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CRITERION FOR BLOW-UP IN THE EULER EQUATIONS VIA CERTAIN PHYSICAL QUANTITIES

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ABSTRACT. We consider the (possible) finite time blow-up of the smooth solutions of the 3D incompressible Euler equations in a smooth domain or in $\mathbf{R}^3$. We derive blow-up criteria in terms of L^∞ of the partial component of Hessian of the pressure together with partial component of the vorticity.

1. INTRODUCTION

Let Ω be $\mathbf{R}^3$ or a smooth bounded domain in $\mathbf{R}^3$. We consider the Euler equations of incompressible fluid in Ω:

$$\begin{cases} u_t + (u\cdot\nabla)u + \nabla p = f, \\ \text{div} u = 0, \end{cases} \tag{1.1}$$

with initial velocity u_0 and boundary condition $u\cdot n = 0$. Here n is the unit outer normal vector to $\partial\Omega$.

It is known that if the initial velocity, $u_0 \in H^m$, $m > 5/2$, then there exists a unique smooth solution for the Euler equations up to some positive time(See [7], [10] and references therein). A natural question is then whether this solution quits to be smooth and thus quits to be a strong solution any more in a finite time. This question is also related with the regularity problem of the 3D incompressible Navier-Stokes equations[6]. There have been developed many criteria whether the solution blows up in a finite time. Especially, blow-up criteria in terms of vorticity, deformation tensor, and the Hessian of the pressure have been developed under various situations(See for example [2, 4, 5, 9] and [3]). Also, localization of these blow-up criteria have been developed[1, 8]. Among others, blow-up criteria by pressure involve all components of the Hessian of the pressure until now.

Our aim here is developing blow-up criteria in terms of L^∞ norm of some components of the Hessian of the pressure together with L^∞ norm of some component of the vorticity. We shall derive differential inequalities for L^q norm of vorticity using certain equations for the product

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of the vorticity and the deformation tensor and apply the energy argument to those inequalities. This approach is already used in [1] to derive a criterion by L^∞ norm of the Hessian of the pressure.

2. BLOW-UP CRITERIA

Let T^* be the blow-up time of a classical solution u for the Euler equations (1.1). The singular set for u at time $t = T^*$ is defined by the set of all $x \in \overline{\Omega}$ such that for any ball B centered at x,

$$\liminf_{t \nearrow T^*} \|u(t)\|_{H^m(B\cap\Omega)} = \infty.$$

Let us denote the singular set by S. It is clear that S is a closed set in $\overline{\Omega}$. If $H \subset S$ is a bounded closed component of S, we call it an isolated singular set for u. Trivially, $dist(H, S - H) > 0$.

Let $\omega = \nabla \times u$ be the vorticity and $J = \omega \cdot \nabla u$. Then, taking a curl to the first equation in (1.1) and by direct calculation, for $0 < t < T^*$

$$\frac{\partial \omega}{\partial t} + (u \cdot \nabla)\omega = J, \tag{2.1}$$

$$\frac{\partial J}{\partial t} + (u \cdot \nabla)J = -(\omega \cdot \nabla)\nabla p. \tag{2.2}$$

Throughout this section, we denote $\tilde{\omega} = (\omega_1, \omega_2)$ and $\tilde{\nabla} = (\partial_1, \partial_2)$. We start with the following lemma first.

