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GEOMETRIC QUANTIZATION OF ODD DIMENSIONAL SPIN^c MANIFOLDS

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Abstract. We prove a Guillemin-Sternberg geometric quantization formula for circle action on odd dimensional spin^c-manifolds. We prove two Kostant type formulas in this case. As a corollary, we get a cutting formula for the odd spin^c quantization.

1. Introduction

In 1982, a fascinating conjecture appeared about group actions. Guillemin and Sternberg [8] gave a precise mathematical formulation of Dirac's idea that "quantization commutes with reduction", in which they defined the former as geometric quantization. This conjecture was first proved by Guillemin-Sternberg in the holomorphic situation for Kähler manifolds. They raised the conjecture for general symplectic manifolds. When group is abelian, this conjecture was first proved by Guillemin [7] in a special case, and later in general by Meinrenken [16] and Vergne [19, 20] independently. The remaining nonabelian case was proved by Meinrenken [17] using the symplectic cut techniques of Lerman [14]. There are also closely related papers by Duistermaat-Guillemin-Meinrenken-Wu [3], where the symplectic cut techniques were applied to the circle action case, and by Jeffery-Kirwan [10], where the authors proved the conjecture under some extra conditions by using the nonabelian localization formula of Witten [21] and Jeffery-Kirwan [11]. In all these works, the equivariant index theorem of Atiyah-Segal-Singer [2], which expressed the analytic equivariant index through topological data on the fixed point sets, play essential roles. In [18], Tian and Zhang gave an analytic localization proof of this conjecture by using the Bismut-Lebeau analytic localization technique. In [1], Cannas, Karshon and Tolman extended this conjecture when manifolds are not symplectic and stated three versions of "quantization commutes with reduction" corresponding to almost complex, stable complex, and spin^c quantizations. For even dimensional spin^c manifolds, Fuchs proved a Kostant type

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formula and got a cutting formula by the Kostant formula in [6]. On the other hand, Freed [5] proved an index theorem for odd spin manifolds with involution and Fang proved an equivariant odd index theorems for Toeplitz operators in [4]. Liu and Wang extended the Freed odd index theorem to the equivariant case and proved the Atiyah-Hirzebruch type theorems for odd spin manifolds in [15]. So a natural question is that whether we can prove an odd geometric quantization formula by equivariant odd index theorems or not. In this paper, we give a positive answer.

In Section 2, we prove spin^c versions of equivariant odd index theorems in [4] and [5] and we prove an odd geometric quantization formula by these equivariant odd index theorems. In Section 3, we prove two Kostant type formulas. As a corollary, we get a cutting formula for the odd spin^c quantization.

2. A Guillemin-Sternberg geometric quantization formula for circle action on odd dimensional spin^c-manifolds

2.1. Equivariant odd index theorems

In this section, we firstly give spin^c versions of the equivariant odd index theorems in [4] and [5]. Let M be a closed oriented spin^c manifold of dimension $2r+1$, with a fixed spin^c structure associated to the complex line bundle L. Let $\triangle(TM)$ be the canonical complex spinors bundle of M. Let D be the associated spin^c Dirac operator which is a self adjoint first order elliptic differential operator acting on $\Gamma(\triangle(TM))$ (the smooth sections space of $\triangle(TM)$), so it induces a spectral decomposition of $L^2(\triangle(TM))$ which is the L^2 completion of $\Gamma(\triangle(TX))$. Denote by $L^2_+(\triangle(TM))$ the direct sum of eigenspaces of D associated to nonnegative eigenvalues, and by P^+ the orthogonal projection operator from $L^2(\triangle(TM))$ to $L^2_+(\triangle(TM))$. Given a trivial complex vector bundle $\mathbb{C}^N$ over M, D and P_+ extend trivially as operators on $\Gamma(\triangle(TM)\otimes\mathbb{C}^N)$. Let $g: M\to U(N)$ be a smooth map. Then g extends to an action on $\Gamma(\triangle(TM)\otimes\mathbb{C}^N)$ as $\mathrm{Id}_{(\triangle(TM))}\otimes g$, still denoted by g.

Definition 2.1. The Toeplitz operator associated to D and g is

$$T_g=(P_+\otimes Id_{\mathbb{C}^N})g(P_+\otimes Id_{\mathbb{C}^N}): L^2_+(\triangle(TM)\otimes\mathbb{C}^N)\to L^2_+(\triangle(TM)\otimes\mathbb{C}^N).$$

It is a classical fact that T_g is a bounded Fredholm operator between the given Hilbert spaces.

