EXACT SOLUTION FOR STEADY PAINT FILM FLOW OF A PSEUDO PLASTIC FLUID DOWN A VERTICAL WALL BY GRAVITY

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ABSTRACT. Here in this paper, the steady paint film flow on a vertical wall of a non-Newtonian pseudo plastic fluid for drainage problem has been investigated. The exact solution of the non-linear problem is obtained for the velocity profile. Also the average velocity, volume flux, shear stress on the wall, force to hold the wall in position and normal stress difference have been derived. We retrieve Newtonian case, when material constant μ_1 and relaxation time λ_1 equal zero. The results for co-rotational Maxwell fluid is also obtained by taking material constant $\mu_1 = 0$. The effect of the zero shear viscosity η_0, the material constant μ_1, the relaxation time λ_1 and gravitational force on the velocity profile for drainage problem are discussed and plotted.

1. INTRODUCTION

Nowadays for different physical problems of Newtonian and non-Newtonian fluids an extensive literature is available and a lot of interest has been shown towards the study of exact, analytic and numerical solutions of these physical problems. The governing equations of non-Newtonian fluids are highly non-linear and it is also accepted now that in general, these equations are higher in order than the Navier-Stokes equations. Therefore, the exact solutions of these equations are very difficult and even for the case of viscous fluids. A lot of work has been carried out regarding the exact solutions for flows of non-Newtonian fluids [1, 2, 3, 4, 5].
Over the past few decades there has been a growing recognition of the fact that many fluids of industrial significance do not obey the Newtonian postulate of linear relationship between the shear stress and shear rate. Therefore, these fluids are known as non-Newtonian fluids. Common examples of such fluids are slurries, pastes, gels, molten plastics and lubricants containing polymer additives. Various food stuffs such as honey and tomato sauce, the biological

fluids like blood and synovial fluid naturally found in cavities of synovial joints also belong
to the general class of the non-Newtonian fluids [6, 7]. The simplest model of non-Newtonian
fluids is the power law model and a special case of the power law equation, with $n < 1$, is the
pseudo plastic model. Commonly encountered non-Newtonian fluids like polymer solutions,
paper pulps, detergents, oils and greases may be classified by the pseudo plastic fluid model
[8]. The pseudo plastic fluids represent shear thinning fluids and, as far as we know, this class
of non-Newtonian fluids received less attention in the literature.

Due to complexity of fluids there is not a single constitutive equation which can describe the
properties of all non-Newtonian fluids. In view of this, several non- Newtonian fluid models
have been proposed. There are many fluid models which describe the non-Newtonian behav-
ior of the fluids but in recent years, the pseudo plastic fluid have attracted many researchers.
Therefore, in the present analysis, we consider the steady, incompressible, parallel, laminar
flow of a pseudo plastic fluid of paint flowing down on an infinite vertical wall. The thickness
of the paint film is assumed to be h and gravity acts in the negative z-direction (downward).
The paint fall under the influence of gravity and there is no applied pressure driving the flow.
Here, we also neglecting the change in the hydrostatic of (atmospheric pressure) the surround-
ing air. Therefore, our aim here, to find the effects of relaxation times on the steady flow of
pseudo plastic paint (fluid) flowing down on an infinite vertical wall.

Nowadays a lot of interest has been shown by researcher towards the study of the thin film
flows of non-Newtonian fluids due to its wide spread applications in nonlinear sciences and
engineering industries. The work on the thin film flows of non-Newtonian fluids under various
configurations is relatively of recent origin. Siddiqui et al. [9] discussed the thin film flows of
a third grade fluid down an inclined plane. In [10] they examined the thin film flows of Phan-
Thein-Tanner (PTT) fluids on a moving belt. Siddiqui et al. [11] also considered the same flow
of non-Newtonian fluids on a moving belt. The thin film flow of a fourth grade fluid down a
vertical cylinder is also analyzed by Siddiqui et al. [12].

To the author's knowledge, no previous attempted has been reported to develop the governing
equations for steady incompressible flow of a pseudo plastic fluid model in case of thin film
paint flow. In this work it is intended to construct the equations for an incompressible pseudo
plastic fluid model. The non-linear differential equations are solved for velocity field by means
of exact method.

This paper is organized as follows: In section 2, the governing equations are developed. Sec-
tion 3 deals with the formulation of the problem, flow equation, average velocity and volume
flow rate. Result and discussion is given in section 4. The conclusion is given in section 5.