Lemma 1. *Let $g \in C[0,a]$ and $A, B \in L^1[0,a]$ be nonnegative and*

$$g(t) - g(0) \le \int_0^t \int_0^s (A(\tau)g(\tau) + B(\tau))\, d\tau ds, \quad t \in [0,a]. \tag{2.3}$$

Then g satisfies

$$g(t) \le \left(g(0) + \int_0^t \int_0^s B\right) \exp \int_0^t \int_0^s A d\tau ds. \tag{2.4}$$

Proof. We shall show that for any $\epsilon > 0$,

$$g(t) \le \left(g(0) + \int_0^t \int_0^s B + \epsilon\right) \exp \int_0^t \int_0^s A(\tau) d\tau ds.$$

For simplicity, let us denote $X(s) = \exp \int_0^s \int_0^r A(\tau) d\tau dr$. Suppose not. Then, since the LHS of the above is smaller than the RHS of the above at $t = 0$, for some $\epsilon > 0$, there would be a first time $b < a$ such that the equality holds in the above. However, at $t = b$, by (2.3),

$$\begin{aligned} g(b) \le & \int_0^b \int_0^t A(s) \left[(g(0) + \epsilon)X(s) + X(s) \int_0^s \int_0^r B \right] \\ & + g(0) + \int_0^b \int_0^t B. \end{aligned}$$

By integration by parts,

$$\begin{aligned}\int_0^b\int_0^t A(s)X(s) &= \left[\int_0^t A(s)X(t) - \int_0^t\left(\int_0^s A(\tau)\right)^2 X(s)\right]\\ &\le \int_0^b\int_0^t A(s)X(t)\\ &= X(b)-X(0) = \exp\int_0^b\int_0^r A(\tau)d\tau dr - 1,\end{aligned}$$

Similarly, by integration by parts,

$$\begin{aligned}\int_0^t A(s)X(s)\int_0^s\int_0^r B = &\int_0^t A(s)X(t)\int_0^t\int_0^r B\\ &-\int_0^t\left(\int_0^s A(\tau)d\tau\right)^2 X(s)\int_0^s\int_0^r B\\ &-\int_0^t\int_0^s A(\tau)d\tau X(s)\int_0^s B\\ &\le \int_0^t A(s)X(t)\int_0^t\int_0^r B.\end{aligned}$$

Therefore,

$$\begin{aligned}\int_0^b\int_0^t A(s)X(s)\int_0^s\int_0^r B &\le \int_0^b\int_0^t A(s)X(t)\int_0^t\int_0^r B\\ &\le X(b)\int_0^b\int_0^r B - \int_0^b X(t)\int_0^t B\\ &\le X(b)\int_0^b\int_0^r B - \int_0^b\int_0^t B\end{aligned}$$

by $X(t)\ge 1$. Gathering these inequalities, we have

$$g(b)\le (g(0)+\epsilon)X(b)-\epsilon + X(b)\int_0^b\int_0^r B = g(b)-\epsilon.$$

Thus, we arrive at a contradiction and finish the proof. □

Theorem 1. *Let u be a smooth solution of (1.1), $f\in L^\infty(0,\infty;H^m)$ a external force, H be an isolated singular set. For any relatively open set G containing H,*

$$\int_0^{T^*}\left[(T^*-t)\|\nabla\partial_3 p(t)\|_{L^\infty(G)} + \sup_{s\le t}\|\tilde\omega\|_{L^\infty(G)}(s)\right]dt = +\infty.$$

Proof. Without loss of generality, we can assume that there exist an (relatively) open smooth set $W\supset G$, $dist(W^c, G) > 0$. Let ξ be a smooth cutoff function such that $\xi = 1$ on G and

$\xi = 0$ on W^c. For convenience, let us denote

$$A_3(t) = (\int_\Omega |\omega_3|^q \xi dx)^{1/q}, \quad B_3(t) = (\int_\Omega |J_3|^q \xi dx)^{1/q},$$