Let H be a compact group of isometries of M preserving the orientation and spin^c structure and there is a lift action on L. There is a lift of $h\in H$ acting on $\Gamma(\triangle(TM))$ which commutes with D, so it commutes with P_+. We also assume that

$$g(hx)=g(x) \text{ for any } h\in H \text{ and any } x\in X. \tag{2.1}$$

Then,

$$T_gh=hT_g.$$

Definition 2.2. Spin^c quantization of M is defined by the virtual complex H-representive space $Q(M, T_g, P) := \ker(T_g) - \mathrm{coker}(T_g)$, where P is the fixed spin^c structure of M. Denote its H-trivial component by $Q(M, T_g, P)^{S^1}$. The equivariant index associated to T_g and h is defined by

$$\mathrm{Ind}_h T_g = \mathrm{tr}(h|\ker(T_g)) - \mathrm{tr}(h|\mathrm{coker}(T_g)). \tag{2.2}$$

Next we assume $H = S^1$ and let $h_0 = e^{2\pi i t}$ be a generator of the circle group which means that a subgroup generated by h_0 is dense in S^1. Let F be the connect component of the fixed point set M^{S^1} under S^1 action. Then the tangent bundle $TM|_F$ has a decomposition into sum:

$$TM|_F = E_1 \oplus \cdots \oplus E_{s_F} \oplus TF, \tag{2.3}$$

where $E_1 \cdots E_{s_F}$ are S^1-complex line bundles. Let h_0 act on E_j by $e^{2\pi i t m_j}$, $m_j > 0 \in \mathbb{Z}$ (we can make $m_j > 0$ by choosing the complex structure of E_j) and $c_1(E_j) = 2\pi\sqrt{-1}x_j$. Let $c_1(L|_F) = 2\pi\sqrt{-1}c$ and h_0 act on $L|_F$ by $e^{2\pi\sqrt{-1}lt}$. Let $\hat{A}(F)$ be the $\hat{A}$ characteristic form on TF and let $\mathrm{ch}(g)$ be the odd Chern character of g (cf. [22]). The complex structure on the normal bundle $N(F)$ and the orientation on TF induced by the equivariant spin^c structure give an orientation on $TM|_F$. We let $(-1)^F$ be $+1$ if this orientation is the given orientation on M, and -1 otherwise. Then similar to Theorem 1.4 in [4], we have:

Proposition 2.3.

$$\begin{aligned}\mathrm{Ind}_{h_0} T_g = \sum_{F \subset M^{S^1}} (-1)^F \cdot (-1)^{s_F} \int_F \mathrm{ch}(g)\widehat{A}(TF) \\ \cdot \prod_{j=1}^{s_F} \frac{1}{e^{\pi\sqrt{-1}(x_j+m_j t)} - e^{-\pi\sqrt{-1}(x_j+m_j t)}} e^{\pi\sqrt{-1}(c+lt)}.\end{aligned} \tag{2.4}$$

Proof. Locally, we can consider D as a Dirac operator twisting a line bundle, then by Theorem 2.3 in [15] and Chern root algorithm, we can get this proposition. □

For any $z = h_0^m$ and an integer m, then z is also a generator of S^1. By (2.4), we have

$$\mathrm{ind}_z(T_g) = \sum_{F \subset M^{S^1}} (-1)^F \cdot (-1)^{s_F} \mathrm{AS}_{F,g}(z), \tag{2.5}$$

where

$$\mathrm{AS}_{F,g}(z) := \int_F \mathrm{ch}(g)\widehat{A}(TF) \cdot \sum_{j=1}^{s_F} \frac{1}{z^{m_j/2} e^{\sqrt{-1}\pi x_j} - z^{-m_j/2} e^{-\sqrt{-1}\pi x_j}} e^{\pi\sqrt{-1}c} z^{\frac{l}{2}}.$$

In the following, we prove the equivariant spin^c version of the odd index theorem for Dirac operators with involution parity (see the following (2.7)) in [5]. As before, let M be a closed oriented spin^c manifold of dimension $2r+1$, with a fixed spin^c structure P associated to the complex line bundle L. A compact group H of isometries of M preserves the orientation and spin^c structure. There is a lift of $h \in H$ acting on $\Gamma(\triangle(TM))$ which commutes with D.

