We will prove that $\pi^{-1}\big(S(D)\big)$ is polynomially convex for D sufficiently small. It follows from the previous step that if D is a small enough bidisk, then $M_1 \cup M_2 \cup M_3 \cup M_4$ is polynomially convex. Hence $\theta_j(M_1 \cup M_2 \cup M_3 \cup M_4)$ is also polynomially convex for $j = 1, 2, 3$ and D small enough. By applying Kallin's lemma to the polynomial

$$\alpha : \mathbb{C}^4 \to \mathbb{C}, \;\; \alpha(z, w, u, v) := zw$$

we can check that $(M_1 \cup M_2 \cup M_3 \cup M_4) \cup \theta_1(M_1 \cup M_2 \cup M_3 \cup M_4)$ is polynomially convex and $\theta_2(M_1 \cup M_2 \cup M_3 \cup M_4) \cup \theta_3(M_1 \cup M_2 \cup M_3 \cup M_4)$ is polynomially convex for D sufficiently small. Finally, by using Kallin's lemma and polynomial

$$\beta : \mathbb{C}^4 \to \mathbb{C}, \;\; \beta(z, w, u, v) := uv,$$

we can conclude that $\pi^{-1}\big(S(D)\big)$ is polynomially convex for D sufficiently small.

Step 3. We show that $P\big(S(D)\big) = C\big(S(D)\big)$. For this, we first show that $P\big(\pi^{-1}(S(D))\big) = C\big(\pi^{-1}(S(D))\big)$ for D sufficiently small. For any closed disk $\tilde{D}$ around $0 \in \mathbb{C}$, we set

$$N := \{(z, \overline{z}) : z \in \tilde{D}\} \cup \{(z, -\overline{z}) : z \in \tilde{D}\}.$$

Notice that $\pi^{-1}(S(D))$ is union of 16 totally real graphs in $\mathbb{C}^4$. Furthermore, for D small enough, $\pi^{-1}(S(D)) \setminus E$ is locally contained in one of these totally graph, where

$$E = \big(N \times \{(0,0)\}\big) \cup \big(\{(0,0)\} \cup N\big).$$

We claim that $P(E) = C(E)$. To see this, we first note that $P(N) = C(N)$. Indeed, by Stone-Weierstrass theorem we have N_1, N_2 are polynomially convex, where

$$N_1 = \{(z, \overline{z}) : z \in \tilde{D}\},\ N_2 = \{(z, \overline{z}) : z \in \tilde{D}\}.$$

By applying Kallin's lemma to $X_1 := N_1$, $X_2 := N_2$ and $p(z, w) = zw$ we get that N is polynomially convex. Note that *outside* the origin of $\mathbb{C}^2$, N is locally contained in a totally real manifold, so by Theorem 2.2 we have $P(N) = C(N)$. It follows that there exists a sequence φ_n of polynomial in z, w such that φ_n converges uniformly to $\overline{z}$ on N. Let π_1, π_2 be projections from $\mathbb{C}^4$ onto $\mathbb{C}^2$ defined by

$$\pi_1(z, w, u, v) := (z, w),\ \ \pi_2(z, w, u, v) := (u, v).$$

It easy to check that $\phi_n \circ \pi_1$ converges uniformly to $\overline{z}$ on E and $\phi_n \circ \pi_2$ converges uniformly to $\overline{w}$. This implies that $\overline{z}, \overline{w}$ belong to $P(E)$. Therefore, in view of the Stone-Weierstrass theorem we get $P(E) = C(E)$ as claimed. Now we apply Theorem 2.2 to obtain that $P\big(\pi^{-1}(S(D))\big) = C\big(\pi^{-1}(S(D))\big)$ for D small enough. Finally, applying Lemma 2.4, we get that $P\big(S(D)\big) = C\big(S(D)\big)$.

Step 4. Completion of the proof. Set $A = \{(z, w) : zw = 0\}$. We claim that $S : D \backslash A \to S(D) \backslash S(A)$ is injective for D small enough. Assume otherwise, then we can find the sequence $\{(z_n, w_n)\}, \{(z'_n, w'_n)\}$ having the following properties:

(a) $z_n \to 0, w_n \to 0$.
(b) $z_n w_n \neq 0, z'_n w'_n \neq 0,\ \forall n$.
(c) $(z_n, w_n) \neq (z'_n, w'_n),\ \forall n$.
(d) $S(z_n, w_n) = S(z'_n, w'_n), \forall n$.

After passing to a subsequence, we can assume, in view of (c) that $z_n \neq z'_n$ for all n. It follows from (d) that for every $n \geq 1$

$$z_n = -z'_n,\ \ (z_n + f(z_n, w_n))^2 = (z'_n + f(z'_n, w'_n))^2.$$

Switching again to a subsequence, there are two cases to consider.

Case 1. $z_n + f(z_n, w_n) = z'_n + f(z'_n, w'_n)\ \forall n \geq 1$. Then we have

$$\begin{aligned}2z_n &= f(z_n, w_n) - f(z'_n, w'_n)\\ &= f(z_n, w_n) - f(-z_n, w'_n)\\ &= p(z_n)\big(|w_n|^m - |w'_n|^m\big) + \hat{f}(z_n, w_n) - \hat{f}(z'_n, w'_n)\\ &= o(z_n).\end{aligned}$$

This is a contradiction.

Case 2. $z_n + f(z_n, w_n) = -z'_n - f(z'_n, w'_n)$ for all $n \geq 1$. Since p satisfies the coefficient condition, we can find a constant $b' > 0$ such that for all n large enough $p(z_n) \geq b'|z_n|^k$. On the other hand, we have

$$\begin{aligned}0 &= f(z_n, w_n) + f(z'_n, w'_n)\\ &= f(z_n, w_n) + f(-z_n, w'_n)\\ &= p(z_n)\big(|w_n|^m + |w'_n|^m\big) + 0(|z_n|^k).\end{aligned}$$

We reach again to a contradiction (when n sufficiently large).