$\tilde{D} = \|\tilde{\nabla}\partial_3 p\|_{L^\infty(W)}$, and $D_3 = \|\partial_3^2 p\|_{L^\infty(W)}$. $\tilde{A}$ is similarly defined with $\tilde{\omega}$. Multiplying the third component of (2.1) by $\xi|\omega_3|^{q-2}\omega_3$ and integrating over Ω, we have

$$\begin{aligned}\frac{1}{q}\frac{d}{dt}\int_\Omega |\omega_3|^q \xi dx &= \frac{1}{q}\int_\Omega |\omega_3|^q (u\cdot\nabla)\xi dx + \int_\Omega |\omega_3|^{q-2}\omega_3 J_3 \xi dx \\ &\le \frac{C}{q}\int_{W_\Omega\setminus G_\Omega} |\omega|^q |u| dx + \int_\Omega |\omega_3|^{q-1}|J_3|\xi dx \\ &\le CC^{q+1}\frac{1}{q}\|u\|^{q+1}_{H^m(W_\Omega\setminus G_\Omega)} + A_3^{q-1}B_3. \qquad (2.5)\end{aligned}$$

Similarly, multiplying (2.2) by $\xi|J_3|^{q-2}J_3$ and integrating over Ω, we have

$$\begin{aligned}\frac{1}{q}\frac{d}{dt}\int_\Omega |J_3|^q \xi dx &= \frac{1}{q}\int_\Omega |J_3|^q (u\cdot\nabla)\xi dx - \int_\Omega |J_3|^{q-2}J_3(\omega\cdot\nabla)\nabla_3 p\xi dx \\ &\le \frac{C}{q}\int_{W_\Omega\setminus G_\Omega} |J|^q|u|dx + \int_\Omega |J_3|^{q-1}[|\omega_3||\partial_3^2 p| + |\tilde{\omega}||\tilde{\nabla}\partial_3 p|]\xi dx \\ &\le \frac{C}{q}C^{2q+1}\|u\|^{2q+1}_{H^m(W_\Omega\setminus G_\Omega)} + \Big[\|\partial_3^2 p\|_{L^\infty(W_\Omega)}(\int_\Omega |\omega_3|^q\xi dx)^{\frac{1}{q}} \\ &+\tilde{A}\tilde{D}\Big](\int_\Omega |J_3|^q \xi dx)^{\frac{q-1}{q}} \\ &\le \frac{C^{2q+2}}{q} + D_3A_3B_3^{q-1} + \tilde{A}\tilde{D}B_3^{q-1}. \qquad (2.6)\end{aligned}$$

From the above equation, we get

$$\begin{aligned}(B_3^q + C^{2q})^{\frac{q-1}{q}}\frac{d}{dt}(B_3^q + C^{2q})^{1/q} &= \frac{1}{q}\frac{d}{dt}B_3^q \\ &\le \frac{C^{2q+2}}{q} + D_3A_3B_3^{q-1} + \tilde{A}\tilde{D}B_3^{q-1} \\ &\le (\frac{C^2}{q} + D_3A_3 + \tilde{A}\tilde{D})(B_3^q + C^{2q})^{\frac{q-1}{q}}.\end{aligned}$$

Therefore, applying the Gronwall lemma, we have

$$(B_3^q + C^{2q})^{1/q}(t) \le (B_3^q + C^{2q})^{1/q}(0) + \int_0^t (\frac{C^2}{q} + D_3A_3 + \tilde{A}\tilde{D}). \qquad (2.7)$$

Plugging the above into (2.5), we have

$$\frac{1}{q}\frac{d}{dt}A_3^q(t) \le \frac{1}{q}C^{2q+2} + A_3^{q-1}\left(C + \int_0^t \tilde{A}\tilde{D} + \int_0^t D_3A_3\right).$$

Then, we again have

$$\begin{aligned}(A_3^q + C^{2q})^{\frac{q-1}{q}} \frac{d}{dt}(A_3^q + C^{2q})^{\frac{1}{q}} &= \frac{1}{q}\frac{d}{dt}A_3^q \\ &\leq \frac{C^{2q+2}}{q} + A_3^{q-1}\left(C + \int_0^t \tilde{A}\tilde{D} + \int_0^t D_3 A_3\right) \\ &\leq \left(\frac{C^2}{q} + \int_0^t \tilde{A}\tilde{D} + \int_0^t D_3(A_3^q + C^{2q})^{1/q}\right) \times \\ &\quad (A_3^q + C^{2q})^{\frac{q-1}{q}}.\end{aligned}$$