Suppose that $\tau: M \to M$ is an orientation-reversing isometric involution which has a lift of τ on L and preserves the Pin^c structure induced by the Spin^c structure and commutes with any $h \in H$. We may take ∇^L as τ-invariant and H-invariant Hermitian connection. As in [5], there exists a self-adjoint lift $\tau: \Gamma(M;\Delta(TM)) \to \Gamma(M;\Delta(TM))$ satisfying

$$\tau^2 = 1;\ D\tau = -\tau D,\ \tau h = h\tau. \tag{2.6}$$

Then the $+1$ and -1 eigenspaces of τ give a splitting of the twisted spinor fields

$$\Gamma(M;\Delta(TM)) \cong \Gamma^+(M;\Delta(TM)) \bigoplus \Gamma^-(M;\Delta(TM)) \tag{2.7}$$

and the Dirac operator interchanges $\Gamma^+(M;\Delta(TM))$ and $\Gamma^-(M;\Delta(TM))$. By (2.6), h preserves $\Gamma^{\pm}_{-}(M;\Delta(TM))$. The associated equivariant index is defined by

$$\mathrm{index}_h[D^+: \ \Gamma^+(M;\Delta(TM)) \to \Gamma^-(M;\Delta(TM))], \tag{2.8}$$

and the associated Spin^c quantization of M is defined by virtual complex H-representive space $\widetilde{Q}(M,P) = \ker(D^+) - \ker(D^-)$. The simplest example is $X = M \times S^1$ with τ the reflection $x \to -x$ on S^1 and $H = S^1$ acting on the even dimensional spin^c manifold M. Then for S^1 acting on M, we have:

Lemma 2.4. *The following statements hold*:

a) *Let $F(\cdot)$ denote a fix point set. Then*

$$F(h^m\tau) = F(h\tau) = F(S^1) \cap F(\tau) \ \textit{for any}\ m \in Z,$$

b) *Let $N(\tau h)$ denote the normal bundle on $F(\tau h)$. Then $N(\tau h)$ has a τS^1-invariant decomposition $N(\tau h) = N_0 \oplus_{1\le j} N_j$, where N_0 is a real vector bundle and N_j is a complex line bundle and τh acts on N_0 by -1 and N_j by $e^{2\pi i(m_j t + a_j)}$, where $m_j > 0 \in Z$ and $a_j = 0$ or $\frac{1}{2}$.*

Proof. a) If $h\tau x = x$, then $(h\tau)^2 x = x$. Since $\tau^2 = \mathrm{id}; \tau h = h\tau$, we know $h^2 x = x$. Since $F(h^m) = F(h) = F(S^1)$, we have $hx = x$ and $\tau x = x$. Thus $F(h\tau) \subset F(S^1) \cap F(\tau)$ and the inverse inclusion is trivial. Similarly we can get $F(h^m\tau) = F(S^1) \cap F(\tau)$.

b) By a), we have $N(h\tau)$ is a S^1-representative bundle, so $N(h\tau)$ has a S^1-invariant decomposition $N_0 \oplus_{1\le j} N_j$ and N_0 is a real vector bundle and N_j is an oriented 2-plane bundle and h acts on N_0 by 1 and N_j by $e^{2\pi i m_j t}$, where $m_j \neq 0 \in \mathbb{Z}$. Since τ is an involution isometry and commutes with h, so τ preserves the above decomposition acting by $+1$ or -1. □

Let $2b_F + 1$ be the dimension for the bundle N_0 and $2c_F$ be the real dimension for the bundle which τh acts on by $e^{2\pi i(m_j t+\frac{1}{2})}$, $m_j > 0$. $2d_F$ is the real dimension for the bundle which τh acts on by $e^{2\pi i m_j t}$, $m_j > 0$. Let the connection matrix of N_0 be given formally as

$$\begin{bmatrix} 0 & x_1 & & & \\ -x_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & x_{b_F} \\ & & & -x_{b_F} & 0 \\ 0 & 0 & & 0 & 0 \end{bmatrix}.$$

Let $y_1, \dots, y_{c_F}$, $w_1, \dots, w_{d_F}$ be the associated Chern roots of above complex bundles respectively by the splitting principle (see [13, p. 227]). Define $s_F = \frac{\mathrm{codim} F(h\tau)-1}{2}$ and similarly define $(-1)^F$ as in (2.4). Let c be the first Chern form of $L|_{F(\tau h)}$ and τh act on $L|_{F(\tau h)}$ by $e^{2\sqrt{-1}(lt+b)}$ where $l \in \mathbb{Z}$ and $b = 0$ or $\frac{1}{2}$. We have the following formula by the method in [12] for any $z = h^m \in S^1$.

