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High School Student-Teachers Attempts to Justify Mathematical Propositions Utilizing Spatial Structuring on Shape Transform[1]

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A group of twenty-nine high school student-teachers were given a set of mathematical propositions focusing on shape-to-shape transformations. Their task was to determine through hands-on manipulation and use of dynamic software that each shape be transformed into an area equivalent rectangular region. This paper reports on a classroom-based research..

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INTRODUCTION AND BACKGROUND

Radical Constructivism

Ernest von Glasersfeld (1995) presented his Radical Constructivism Theory known as "*Radical Constructivism: A Way of Knowing & Learning*", by stating that when individu-

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als deal with the physical world, their minds construct, through certain mental mechanisms and collections of cognitive structures, their conceptualization, reason, and coordination of their engagements (von Glasersfeld, 1995; 1984; 1974). Within the von Glasersfeld's (1995) list of mental mechanisms indicated above, Glasersfeld has included the notions of ***abstraction and reflective abstraction*** mechanisms as among the most fundamental mental mechanisms being utilized when individuals deal with the physical world around them (1995, p. 69).

Battista's Abstraction Levels

A few years later, Battista (1999, p. 418) has added an interesting elaboration describing the von Glasersfeld's notion of *abstraction* as the process through which the mind selects, coordinates, unifies, and registers in memory a collection of mental acts that appear in the attention field. Further, Battista (1999) has elaborated on von Glasersfeld's ideas of *abstraction* stating that abstraction essentially has three levels:

Perceptual Level – This level represents abstraction at its *perceptual level* (most basic), it isolates an item in the stream of an experience and seizes it as a unit;

Internalized Level – Battista added that material or entity is said to have reached the *internalized level* whenever it has been sufficiently abstracted (underline is added) so that it can be re-presented (re-created) in the absence of its perceptual input;

Interiorized Level – In this level, material or entity is said to have reached *interiorized level* whenever it has been disembodied (underline is added) from its original perceptual context and it can be freely operated on in imagination, including being "projected" into other perceptual material and utilized in novel situations (Battista, 1999, p. 418).

Earlier, Steffe & Cobb (1988, p. 337) asserted that *interiorization* is:

> "the most general form of abstraction; it leads to the isolation of structure (form), pattern (coordination), and operations (actions) from experiential things and activities."

On the notion of understanding, von Glasersfeld (1995) elaborated further stating that *understanding* requires more than abstraction; it requires *reflection* which is the conscious process of re-presenting experiences, actions, or mental processes and considering their results or how they are composed. *Reflective abstraction* takes mental operations performed on previously abstracted items as elements and coordinates them into new forms or structures that, in turn, can become the content (what is acted upon) in future acts of abstraction (von Glasersfeld, 1995, p. 69).

Battista's Levels of Students' Reasoning

Battista (1999), in reporting his 3D cube arrays' study, has come with another interesting elaboration for von Glasersfeld's notion of abstraction and reflective abstraction; he suggested that besides von Glasersfeld's (1995) list of mental mechanisms that includes abstraction and reflection mechanisms there are three additional mechanisms that are fundamental to understanding students' reasoning. They are (1) *spatial structuring,* (2) *mental models, and* (3) *schemes* (Battista, 1999; Battista & Clements 1996).

Spatial structuring – It is the mental act of constructing an organization or form for an object or set of objects. It determines an object's nature or shape by identifying its spatial components, combining components into spatial composites, and establishing interrelationships between and among components and composites.

Mental models – Are nonverbal recall-of-experience-like mental versions of situations; they have structures isomorphic to the perceived structures of situations they represent (Battista, 1994; 1999, p. 418). Mental models consist of integrated sets of abstractions that are activated to interpret and reason about situations that one is dealing with in action or thought.

Schemes – A *scheme* is an organized sequence of actions or operations that has been abstracted from experience and can be applied in response to similar circumstances. It consists of a mechanism for recognizing a situation, a mental model that is activated to interpret actions within the situation, and a set of expectations (usually embedded in the behaviour of the model) about the possible results of those actions (Battista, 1999).

Meaningful Teaching and Training

Meaningful learning occurs as students make adoptions to their current cognitive structures as a result of their reflection on an experience (Steffe, 1988; Battista, 1999). An accommodation is triggered by a *perturbation* which is described as a disturbance in mental equilibrium caused by an unexpected result or a realization that something is missed or does not work (von Glasersfeld, 1995, p. 67). Perturbation arises when students interact with other individuals or with the physical world (Battista, 1999).

von Glasersfeld (1995, p. xvi) made a clear distinction between teaching and training stating that,

> "From an educator point of view one of the most important features of radical constructivism is the sharp distinction it draws between teaching and training. The first aims at generating understanding, the second at competent performance."