2. Governing Equations

The basic equations, governing the motion of an isothermal, homogeneous, incompressible
fluid are:

$$div\mathbf{V} \quad = \quad 0, \qquad\qquad\qquad (2.1)$$

$$\rho\frac{D\mathbf{V}}{Dt} \quad = \quad \rho\mathbf{f} - \mathrm{grad}P + div\mathbf{S}, \qquad\qquad\qquad (2.2)$$

where ρ is the constant fluid density, $\mathbf{V}$ is the velocity vector and $\mathbf{f}$ is the body force per unit mass, P denotes the dynamics pressure, the $\dfrac{D}{Dt}$ denotes the material derivative defined as

$$\frac{D(*)}{Dt} = \frac{\partial}{\partial t}(*) + (\mathbf{V} \cdot \nabla)(*).$$

$\mathbf{S}$ is the extra stress tensor which for pseudoplastic fluid model is defined as [13]:

$$\mathbf{S} + \lambda_1 \overset{\nabla}{\mathbf{S}} + \frac{1}{2}(\lambda_1 - \mu_1)(\mathbf{A_1}\mathbf{S} + \mathbf{S}\mathbf{A_1}) = \eta_0 \mathbf{A}_1, \qquad (2.3)$$

where η_0 is the zero shear viscosity, λ_1 is the relaxation time and μ_1 is the material constant. The first Rivlin-Ericksen tensor $\mathbf{A_1}$ is defined as

$$\mathbf{A}_1 = (\mathrm{grad}\mathbf{V}) + (\mathrm{grad}\mathbf{V})^T. \qquad (2.4)$$

The contravariant convected derivative denoted by super imposed ∇ over $\mathbf{S}$, is defined as

$$\overset{\nabla}{\mathbf{S}} = \frac{D\mathbf{S}}{Dt} - \left\{ (\mathrm{grad}\mathbf{V})^T\mathbf{S} + \mathbf{S}(\mathrm{grad}\mathbf{V}) \right\}, \qquad (2.5)$$

3. FORMULATION OF THE PAINT FILM FLOW DOWN A VERTICAL WALL BY GRAVITY

Consider steady, incompressible, parallel, laminar flow of paint flowing down on an infinite vertical wall. The thickness of the paint film is assumed to be h and gravity acts in the negative z-direction (downward) which means that $g_z = -g$. The paint fall under the influence of gravity and there is no applied pressure driving the flow as shown in figure I. Neglecting the change in the hydrostatic of (atmospheric pressure) the surrounding air.
The boundary conditions for the problem are
(i) There is no-slip at the wall

$$\text{at} \quad x = 0, \quad w = 0. \qquad (3.1)$$

(ii) At the free surface, there is negligible shear

$$\text{at} \quad x = h, \quad S_{xz} = 0. \qquad (3.2)$$

where S_{xz} is the shear stress component of pseudoplastic fluid.
We seek accordingly the velocity profile of the form

$$\mathbf{V} = (0, 0, w(x)), \qquad (3.3)$$

and extra stress tensor as
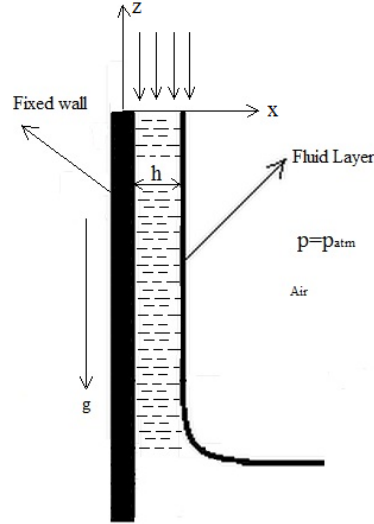
$$\mathbf{S} = \mathbf{S}(x). \qquad (3.4)$$

Fig I. A pseudo plastic film of paint falling by gravity along a vertical wall

By substituting (3.3) in (2.1) and (2.2), the continuity equation (2.1) is identically satisfied and the momentum equation (2.2) reduces to

$$0 = \frac{d\,S_{xx}}{dx} + \rho f_1, \tag{3.5}$$

$$0 = \frac{d\,S_{xz}}{dx} + \rho f_3, \tag{3.6}$$

where f_1 and f_3 are components of body force in x and z directions, respectively.
Since $z-$ axis is in upward direction and gravity acts in the negative z-direction (downward) which means that $g_z = -g$, so the above equations become