Proposition 2.5.

$$\mathrm{ind}_z(D^+) = \sum_{F\subset M^{S^1}\cap F(\tau)} (-1)^F \cdot (-1)^{s_F} \int_{F(S^1)\cap F(\tau)} i^{-(b_F+c_F+1)} \widehat{A}(F(S^1)\cap F(\tau))$$

$$\cdot z^{\frac{l}{2}} e^{\pi\sqrt{-1}b+\frac{1}{2}c} \sum_{j=1}^{b_F} \frac{1}{e^{i\pi x_j} + e^{-i\pi x_j}} \sum_{\alpha=1}^{c_F} \frac{1}{z^{m_\alpha/2} e^{i\pi y_\alpha} + z^{-m_\alpha/2} e^{-i\pi y_\alpha}}$$

$$\cdot \sum_{\beta=1}^{d_F} \frac{1}{z^{m_{c_F+\beta}/2} e^{i\pi w_\beta} - z^{-m_{c_F+\beta}/2} e^{-i\pi w_\beta}}. \tag{2.9}$$

Proof. We can consider D as a Dirac operator twisting a line bundle, then by Theorem 4.1 in [15] and Chern root algorithm, we can get this proposition. □

2.2. A geometric quantization formula

We firstly recall some definitions and properties on reduction in [1].

Definition 2.6. A reducible hypersurface in M is a co-oriented submanifold Z of codimensional one that is invariant under the S^1 action and on which this action is free. The reduction of M at Z by the circle action, $M_{\mathrm{red}} := Z/S^1$.

Definition 2.7. A reducible hypersurface Z is splitting if its complement, $M\backslash Z$, is a disjoint union of two (not necessarily connected) open pieces, M_+ and M_-, such that positive normal vectors to Z point into M_+ and negative normal vectors point into M_-. We then say that Z splits M into M_+ and M_-.

Example. Let $\Phi : M \to \mathbb{R}$ be a smooth S^1-invariant function. Assume that 0 is a regular value for Φ and that S^1 acts freely on the level set $\Phi^{-1}(0)$. Then $Z = \Phi^{-1}(0)$ is a reducible hypersurface, and it splits M into $M_+ := \Phi^{-1}(0, \infty)$ and $M_- := \Phi^{-1}(-\infty, 0)$. Conversely, every reducible splitting hypersurface can be obtained in this way.

The orientation, the Riemannian metric and the spin^c structure on M determine the reduced orientation, the reduced Riemannian metric and the reduced spin^c structure $P_{\rm red}$ on $M_{\rm red}$ (for details, see [1]). By h_0 acts on L_F by h_0^l in Definition 2.2, so the fibre weight (see Remark 4.3 in [1]) is l. Since the unitary matrix g is S^1-invariant, it determines a unitary matrix $g_{\rm red}$ on $M_{\rm red}$. Now we can state our quantization formula.

Theorem 2.8. *Let the circle act on a compact oriented odd dimensional Riemannian manifold. Let P be the equivariant spin^c structure on M. Let Z be a reducible hypersurface that splits into M_+ and M_-. Assume the following conditions are satisfied for every component F of the fixed point set*:

$$l \geq \sum |m_j| \implies F \subset M_+; \; l \leq -\sum |m_j| \implies F \subset M_-, \tag{2.10}$$

then

$$\dim Q(M, T_g, P)^{S^1} = \dim Q(M_{\rm red}, T_{g_{\rm red}}, P_{\rm red}). \tag{2.11}$$

Example 1. Let S^1 act on any odd dimensional spin manifolds, consider the spin^c bundle with trivial associated line bundle. The criteria are automatically satisfied on every fixed point set, so the quantization formula is true. In fact, by the odd dimensional Aiyah-Hirzebruch theorem, we know that both are trivial.

Example 2. Let $M = N_1 \times N_2$ where N_1 and N_2 are even and odd dimensional spin^c manifolds respectively. Let S^1 act on N_1 satisfying the condition (2.10) (for such examples, see [1]) and S^1 act on N_2 trivially.