Further, von Glasersfeld (1995, p. xvi) in referring to learning mathematics stated that

> "To know mathematics is to know how and why one operates in specific ways and not in others, how and why the results one obtains are derived from the operations one carries out."

DISSECTION-MOTION-OPERATIONS (DMO) FRAMEWORK

Rahim & Siddo (2010), in classroom research, have elaborated on classroom attempts of high school student-teachers to justify mathematical propositions and solving problems contained in certain shape transforms tasks utilizing spatial structuring. Battista (1999) has described spatial structuring as Decomposites-Composites-Operations (DCO) when dealing with shape-to-shape transformation. The Battista's notion of Decomposites-Composites-Operations echoes Rahim & Sawada's (1986) notion of Dissection-Motion-Operations (DMO). The spatial structuring in Rahim & Siddo's (2010) report was virtually based on the Decomposites-Composites-Operation's (Battista, 1999) concept, abbreviated as DCO, and is briefly described below.

There are three components that constitute a DCO (or DMO) operation:

Dissection – See for example Eves, 1972, pp. 194–239. Specifically, a polygonal region is dissected into a finite number of smaller certain sub-regions.
Composition (or Motion) –Through which one or more of the sub-regions are moved to another location without overlapping forming a new shape; and
Recursion – Through which the two above operations or one of them may be repeated.

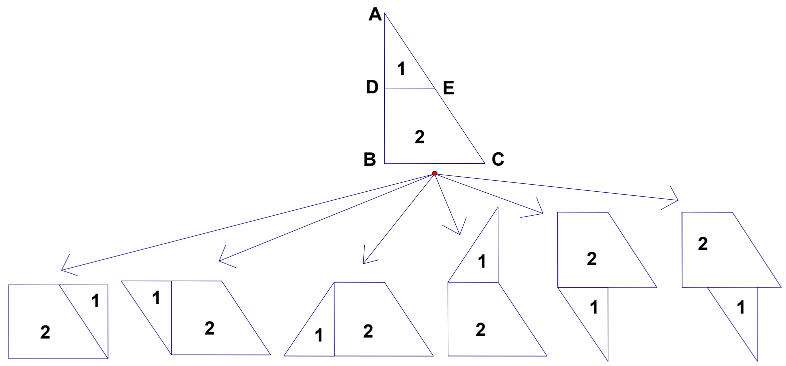

Figure 1. DCO is a relation

Together, these three components constitute the operation called Dissection-Motion-Operations (DMO), or in Battista's terms, Decomposites-Composites-Operations (DCO). Under such operation, a polygonal region can be transformed into another polygonal region of equivalent area. Note that DCO (or DMO for that matter) is a *relation,* from the space P, of all polygonal regions into P. DCO may take a single polygonal region to several distinct polygonal regions (see Figure 1), all the while, the area remains unchanged throughout.

Technologically, through the use of Geometer's Sketchpad or Cabri, problems focused on interrelationships among shapes or objects, among shapes and their parts or among the parts themselves, would be suitable. Clearly, such activities will not be difficult for students to deal with nor will be out of the reach for teachers given an appropriate training on these tools.

METHODOLOGY

This article reports on a classroom research work focusing on Decomposites-Composites-Operations applied on polygonal regions by the high school student-teachers. The students were supplied with a set of geometrical tools. The student were already introduced and be acquainted with the Geometer's Sketchpad software through a series of computer lab. Three tasks were used in the classroom research project. Below is a brief description for each task.

Description of the Tasks

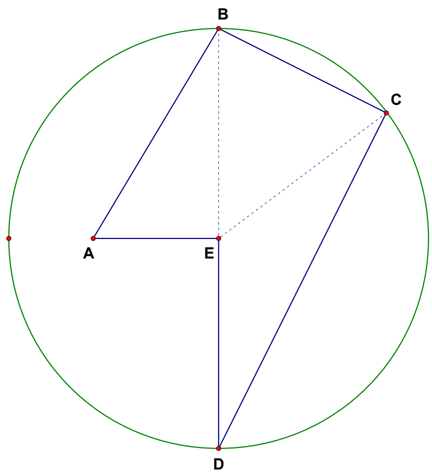

Figure 2. Non-convex non-regular pentagonal region ABCDE

The focus in each of these tasks was to transform a given polygonal region into an area equivalent rectangular region applying decomposites-composites-operations. Each student-teacher was required to provide a step-by-step explanation for their actions on each task. Specifically, in addition of the use of geometric tools, they were required to provide, in step-by-step fashion, appropriate Geometer's Sketchpad commands.