$$0 = \frac{d\,S_{xx}}{dx}, \tag{3.7}$$

$$0 = \frac{d\,S_{xz}}{dx} - \rho g, \tag{3.8}$$

since S_{xx} is constant and integrating equation (3.8), we find that

$$S_{xz} = \rho g x + C_1, \tag{3.9}$$

where C_1 is a constant of integration. Applying the free surface condition (3.2) on equation (3.9), we get

$$S_{xz} = \rho g(x - h). \tag{3.10}$$

Using equation (2.4), (2.5), (3.3) and (3.4) in equation (2.3), we obtain the non-zero components of **S** as

$$S_{xx} = -(\lambda_1 - \mu_1)\left(\frac{dw}{dx}\right)S_{zx}, \tag{3.11}$$

$$S_{zz} = (\lambda_1 + \mu_1)\left(\frac{dw}{dx}\right)S_{zx}, \tag{3.12}$$

$$S_{zx} = \frac{\eta_0\left(\dfrac{dw}{dx}\right)}{1 + \left(\lambda_1^2 - \mu_1^2\right)\left(\dfrac{dw}{dx}\right)^2}. \tag{3.13}$$

On implementing one value of S_{zx}, we rewrite equation (3.10) as

$$\frac{\eta_0\left(\dfrac{dw}{dx}\right)}{1 + \left(\lambda_1^2 - \mu_1^2\right)\left(\dfrac{dw}{dx}\right)^2} = \rho g(x - h). \tag{3.14}$$

or

$$\left(\frac{\rho g}{\eta_0}\right)\left(\lambda_1^2 - \mu_1^2\right)(h - x)\left(\frac{dw}{dx}\right)^2 + \left(\frac{dw}{dx}\right) + \frac{\rho g}{\eta_0}(h - x) = 0. \tag{3.15}$$

This is a first order non-linear differential equation which has to be solved with the help of no-slip condition (3.1), that is

$$w = 0, \quad \text{at} \quad x = 0.$$

The above equation (3.15) is a quadratic equation in $\dfrac{dw}{dx}$, on solving this equation, we arrive at

$$\frac{dw}{dx} = \frac{-1 \pm \sqrt{1 - 4\left(\dfrac{\rho g}{\eta_0}\right)^2\left(\lambda_1^2 - \mu_1^2\right)(h - x)^2}}{2\left(\dfrac{\rho g}{\eta_0}\right)(h - x)\left(\lambda_1^2 - \mu_1^2\right)}. \tag{3.16}$$

On defining $m = \left(2\dfrac{\rho g n}{\eta_0}\right)^2$, where $n^2 = \left(\lambda_1^2 - \mu_1^2\right)$ or $n\sqrt{m} = 2\dfrac{\rho g}{\eta_0}n^2$. Now considering the positive real part of equation (3.16), we have

$$\frac{dw}{dx} = \frac{\sqrt{1 - m(h - x)^2} - 1}{n\sqrt{m}(h - x)}. \tag{3.17}$$

On integrating equation (3.17), we get

$$w(x) = \frac{1}{n\sqrt{m}}\int \frac{\sqrt{1 - m(h - x)^2} - 1}{(h - x)}dx. \tag{3.18}$$

The solution of the above integration yields the general form as

$$w(x) = \frac{1}{n\sqrt{m}} \left[\sqrt{1 - m(h-x)^2} - \ln\left(\sqrt{1 - m(h-x)^2} + 1\right) \right] + C_2. \qquad (3.19)$$

where C_2 is a constant of integration and the above solution is the general form of equation (3.18). To find the integration constant C_2 we apply the boundary condition (3.1), the velocity profile $w(x)$ takes the form

$$w(x) = \frac{1}{n\sqrt{m}} \left[\sqrt{1 - m(h-x)^2} - \sqrt{1 - mh^2} + \frac{\ln\left(\sqrt{1 - mh^2} + 1\right)}{\ln\left(\sqrt{1 - m(h-x)^2} + 1\right)} \right], \qquad (3.20)$$

substituting the values of m and n, we arrive at the following form

$$w(x) = \frac{\eta_0}{2\rho g \left(\lambda_1^2 - \mu_1^2\right)} \left[\sqrt{1 - \frac{(2\rho g)^2 \left(\lambda_1^2 - \mu_1^2\right)(h-x)^2}{\eta_0^2}} - \sqrt{1 - \frac{(2\rho g)^2 \left(\lambda_1^2 - \mu_1^2\right)}{\eta_0^2}h^2} \right.$$
$$\left. + \frac{\ln\left(\sqrt{1 - \frac{(2\rho g)^2 \left(\lambda_1^2 - \mu_1^2\right)}{\eta_0^2}h^2} + 1\right)}{\ln\left(\sqrt{1 - \frac{(2\rho g)^2 \left(\lambda_1^2 - \mu_1^2\right)}{\eta_0^2}(h-x)^2} + 1\right)} \right]. \qquad (3.21)$$

which is the velocity of the paint flow down on an infinite vertical wall. Here it should be noted that for $\lambda_1 = \mu_1 = 0$, we get solution for Newtonian fluid. Also for $\mu_1 = 0$, the co-rotational Maxwell fluid [6, 7] recorded as shown below.