We recall the definitions of the cut spaces. We can assume that Z is the zero level set of an invariant function $\Phi : M \to \mathbb{R}$ for which 0 is a regular value, and that $M_+ := \Phi^{-1}(0, \infty)$ and $M_- := \Phi^{-1}(-\infty, 0)$. Consider the product $M \times \mathbb{C}$ with the circle action, $a \cdot (m, z) = (a \cdot m, a^{-1} \cdot z)$, and with the function $\widetilde{\Phi}(m, z) := |z|^2 - \Phi(m)$. It is easy to check that 0 is a regular value for $\widetilde{\Phi}$ and that S^1 acts freely on the zero level set of $\widetilde{\Phi}$, $\widetilde{Z} = \{(m, z)| \ \Phi(m) = |z|^2\}$. The cut space defined by $M^+_{\rm cut} := \widetilde{Z}/S^1$ is a smooth compact manifold. Let S^1 act on $M^+_{\rm cut}$ by $\lambda \cdot [m, z] = [\lambda \cdot m, z] = [m, \lambda \cdot z]$. The cut space $M^-_{\rm cut}$ is defined similarly, using the diagonal action on $M \times \mathbb{C}$, $a \cdot (m, z) = (a \cdot m, a \cdot z)$, and $M^-_{\rm cut} := \{(m, z)| \ \Phi(m) = |z|^2\}/S^1$. Let S^1 act on $M^-_{\rm cut}$ by $\lambda \cdot [m, z] = [\lambda \cdot m, z] = [m, \lambda^{-1} \cdot z]$. We let $g(m, z) = g(m)$. By g is S^1-invariant, we have $g(a \cdot (m, z)) = g(m, z)$, so we can get a $g^+_{\rm cut}$ on $M^+_{\rm cut}$ by descending g. It is obvious that $g^+_{\rm cut}$ is S^1-invariant. Similarly, we can get a $g^-_{\rm cut}$ on $M^-_{\rm cut}$ which is S^1-invariant. Define the embedding $i_+ : M_+ \to M^+_{\rm cut}$ by $i_+(m) := [(m, \sqrt{\Phi(m)})]$, and the map $i_{\text{red}} : M_{\text{red}} \to M^+_{\text{cut}}$ by $i_{\text{red}}([m]) := [(m,0)]$. It is obvious that

$$g|_{M^+} = i_+^* g^+_{\text{cut}};\ i^*_{\text{red}} g^+_{\text{cut}} = g_{\text{red}}. \tag{2.12}$$

We can define g^-_{cut} similarly.

We extend the both hands of (2.5) to the meromorphic functions on the complex plane. Since they are equal on the group generated by h, they must be equal on the complex plane. $\dim Q(M, T_g, P)^{S^1}$ is equal to the coefficient of 1 in the Taylor expansion of this meromorphic function. As in [1], we now expand (2.5) into a formal power series in z^{-1}. If $m_j > 0$, then

$$\begin{aligned} &\frac{1}{z^{m_j/2} e^{\sqrt{-1}\pi x_j} - z^{-m_j/2} e^{-\sqrt{-1}\pi x_j}} \\ &= z^{-m_j/2} e^{-\sqrt{-1}\pi x_j} (1 + e^{-2\sqrt{-1}\pi x_j} z^{-m_j} + e^{-4\sqrt{-1}\pi x_j} z^{-2m_j} + \cdots). \end{aligned} \tag{2.13}$$

If $m_j < 0$, then

$$\begin{aligned} &\frac{1}{z^{m_j/2} e^{\sqrt{-1}\pi x_j} - z^{-m_j/2} e^{-\sqrt{-1}\pi x_j}} \\ &= -z^{m_j/2} e^{\sqrt{-1}\pi x_j} (1 + e^{2\sqrt{-1}\pi x_j} z^{m_j} + e^{4\sqrt{-1}\pi x_j} z^{2m_j} + \cdots). \end{aligned} \tag{2.14}$$

So

$$\mathrm{As}_{F_i,g}(z) = z^{\frac{1}{2}(l - \sum |m_j|)} \int_{F_i} (c + c' z^{-1} + c" z^{-2} + \cdots), \tag{2.15}$$

where $c', c"$ denotes some cohomology classes. A similar computation, when we expand in power of z, give the

$$\mathrm{As}_{F_i,g}(z) = z^{\frac{1}{2}(l + \sum |m_j|)} \int_{F_i} (c + c' z + c" z^2 + \cdots). \tag{2.16}$$

By (2.15) and (2.16), we have:

Lemma 2.9. *Let S^1 act on a smooth odd dimensional spin^c manifold. Let P be an equivariant spin^c structure on M.*

1. *When we expand the index formula* (2.5) *as a formal power series in z, the fixed point component F does not contribute to* $\dim Q(M, T_g, P)^{S^1}$ *if $l > -\sum |m_j|$.*
2. *When we expand the index formula* (2.5) *as a formal power series in z^{-1}, the fixed point component F does not contribute to* $\dim Q(M, T_g, P)^{S^1}$ *if $l < \sum |m_j|$.*

Proof of Theorem 2.8. Let Z be a reducible hypersurface that splits M into M_+ and M_-. By S^1 acts on Z freely, so $F \subset M_+ \cup M_-$. So the condition (2.10) is equivalent to the following condition:

$$F \subset M_+ \implies l > -\sum |m_j|;\ F \subset M_- \implies l < \sum |m_j|. \tag{2.17}$$

By Lemma 2.9 and (2.17), fix points in M_- contribute nothing to $Q(M,T_g,P)^{S^1}$ when we take the expansion in z^{-1}; that is
(2.18)

$$\mathrm{dim}Q(M,T_g,P)^{S^1}=\sum_{F\subset M_+}\begin{array}{c}\text{coefficient of 1 in the expansion of } AS_{F,g}(z)\\ \text{as a formal power series in } z^{-1}.\end{array}$$