Task 1: **A Non-Convex and Non-Regular Pentagon Region**

Given the shape (Figure 2), use your geometric tools as an aid to indicate and justify:

(a) Your decomposition of the given shape into smaller pieces of your choice;
(b) Your composition steps for rearranging your pieces into a rectangular region of equal area.

Task 2: **A Convex Regular Pentagon Region**

Given the shape (Figure 3), use your geometric tools as an aid to indicate and justify:

(a) Your decomposition of the given shape into smaller pieces of your choice;
(b) Your composition steps for rearranging your pieces into a rectangular region of equal area.

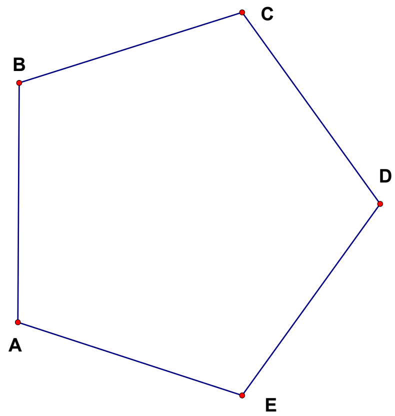

Figure 3. A convex regular pentagon region ABCDE

Task 3: **A Convex Non-Regular Quadrilateral Region**

Given the shape (Figure 4), use your geometric tools as an aid to indicate and justify:

(a) Your decomposition of the given shape into smaller pieces of your choice;
(b) Your composition steps for rearranging your pieces into a rectangular region of equal area.

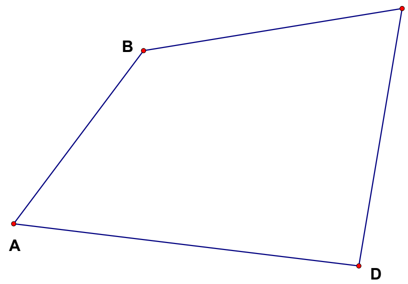

Figure 4. A convex non-regular quadrilateral region ABCD

RESULTS

In this classroom research, a series of student-teachers performances on the three tasks described above has been closely examined. Descriptions of a group of students' performances on each task are provided below. Throughout these performances, it is evidenced and clearly apparent that the Battista's (1999) concept of *special structuring was present*. Furthermore, based on these students' detailed performances reported below, it is safe to state that von Glasersfeld's (1995) Radical Constructivism Theory known as "*Radical Constructivism: A Way of Knowing & Learning"* was present throughout.

Task 1

Some interesting performances on Task 1 throughout the student-teachers responses are reported below.

(I) A student identified as 'L' has made the following response. Student 'L' has, in a step-by-step fashion, has transformed the non-convex non-regular pentagon region ABCDE into a rectangular region of equal area successfully (see Figures 5a & 5b); in his words:

(1) Take triangle ABC; construct midpoint on AE and AB to be F & G.

(2) Connect FG & mark G as a center of center and rotate triangle AFG by 180° and we get rectangle FF'DE.

(3) Take triangle BCE, construct midpoint H & I on BE and BC.

(4) Connect HI and construct a perpendicular line BJ to HI from B.

(5) Mark I as center and rotate triangle BJI by 180°.

(6) Mark H as center and rotate triangle BJH by 180°.