Thus $S : D \setminus A \to S(D) \setminus S(A)$ is injective for D small enough. It then follows that h is a homeomorphism between $D \setminus A$ and $S(D) \setminus A$ for D small enough.

We are going to show the equality

$$\begin{aligned}&[z^2, (\overline{z} + f)^2, w^2, (\overline{w} + g)^2; D]\\ &= \{h \in C(D) : h(z, 0) = h(-z, 0) = h(0, -w) = h(0, w)\}.\end{aligned}$$

Given $h \in C(D)$ satisfying

$$f(z, 0) = f(-z, 0) = f(0, w) = f(0, -w)$$

for every $(z, w) \in D$. We define

$$k(z, w, u, v) = \begin{cases}(h \circ S^{-1})(z, w, u, v) & (z, w, u, v) \in S(D) \setminus S(A)\\ h(z', 0) & z = z'^2, w = z'^2, u = v = 0\\ h(0, w') & z = w = 0, u = w'^2, v = w'^2.\end{cases}$$

It is easy to check that $k \in C\big(S(D)\big)$ and $k \circ S \equiv h$ on D. By the result proven in Step 3, we have

$$h \in [z^2, (\overline{z} + f)^2, w^2, (\overline{w} + g)^2; D].$$

This is completion of the proof of Theorem 3.1. □

Proof of Theorem 3.2. We retain the notation used in the proof of Theorem 3.1. The proof of this case is quite similar to Theorem 3.1, we only indicate the necessary changes. As before, we proceed through four steps.

Step 1. $M_1 \cup M_2 \cup M_3 \cup M_4$ is polynomially convex. The proof is the same as the proof of Theorem 3.1. There are two changes that have to consider the polynomials

$$\varphi(z,w,u,v) = \left(a_{l_0}u^{k'-2l_0+1} + \overline{a_{l_0}}v^{k'-2l_0+1}\right)$$

and

$$\psi(z,w,u,v) = \left(b_{l_1}z^{k-2l_1+1} + \overline{b_{l_1}}w^{k-2l_1+1}\right)$$

when proving polynomial convexity of $M_1 \cup M_2$, $M_3 \cup M_4$ and $M_1 \cup M_2 \cup M_3 \cup M_4$.

Step 2. $\pi^{-1}\big(S(D)\big)$ is polynomially convex for D small enough. The proof is very similar to the proof of Theorem 3.1. We omit the details.

Step 3. We show that $P(S(D)) = C(S(D))$. For this, we first show that $P\big(\pi^{-1}(S(D))\big) = C\big(\pi^{-1}(S(D))\big)$ for D sufficiently small. For any closed disk $\tilde{D}$ around $0 \in \mathbb{C}$, we set

$$N_1 := \{(z, \overline{z} + p(z)) : z \in \tilde{D}\},\ N_2 := \{(z, \overline{z} - p(z)) : z \in \tilde{D}\};$$

$$N_3 := \{(w, \overline{w} + q(w)) : z \in \tilde{D}\},\ N_4 := \{(w, \overline{w} - q(w)) : z \in \tilde{D}\}.$$

Since p and q are general polynomials of even degree we deduce that $\pi^{-1}(S(D))$ is locally contained in totally real manifold outside the compact

$$E := \big((N_1 \cup N_2) \times \{(0,0)\}\big) \cup \big(\{(0,0)\} \times (N_3 \cup N_4)\big).$$

We claim that $P(E) = C(E)$. To do this, we first apply Lemma 2.3 to get that $N_1 \cup N_2$ and $N_3 \cup N_4$ are polynomially convex for $\tilde{D}$ small enough. Now it suffices to apply Kallin's lemma to polynomial map $\lambda(z,w,u,v) = zw$ and two polynomially convex compact sets $(N_1 \cup N_2) \times \{(0,0)\}$, $\{(0,0)\} \times (N_3 \cup N_4)$ to get that E is polynomially convex for D small enough. Note that E is locally contained in a totally real manifold outside the origin (in $\mathbb{C}^4$). Therefore, by Theorem 2.2 $P(E) = C(E)$ as claimed. Next, by Step 2, $\pi^{-1}\big(S(D)\big)$ is polynomially convex for D sufficiently small. Using again Theorem 2.2 we have $P\big(\pi^{-1}\big(S(D)\big)\big) = C\big(\pi^{-1}\big(S(D)\big)\big)$. It follows from Lemma 2.4 that $P\big(S(D)\big) = C\big(S(D)\big)$ for D sufficiently small.

Step 4. Completion of the proof. By the coefficient condition of p and q, it is easy to check that the functions $z^2, (\overline{z} + f)^2, w^2, (\overline{w} + g)^2$ separate points near the origin of $\mathbb{C}^2$. This follows that S is a homeomorphism from D onto $S(D)$. Hence the theorem follows from the result proven in Step 3. The proof is complete. $\square$

Remark 3.3. By Theorem 11 in [5], we have $[z^2, F^2; \tilde{D}] \neq C(\tilde{D})$ for

$$F(z) = \frac{\overline{z}}{1 + \overline{z}} = \overline{z} - \overline{z}^2 + o(|z|^2)$$

and $\tilde{D}$ is a closed disk around $0 \in \mathbb{C}$ with radius < 1. It follows that

$$[z^2, F(z)^2, w^2, F(w)^2; \tilde{D}] \neq C(D)$$

for every closed bidisk D around the origin in $\mathbb{C}^2$ with radius < 1. Therefore the coefficient condition on f and g in Theorem 3.2 can not be entirely dropped.

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