Now consider the cut space, M^+_{cut} (In [1], they use M_{cut}). By Proposition 6.1 in [1], the set of fixed points in M^+_{cut} is the union $M^{+,S^1}_{\mathrm{cut}}=i_+(M^{S^1}_+)\cup i_{\mathrm{red}}(M_{\mathrm{red}})$, where $M^{S^1}_+$ is the set of fixed points in M_+ and for $i_+:M_+\to M^+_{\mathrm{cut}}$ and $i_{\mathrm{red}}:M_{\mathrm{red}}\to M^+_{\mathrm{cut}}$, see p. 544 in [1]. Let us denote by F' and X', respectively, the images in M_{cut} of connected components F of $M^{S^1}_+$ and X of M_{red}. Then

$$\begin{aligned}&\mathrm{dim}Q(M^+_{\mathrm{cut}},T_{g^+_{\mathrm{cut}}},P^+_{\mathrm{cut}})^{S^1}\\ &=\sum_{F\subset M_+}\begin{array}{c}\text{coefficient of 1 in the expansion of } AS_{F',g^+_{\mathrm{cut}}}(z)\\ \text{as a formal power series in } z^{-1}\end{array}\\ &\quad+\sum_{X\subset M_{\mathrm{red}}}\begin{array}{c}\text{coefficient of 1 in the expansion of } AS_{X',g^+_{\mathrm{cut}}}(z)\\ \text{as a formal power series in } z^{-1},\end{array}\end{aligned}\tag{2.19}$$

where $AS_{F',g^+_{\mathrm{cut}}}(z)$ denotes a expansion term of the equivariant index of the Toeplitz operator on M^+_{cut} associated to g^+_{cut}. By (2.12), we have $\mathrm{ch}(g)=i^*_+\mathrm{ch}g_{\mathrm{cut}}$. Since i_+ is a S^1 equivariant embedding, so we use the definition of $\mathrm{AS}_{F,g}(z)$ and the Pincaré dual, we get $\mathrm{AS}_{F',g^+_{\mathrm{cut}}}(z)=\mathrm{AS}_{F,g}(z)$. Let X be a connected component of M_{red}, let $X':=i_{\mathrm{red}}(X)$ be its image in M^+_{cut}. Let $N:=Z\times_{S^1}\mathbb{C}$ be a line bundle over X. Let L be a complex line bundle associated to the spinc structure on M and $L_{\mathrm{red}}:=i^*L/S^1$ over $M_{\mathrm{red}}:=Z/S^1$, where $i:Z\to M$ is the conclusion of a reducible hypersurface. Let L_{cut} be the associated line bundle to the induce spinc structure over M^+_{cut}, then by Proposition 6.1 4) in [1], we have $i^*_{\mathrm{red}}L_{\mathrm{cut}}\cong L_{\mathrm{red}}\otimes N$. By (2.12), we have $\mathrm{ch}(g_{\mathrm{red}})=i^*_{\mathrm{red}}\mathrm{ch}g_{\mathrm{cut}}$. So by Proposition 6.1 2) in [1] and the Pincaré dual, similar to the computation of (7.14) in [1], we get
(2.20)

$$AS_{X',g^+_{\mathrm{cut}}}(z)=-\int_X\exp(\frac{1}{2}c_1(L_{\mathrm{red}}))\widehat{A}(X)\mathrm{ch}(g_{\mathrm{red}})(1+cz^{-1}+c'z^{-2}+\cdots),$$

where $c_1(L_{\mathrm{red}})$ denotes the first Chern class. By applying the (non-equivariant) odd Atiyah-Singer formula to M_{red}, we get

$$\begin{aligned}&-\mathrm{dim}Q(M_{\mathrm{red}},T_{g_{\mathrm{red}}},P_{\mathrm{cut}})\\ &=\sum_{X\subset M_{\mathrm{red}}}\begin{array}{c}\text{coefficient of 1 in the expansion of } AS_{X',g^+_{\mathrm{cut}}}(z)\\ \text{as a formal power series in } z^{-1}.\end{array}\end{aligned}\tag{2.21}$$

By (2.18), (2.19) and (2.21), we get

(2.22)
$$\dim Q(M^+_{\rm cut}, T_{g^+_{\rm cut}}, P^+_{\rm cut})^{S^1} = \dim Q(M, T_g, P)^{S^1} - \dim Q(M_{\rm red}, T_{g_{\rm red}}, P_{\rm red}).$$