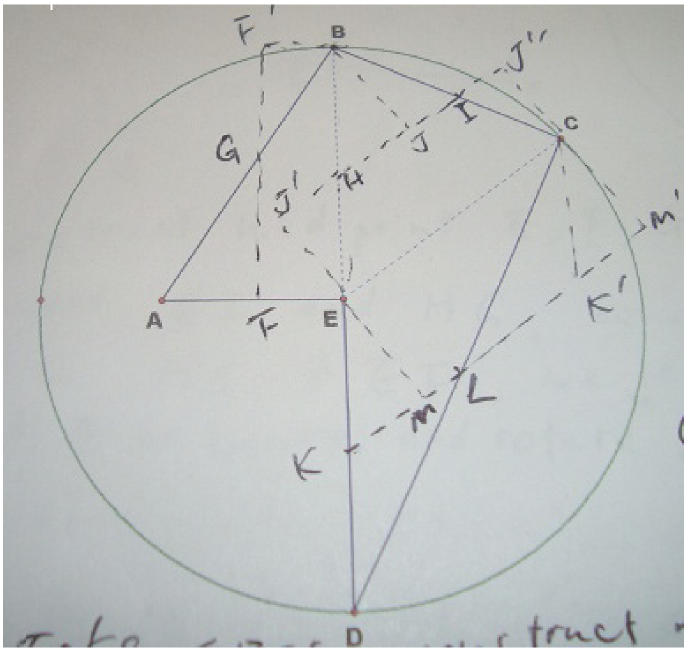

Figure 5a. Decomposites-composites-operations of student ‘L’ for the non-convex non-regular pentagon region ABCDE

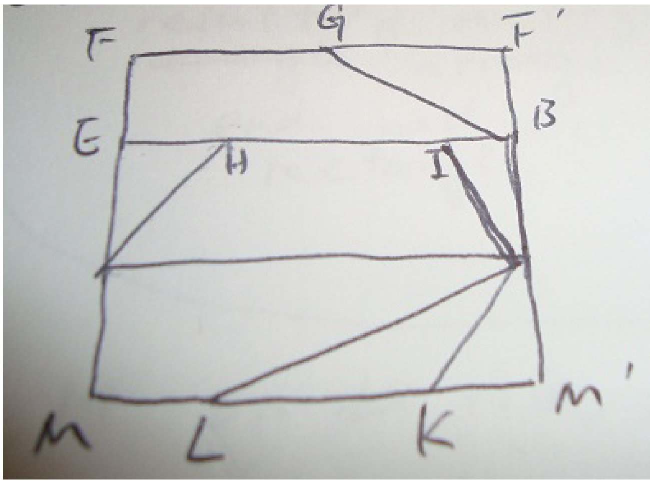

Figure 5b. Composition of student ‘L’ of a rectangular region FF′M′M

(7) We get rectangle J′J″CE.

(8) Take triangle DEC construct midpoint K & L on ED and CD.

(9) Connect KL and construct perpendicular line from E to KL intersection at M.

(10) Mark EC as a vector & translate triangle EKM to triangle CK′M′.

(11) Mark Las center and rotate triangle DLK by 180° to triangle CLK', we get rectangle EMM'C.

(13) Mark angle BHI as the angle and rotate rectangle FF′BE by angle BHI and translate vertex B to J″.

(14) We get the big rectangle FF'M'M.

(II) Another student identified by 'I' has made the following response. 'I' has, in a step-by step manner, has successfully transformed the pentagon region ABCDE into a rectangular region of equal area shown in Figure 6. In her own words:

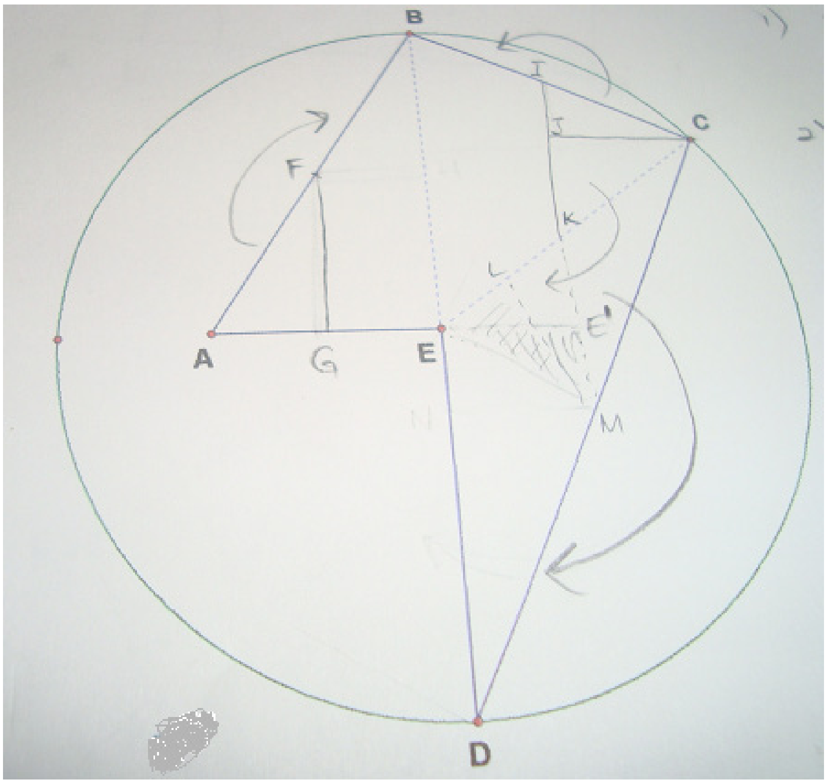

Figure 6. Composition of student 'I' for the pentagonal region ABCDE

(1) Split the shape into three triangles: ΔABE, ΔBCE, ΔCED.

(2) Create midpoint of AB, call it F. Create perpendicular FG to AE. Rotate triangle AFG about point F so that A aligns with B.

(3) Create midpoint I and K of BC and CE respectively. Create IK and line perpendicular to C (CJ). Rotate ΔCIJ about point I so that point C meets B. Rotate ΔCJK about point K so that point C meets E.

(4) Create midpoint of CD call it M. Create EM. Create perpendicular line ML to EC. Rotate ΔELM about point E so that EM is horizontal and at the top of ΔMLE′. Translate to the right. Rotate ΔCLM about M so C align with D. Translate rectangle EDLE′ next to the other two rectangles.

(III) ‘M’, another student, when dealing with the transformation of the shape ABCDE, has a good start converting the right triangular piece into a rectangular sub-region yet ended up with an inappropriate choice of dissecting the shape as a whole – he seemed missing to maintain a *common side* among the three resulting rectangular sub-regions and hence his attempts resulted into an incomplete transformation.

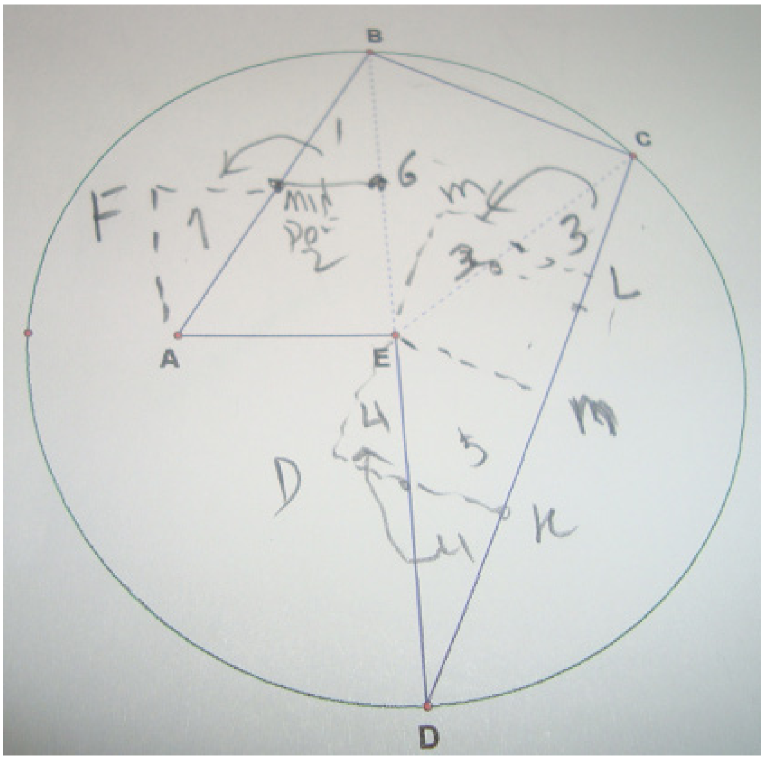

Figure 7. Incomplete shape transformation of student ‘M’ for the pentagon region ABCDE

However, when dealing with the three sub-regions representing right, acute and obtuse triangular shapes, 'M' was unsuccessful to convert each of them into an appropriate corresponding area equivalent rectangular sub-region where a *common* side among the three resulting shapes has to be maintained. He continued incorrectly applying the same strategy used on the right triangular sub-region on the other parts with no awareness of the common side property as students 'L' and 'I' had maintained (see Figures 5a, 5b, and 6). Student 'M' has dissected the obtuse triangular region into two right triangular regions and then applied his initial strategy on each part – a sign of inconsistency. 'M' left the acute triangular piece as is (see Figure 7).