$$w(x) = \frac{\eta_0}{2\rho g \lambda_1^2} \left[\sqrt{1 - \frac{(2\rho g)^2 \lambda_1^2 (h-x)^2}{\eta_0^2}} - \sqrt{1 - \frac{(2\rho g)^2 \lambda_1^2}{\eta_0^2}h^2} + \frac{\ln\left(\sqrt{1 - \frac{(2\rho g)^2 \lambda_1^2}{\eta_0^2}h^2} + 1\right)}{\ln\left(\sqrt{1 - \frac{(2\rho g)^2 \lambda_1^2}{\eta_0^2}(h-x)^2} + 1\right)} \right].$$

3.1. **Normal Stress Difference.** The expressions (3.11) to (3.13) show that the normal stress difference is given by

$$S_{xx} - S_{yy} = -\frac{2\lambda_1 \eta_0 \left(\frac{dv}{dx}\right)^2}{1 + \left(\lambda_1^2 - \mu_1^2\right)\left(\frac{dv}{dx}\right)^2} = -2\lambda_1 \left(\frac{dv}{dx}\right) S_{xy} \qquad (3.22)$$

which, with the help of (3.10) and (3.21), yields

$$S_{xx} - S_{yy} = \frac{\lambda_1 \eta_0}{\left(\lambda_1^2 - \mu_1^2\right)} \left[1 - \sqrt{1 - \left(\frac{2\rho g}{\eta_0}\right)^2 \left(\lambda_1^2 - \mu_1^2\right)(\delta - x)^2} \right]. \qquad (3.23)$$

The expression (3.23) reveals that the normal stress difference vanishes at the free surface $(x = \delta)$.

3.2. Force to Hold the wall in Position.

The force F per unit width to hold the wall in position can also be determined using the expression for shear stress at the belt surface. In fact, we have the formula

$$\frac{F}{W} = \int_0^H (S_{xy})_{x=0}\, dx \tag{3.24}$$

where H is the length of the belt. Using (3.10) and (3.24), we obtain

$$\frac{F}{W} = -S_t H. \tag{3.25}$$

Equation (3.25) can also be used to determine the length H of the belt, once the force per unit width is known.

3.3. Flow rate and average velocity of paint film flow.

The flow rate per unit width is given by the formula

$$Q = \int_0^h w(x)dx. \tag{3.26}$$

Substituting (3.21) in (3.26) and then integrating, we obtain the flow rate of pseudoplastic fluid as

$$
\begin{aligned}
Q &= h + \frac{\left(\sqrt{2}-1\right)\eta_0}{\sqrt{2\rho g\left(\lambda_1^2 - \mu_1^2\right)}}\mathrm{Arcsin}\left[\left(\sqrt{\frac{2\rho g\left(\lambda_1^2 - \mu_1^2\right)}{\eta_0}}\right)h\right] - \frac{3h}{4}\sqrt{1 - \frac{2\rho g\left(\lambda_1^2 - \mu_1^2\right)h^2}{\eta_0}} \\
&+ (h-1)\log\left[1 + \sqrt{1 - \frac{2\rho g\left(\lambda_1^2 - \mu_1^2\right)h^2}{\eta_0}}\right].
\end{aligned}
\tag{3.27}
$$

The average velocity $\overline{W}$ is given by

$$
\begin{aligned}
\overline{W} &= \frac{Q}{h}, \\
&= 1 + \frac{\left(\sqrt{2}-1\right)\eta_0}{h\sqrt{2\rho g\left(\lambda_1^2 - \mu_1^2\right)}}\mathrm{Arcsin}\left[\left(\sqrt{\frac{2\rho g\left(\lambda_1^2 - \mu_1^2\right)}{\eta_0}}\right)h\right] - \frac{3}{4}\sqrt{1 - \frac{2\rho g\left(\lambda_1^2 - \mu_1^2\right)h^2}{\eta_0}} \\
&+ \left(1 - \frac{1}{h}\right)\log\left[1 + \sqrt{1 - \frac{2\rho g\left(\lambda_1^2 - \mu_1^2\right)h^2}{\eta_0}}\right].
\end{aligned}
\tag{3.28}
$$