Now we consider the expansion in z. For every fixed component of the form $F' = i_+(F)$, for $F \subset M_+$, the first condition of (2.17) and Lemma 2.9 implies that this fixed point set does not contribute. Similarly, consider a fixed component of form $X' = i_{\rm red}(X)$ for $X \subset M_{\rm red}$. Its fibre weight is 1 by Proposition 6.1 2) in [1]. By part 1 of Lemma 2.9, X' does not contribute either. Therefore $\dim Q(M^+_{\rm cut}, T_{g^+_{\rm cut}}, P^+_{\rm cut})^{S^1} = 0$. So we have

$$\dim Q(M, T_g, P)^{S^1} = \dim Q(M_{\rm red}, T_{g_{\rm red}}, P_{\rm red}). \qquad \square$$

3. The Kostant type formulas

Let $\delta \in \mathbb{Z}$ be an integer as a weight of circle group and denote by $\mathbf{n}(\delta, Q(M, T_g))$ the multiplicity of this weight in $Q(M, T_g)$. If we assume that $\dim F = 1$, then $s_F = r$. We define the partition function $\overline{N}_F : \{m + \frac{1}{2}n |\ m, n \in \mathbb{Z}\} \to \mathbb{Z}^+$ by setting

$$\overline{N}_F(\beta) = \left| \left\{ (k_1, \ldots, k_r) \in (\mathbb{Z} + \tfrac{1}{2})^r \ : \ \beta + \sum_{j=1}^{r} k_j m_{j,F} = 0,\ k_j > 0 \right\} \right|.$$

The right hand side is always finite since $m_{j,F} > 0$.

Theorem 3.1 (Kostant formula)**.**

$$\mathbf{n}(\delta, Q(M, T_g)) = (-1)^r \sum_{F \subset M^{S^1}} (-1)^F \overline{N}_F(\delta - \frac{1}{2} l_F) \int_F \mathrm{ch}(g). \tag{3.1}$$

Proof. It is obvious that $\mathbf{n}(\delta, Q(M, T_g))$ is the coefficient of z^δ in the Taylor expansion of $\sum_{F \subset M^{S^1}} (-1)^F \cdot (-1)^{s_F} \mathrm{AS}_{F,g}(z)$. By $\dim F = 1$ and the formula (2.5), we get

$$\mathrm{AS}_{F,g}(z) = z^{\frac{1}{2}(l_F - \sum_{j=1}^r m_{j,F})} \prod_{j=1}^{r} \frac{1}{1 - z^{-m_{j,F}}} \int_F \mathrm{ch}(g). \tag{3.2}$$

Note that when $|z| > 1$, we have

$$\prod_{j=1}^{r} \frac{1}{1 - z^{-m_{j,F}}} = \sum N_F(\delta) \cdot z^\delta,$$

where $N_F(\delta)$ is the number of non-negative integer solutions $(k_1, \ldots, k_r) \in (\mathbb{Z}^+)^r$ to $\delta + \sum_{j=1}^r k_j m_{j,F} = 0$. By (3.2) and Lemma 5.0.1 in [6], then

$$\sum_{F \subset M^{S^1}} (-1)^F \cdot (-1)^{s_F} \mathrm{AS}_{F,g}(z)$$

$$= (-1)^r \sum_{F\subset M^{S^1}} (-1)^F \sum_\delta N_F(\delta) z^{\delta+\frac{1}{2}(l_F-\sum_{j=1}^r m_{j,F})} \int_F \mathrm{ch}(g)$$
$$= (-1)^r \sum_{F\subset M^{S^1}} (-1)^F \sum_\delta N_F(\delta - \frac{1}{2}l_F + \frac{1}{2}\sum_{j=1}^r m_{j,F}) z^\delta \int_F \mathrm{ch}(g)$$
$$= (-1)^r \sum_{F\subset M^{S^1}} (-1)^F \sum_\delta \overline{N}_F(\delta - \frac{1}{2}l_F) z^\delta \int_F \mathrm{ch}(g).$$

So we get the Kostant formula. □

Remark. By $\mathrm{dim}F = 1$, so $F = S^1$. We note that

$$\int_F \mathrm{ch}(g) = \int_F \widehat{A}(F)\mathrm{ch}(g) = \mathrm{ind}T_g,$$

so $\int_F \mathrm{ch}(g)$ is an integer. For example, we take $g : S^1 \to S^1;\ e^{i\theta} \to e^{in\theta}$, then $\int_F \mathrm{ch}(g) = n$.