Clearly, as Figure 7 shows, student 'M' was not having an obvious visualization of the shape that he was supposed to transform ABCDE into. He was stuck with the idea of converting a right angle triangle into a rectangle.

Task 2

A set of performances on Task 2 are reported below.

(IV) A student identified as 'A', has offered an interesting response as shown in Figures 8a and 8b. He proceeded as follows:

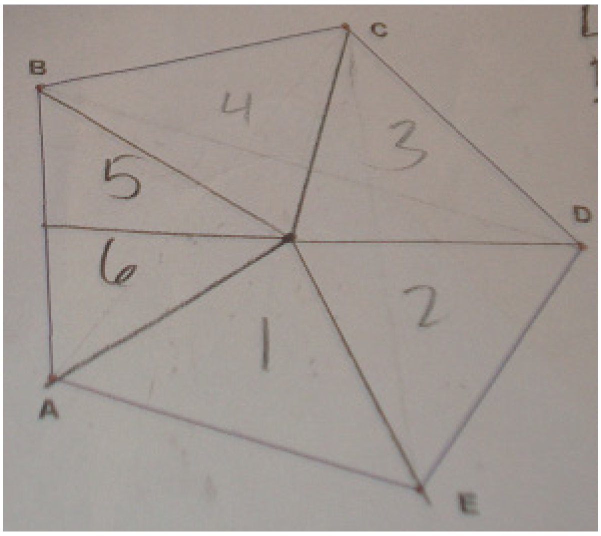

Figure 8a. Decomposition operations of student 'A' for the regular pentagon region ABCDE

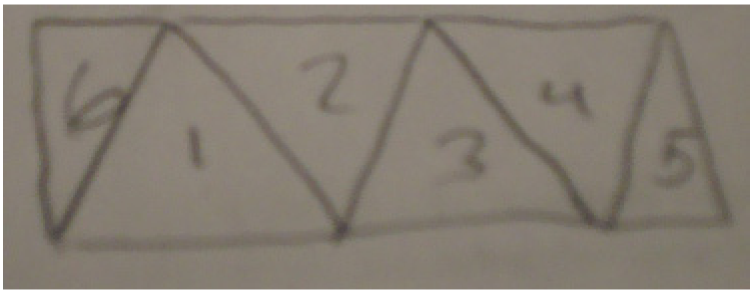

Figure 8b. Composition of the sub-regions by student 'A' into an area equivalent rectangular region

(1) Divided the regular pentagon region ABCDE into five congruent triangular regions (see Figure 8a);

(2) Arranging four of the resultant triangles (Pieces 1, 2, 3, and 4) into a parallelogram region (see Figure 8b);

(3) Dissecting the fifth triangular sub-region into two halves (right angle triangles, Pieces 5 and 6); and

(4) Attaching a half at each side of the parallelogram region making a rectangular region whose area is equivalent to the given original pentagonal region ABCDE (see Figure 8b).

Note that student 'A' was conceptually correct in making the two halves in the last step; however, he was inaccurate in producing a final visual representation for the rectangle as his final product – Piece 5 shown in Figure 8b has to be identical to Piece 6.

(V) Student labeled, student 'Y', has come with a different method in converting the given pentagon region ABCDE into a rectangular region. The following is a summary of his steps:

(1) Dividing the pentagon region ABCDE into five congruent triangles,

(2) Dividing each triangle into two halves each of which is a right triangle and,

(3) Arranging the ten halves into a rectangular shape as shown in Figures 9a & 9b.

Through a close examination of Figure 9b, it is clear that student 'Y' has a counting error of the *number* of the sub-regions he used.

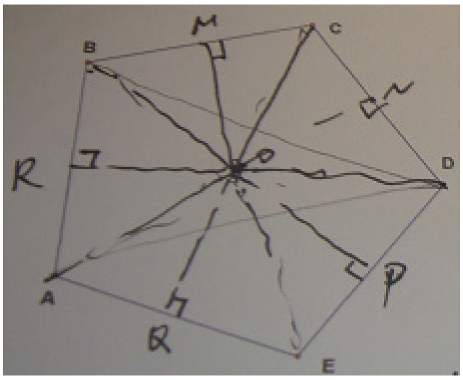

Figure 9a. Decomposition of student 'Y' for the regular pentagon region ABCDE into right triangular sub-regions