Subsequently, by the Gronwall lemma,

$$(A_3^q + C^{2q})^{1/q}(t) \leq (A_3^q + C^{2q})^{1/q}(0) + \int_0^t \int_0^s (\frac{C^2}{q} + D_3 A_3 + \tilde{A}\tilde{D}).$$

Here, we redefine C suitably large constant. C depends on u_0, T^*, and $\sup_t \|u\|_{H^m(W_\Omega \setminus G_\Omega)}(t)$ but can be chosen independent of q since $q > 1$. Integrating the above and using (2.4), we have

$$A_3(t) \leq (C + A_3(0) + \int_0^t \int_0^s \tilde{A}\tilde{D}) \exp \int_0^t \int_0^s D_3.$$

Letting $q \to \infty$, we arrive at

$$\begin{aligned}\|\omega_3\|_{L^\infty(G)}(t) \leq &(C + \|\omega_3\|_{L^\infty(W)}(0) + \int_0^t \int_0^s \|\tilde{\omega}\|_{L^\infty(W)} \|\tilde{\nabla}\nabla_3 p\|_{L^\infty(W)}) \times \\ &\exp \int_0^t \int_0^s \|\nabla_3^2 p\|_{L^\infty(W)}.\end{aligned}$$

Since $\nabla^2 p$ is also uniformly bounded on $W \setminus G$,

$$\int_0^t \int_0^s \|\tilde{\omega}\|_{L^\infty(W)} \|\tilde{\nabla}\nabla_3 p\|_{L^\infty(W)} \leq \sup_{s<t} \|\tilde{\omega}\|_{L^\infty(W)}(s) \int_0^t \int_0^s \|\tilde{\nabla}\nabla_3 p\|_{L^\infty(W)},$$

and $\int_0^t \int_0^s h(\tau) d\tau = \int_0^t (t-s)h(s)$, we finish the proof by the local blow-up criterion by vorticity(see theorem 1 in [1]). □

Repeating the same argument in the proof of the above theorem, we have the following theorem.

Theorem 2. *Let u, f, H as in the previous theorem. For any relatively open set G containing H,*

$$\int_0^{T^*} \left[(T^* - t)\|\nabla\tilde{\nabla} p(t)\|_{L^\infty(G)} + \sup_{s \leq t} \|\omega_3\|_{L^\infty(G)}(s)\right] dt = +\infty.$$

Proof. Multiplying (2.1) by $\xi|\tilde\omega|^{q-2}\tilde\omega$ and (2.2) by $\xi|\tilde J|^{q-2}\tilde J$ and integrating over Ω, we have

$$\frac{1}{q}\frac{d}{dt}\tilde A^q \le \frac{C^{2q+2}}{q} + \tilde A^{q-1}\tilde B.$$

$$\frac{1}{q}\frac{d}{dt}\tilde B^q \le \frac{C^{2q+2}}{q} + \tilde D A_3 \tilde B^{q-1} + \tilde A\|\tilde\nabla^2 p\|_{L^\infty(W)}\tilde B^{q-1}.$$

Here, $\tilde B = (\int_\Omega |\tilde J|^q \xi dx)^{1/q}$ and A and D are as before. Following the same calculation as in the previous proof, we have

$$\tilde A(t) \le (C + \tilde A(0) + \int_0^t\int_0^s A_3\tilde D)\exp\int_0^t\int_0^s \|\tilde\nabla^2 p\|_{L^\infty(W)}.$$

Then, sending $q\to\infty$ again and using theorem 1 in [1], we arrive at the conclusion. □

Theorem 1 and 2 are different criteria from theorem 2 in [8] where blow-up criterion by stronger norm of $\tilde\omega$ was developed.

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