3.4. **Shear stress on the wall.** The expression for the shear stress $S_{xz}|_{x=0}$ on the wall is obtained by using the equation (3.13) as:

$$S_{xz}|_{x=0} = -\frac{h\eta_0^2\left(1 + \sqrt{\dfrac{\eta_0 + 2gh^2\rho\left(\mu_1^2 - \lambda^2\right)}{\eta_0}}\right)}{2gh^2\rho\left(\mu_1^2 - \lambda^2\right) + \eta_0\left(2 + h^2\lambda^2 - h^2\mu_1^2 + 2\sqrt{\dfrac{\eta_0 + 2gh^2\rho\left(\mu_1^2 - \lambda^2\right)}{\eta_0}}\right)} \quad (3.29)$$

3.5. **Vorticity.** The vorticity function $\tilde{w}$ (z-component of vorticity vector) of the flow is given by

$$\tilde{w} = \frac{dw}{dx} = \frac{-1 + \sqrt{1 - \dfrac{4\rho^2 g^2}{\eta_0^2}(h-x)^2\left(\lambda_1^2 - \mu_1^2\right)}}{2\left(\dfrac{\rho g}{\eta_0}\right)\left(\lambda_1^2 - \mu_1^2\right)(h-x)}. \quad (3.30)$$

We can recover the expression for vorticity of a Newtonian fluid upon taking the limits $\lambda_1 \to 0$, $\mu_1 \to 0$ of equation (3.30).

4. RESULTS AND DISCUSSION

Here, we consider steady, incompressible, parallel and laminar flow of paint flowing down on an infinite vertical wall. The thickness of the paint film is assumed to be h and gravity acts in the negative z-direction (downward). The paint fall under the influence of gravity and there is no applied pressure driving the flow. Figures 2 to 6 show the behavior of the velocity profile for the pseudo plastic paint film fluid given in expression (3.21). The graphs in these figures are plotted to show the effect of variation of the non-Newtonian parameters on the velocity of the fluid. All these graphs are plotted by considering the parameters δ, U_0, g, ρ, λ_1, μ_1 and η_0. The pseudo plastic fluid velocity in expression (3.21) is valid only if the condition $0 < \left(\dfrac{\rho g \delta}{\eta_0}\right)^2\left(\lambda_1^2 - \mu_1^2\right) < \dfrac{1}{4}$ holds. Therefore, figures 3 and 4 show graphs that are plotted when $0 < \left(\dfrac{\rho g \delta}{\eta_0}\right)^2\left(\lambda_1^2 - \mu_1^2\right) < \dfrac{1}{4}$. Since the value of $\left(\dfrac{\rho g \delta}{\eta_0}\right)^2$ is always positive, the two cases are of $\lambda_1 > \mu_1$ and $\lambda_1 < \mu_1$. Figure 3 shows the effect of increasing the values of μ_1 on the velocity profile (3.21) when λ_1 is fixed. It is obvious from the figure that the downward velocity decreases with increasing the value of μ_1 and the fluid with lesser μ_1 moves more freely under the action of gravity. But figure 4 reveals that when the value of μ_1 is fixed and that of λ_1 is varied, the fluid with greater λ_1 moves downward more freely and increasingly as compared to the one with a higher value of μ_1. In figure 5, the value of η_0 is varying and the effect of varying the value of η_0 on fluid velocity is observed by fixing other parameters. Figure 6 shows the behavior of velocity when the value of h is increased and that of λ_1, μ_1 and other parameters are kept fixed.
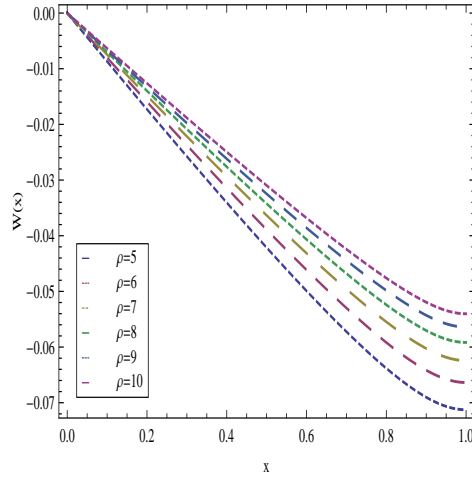
FIGURE 1. The effect of constant density ρ keeping $g = 9; h = 0.1; \eta_0 = 0.2; \mu_1 = 0.6; \lambda_1 = 0.4$ on velocity profile for drainage problem.
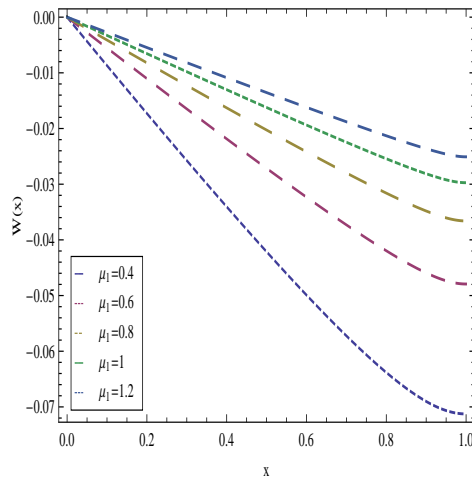