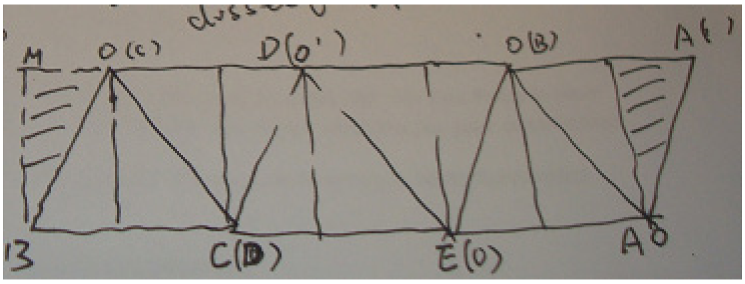

Figure 9b. Composition by student 'Y' for the sub-regions of the pentagon region ABCDE into a rectangular region of equal area

Task 3

Some students' performances on task 3 are described below.

(VI) Student, 'G', has shown the shape transform of the quadrilateral ABCD into an area equivalent rectangle by:

(1) Decomposing the shape ABCD by the diagonals BD (Figure 10) into two triangular regions ABD and CBD;

(2) Maintaining the diagonal BD as a common base and transforming each resultant triangle into a rectangle with BD being one of its sides.

He then considered the result as the required result as shown in Figure 10 below. However, student 'G' has not explain why triangle ABD is a right triangle at B?

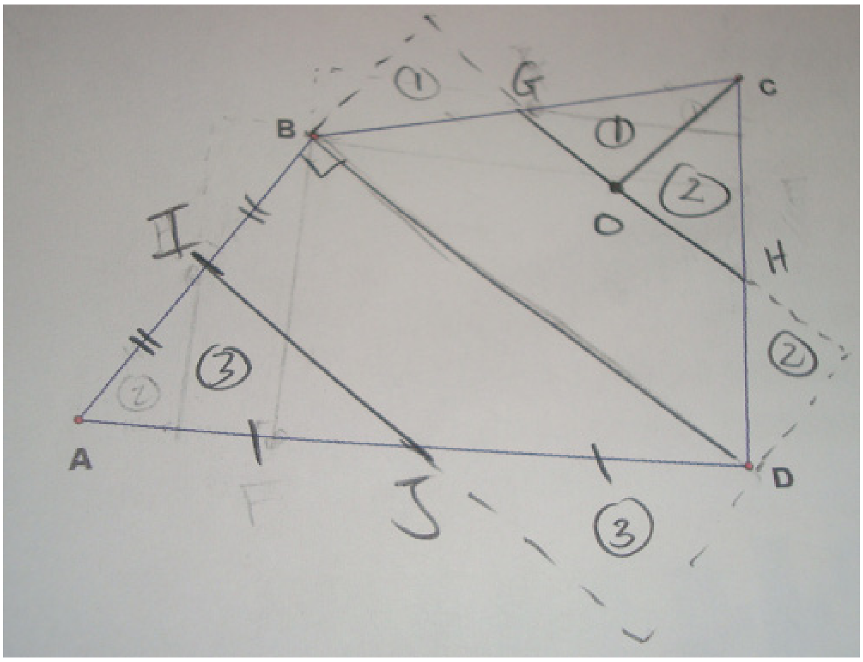

Figure 10. Decomposites-composites-operations of student 'G' for the convex non-regular quadrilateral ABCD

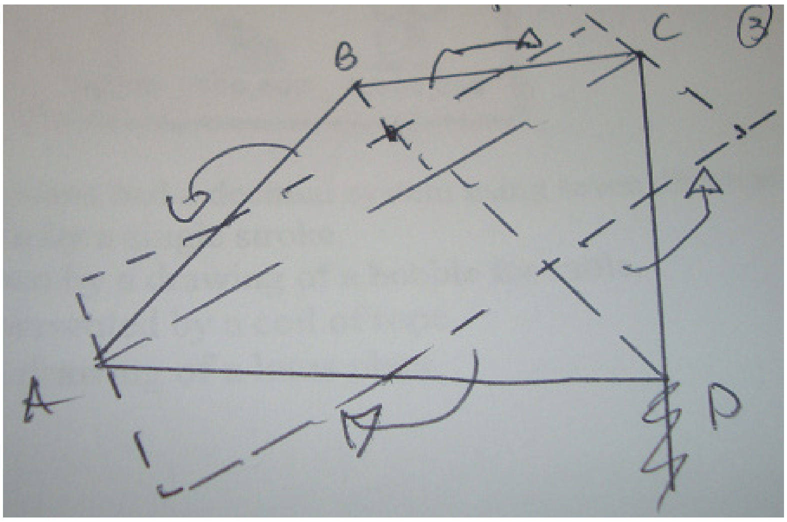

Figure 11. 'D' horizontal decomposites-composites-operations of the convex non-regular quadrilateral ABCD

(VII) A student ‘D’, has shown the shape transform in a way similar to the student ‘G’ method but selecting the long diagonal in dissecting the shape into two triangular regions. Then he transformed each resultant triangle into a rectangle with the common base being the long diagonal AC of the shape ABCD and resultant is a rectangle of area equivalent to the quadrilateral ABCD as shown in Figure 11.