By the formula (2.9), we can get another Kostant formula. Assume that the fixed points $F(S^1)\cap F(\tau)$ are isolated. So by the formula (2.9) and using the geometric series

$$\frac{1}{1-z} = \sum_{l=0}^\infty z^l;\ \frac{1}{1+z} = \sum_{l=0}^\infty (-1)^l z^l,$$

we get

(3.3)

$$\mathrm{ind}_z(D^+) = (-1)^r \sum_{F\subset M^{S^1}\cap F(\tau)} (-1)^F i^{-(b_F+c_F+1)} e^{\pi\sqrt{-1}b_F} \sum_\delta z^{\delta+\frac{l}{2}-\frac{1}{2}\sum_{j=1}^{c_F+d_F} m_{j,F}} \widetilde{N}(\delta),$$

where $\widetilde{N}(\delta) = \sum(-1)^{k_1+\cdots+k_{c_F}}$ and the sum is taken over $\beta+\sum_{j=1}^{c_F+d_F} k_j m_{j,F} = 0$ and $k_j \geq 0$. Let $\widetilde{N}_0(\delta) = \sum(-1)^{k_1+\cdots+k_{c_F}}$ where the sum is taken over $\beta + \sum_{j=1}^{c_F+d_F}(k_j+\frac{1}{2})m_{j,F} = 0$ and $k_j \geq 0$. So we get:

Lemma 3.2 (Kostant formula)**.**

$$\text{(3.4)}\quad \mathbf{n}(\delta, \widetilde{Q}(M)) = (-1)^r \sum_{F\subset M^{S^1}\cap F(\tau)} (-1)^F i^{-(b_F+c_F+1)} e^{\pi\sqrt{-1}b_F} \widetilde{N}_0(\delta - \frac{l_F}{2}).$$

Next we consider the non-isolated fixed points cases. Define the following set

$$\mathbf{S}_{\delta,s_F} = \left\{ (k_1,\ldots,k_{s_F}) \in (\mathbb{Z}+\tfrac{1}{2})^{s_F}\ :\ \delta + \sum_{j=1}^{s_F} k_j m_{j,F} = 0,\ k_j > 0 \right\}$$

and for each tuple $k = (k_1, \ldots, k_{s_F})$, let

$$p_{k,F} = (-1)^{s_F} \int_F \mathrm{ch}(g)\widehat{A}(F)e^{\pi\sqrt{-1}c - \sum \pi\sqrt{-1}x_j} e^{-2\sum_j k_j \pi\sqrt{-1}x_j}.$$

Now define $\overrightarrow{N}_F(\delta) = \sum_{k\in \mathbf{S}_{\delta,s_F}} p_{k,F}$. Using the same trick as in the isolated fixed points case, we get:

Theorem 3.3 (Kostant formula)**.**

$$\mathbf{n}(\delta, Q(M, T_g)) = (-1)^r \sum_{F\subset M^{S^1}} (-1)^F \overrightarrow{N}_F(\delta - \frac{1}{2} l_F). \tag{3.5}$$

Let

$$\overline{p}_{k,F} = (-1)^{s_F} \int_F i^{-(b_F+c_F+1)} \widehat{A}(F) e^{\pi\sqrt{-1}b + \frac{1}{2}c} \sum_{j=1}^{b_F} \frac{1}{e^{i\pi x_j} + e^{-i\pi x_j}}$$
$$\cdot(-1)^{k_1+\cdots+k_{c_F}} e^{-2\sum_\alpha k_\alpha \pi\sqrt{-1}y_\alpha - 2\sum_\beta k_{\beta+c_F}\pi\sqrt{-1}w_\beta},$$

and $\overrightarrow{N}^0_F(\delta) = \sum_{k\in \mathbf{S}_{\delta,c_F+d_F}} \overline{p}_{k,F}$, then we get:

Theorem 3.4 (Kostant formula)**.**

$$\mathbf{n}(\delta, \widetilde{Q}(M)) = \sum_{F\subset M^{S^1}\cap F(\tau)} (-1)^F \overrightarrow{N}^0_F(\delta - \frac{l_F}{2}). \tag{3.6}$$

By Theorem 3.3, repeating the discussions in [6], we can get:

Theorem 3.5 (Cutting formula)**.**

$$\mathbf{n}(\delta, Q(M, T_g)) = \mathbf{n}(\delta, Q(M^+_{\mathrm{cut}}, T_{g^+_{\mathrm{cut}}})) + \mathbf{n}(\delta, Q(M^-_{\mathrm{cut}}, T_{g^-_{\mathrm{cut}}})). \tag{3.7}$$

Remark. If we assume that the involution can descend to an involution on M_{red}, then we can get a cutting formula for $\widetilde{Q}$. We may also consider the odd signature quantization as in [9].

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