FIGURE 2. The effect of material constant μ_1, keeping $\rho = 5; g = 9; h = 1; \eta_0 = 0.1; \lambda_1 = 2$ on velocity profile for drainage problem.
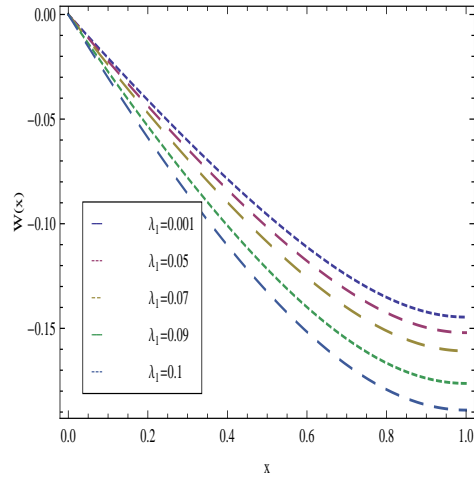
FIGURE 3. The effect on velocity profile of λ_1 keeping $\rho = 5; g = 9; h = 1; \eta_0 = 0.1; \mu_1 = 0.012$ for drainage problem.
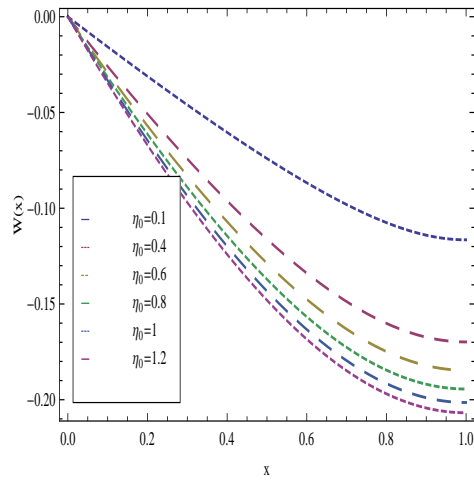

FIGURE 4. The effect on velocity profile of η_0 keeping $\rho = 5; g = 9; h = 1; \mu = 0.2; \lambda_1 = 0.1$ for drainage problem.
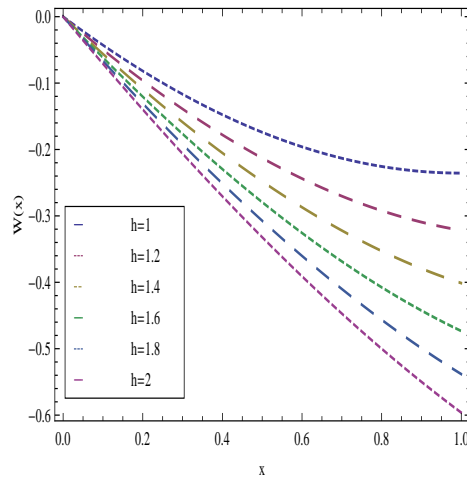
FIGURE 5. The effect on velocity profile of η_0 keeping $\rho = 5; g = 9; h = 1; \mu = 0.2; \lambda_1 = 0.1$ for drainage problem.

5. CONCLUSION

Thin film flow of pseudo plastic fluid down a vertically wall is analyzed. Exact analytical solution for fluid velocity is obtained. The condition under which this solution is valid is also stated. Expressions for some other useful quantities related to mechanics of the problem are derived. Graphs show the trend of the velocity profile with reference to the coordinate axes. Effects of non-Newtonian parameters on velocity profile are also presented through graphs.

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