CONCLUDING REMARKS

Throughout this research report, the actions the student-teachers have exhibited can be considered as evidences of what von Glasersfeld and Battista have provided us. The notion of *mental mechanisms* that includes *abstraction and reflective abstraction,* as von Glasersfeld’s theory of Radical Constructivism indicates, was present throughout the students’ actions reported in this study. Further, as Battista stated, the students’ actions in this study show that there are three additional mechanisms that are fundamental in the students’ reasoning; they are

(1) *spatial structuring,* (2) *mental models, and* (3) *schemes.*

In particular, spatial structuring has been repeatedly utilized by the student-teachers action on the three tasks. We recall that spatial structuring is the mental act of constructing an organization or form for an object or set of objects; it determines an object’s nature or shape by identifying its spatial components, combining components into spatial composites, and establishing interrelationships between and among components and composites. That what has been happing through the students’ performances reported.

Earlier, Rahim & Sawada (1986; 1990) have introduced “Dissection-Motion-Operations” approach explicitly

(1) as a way to explore properties of geometric shapes through shape-to-shape, shape-to-part and part-to-part interrelationships; and

(2) to deal with the dilemma of focusing too early on quantitative knowing at the expense of qualitative understanding (Piaget, 1962; Wirszupe, 1976).

The Decomposies-Composites-Operations’ concept (in Battista’s terms, 1999) or Dissection-Motion-Operations’ idea (in Rahim & Sawada’s terms, 1986) have been evidently essential for the student-teachers to approach the tasks in this study.

Finally, the observations reported throughout this work on the use of Decomposites-Composites-Operations through hands-on manipulation with the aid of using geometric tools and the aid of technology, are essential for consideration. We recommend that student-teachers and school teachers would consider using the hands-on manipulation environment through Decomposites-Composites-Operations in the classrooms.

REFERENCES

Battista, M. T. (1994). On Greeno's environmental/model view of conceptual domain: A spatial/geometric perspective. *J. Res. Math. Educ*. **25(1)**, 86–94. ME **1995c.**01406

_____ (1999). Fifth graders' enumeration of cubes in 3D arrays: Conceptual progress in an inquiry-based classroom. *J. Res. Math. Educ*. **30(4)**, 417–448. ME **2000a.**00410

Battista, M. T., & Clements, D. H. (1996). Student's understanding of three-dimensional rectangular arrays of cubes. *J. Res. Math. Educ*. **27(3)**, 258–292. ME **1997a.**00114.

Eves, H. (1972). *A Survey of geometry: Revised Edition*. Boston, MA: Allyn and Bacon, Inc.

Hoffmann, D. (1998). *Visual intelligence: How we create what we see?* New York: Norton.

Kosslyn, S. (1996). *Image and Brain*. Boston, MA: MIT Press.

National Council of Teachers of Mathematics (NCTM) (2006). *Curriculum focal points for prekindergarten through grade 8 mathematics.* Reston, VA: Author. ERIC ED495271

_____ (2008). *Focus in high school mathematics: Reasoning and sense making.* Reston, VA: Author. ERIC ED507511

O'Daffer, P. G. & Clemens, S. R. (1992). *Geometry: An Investigative Approach, Second Edition.* New York: Addison-Wesley Publishing Company.

Piaget, J. (1962). Comments on Vygotsky's critical remarks concerning the thought and language of the child. Boston, MA: MIT Press.

Rahim, M. H. (1986). Laboratory investigations in geometry: a piece-wise congruence approach. *Int. J. Math. Educ. Sci. Technol.* **17(4)**, 425–447. ME **1986i.**02532

Rahim, M. H. & Sawada, D. (1986). Revitalizing school geometry through Dissection Motion Operations. *Sch. Sci. Math*. **8(3)**, 235–246. ME **1986x.**00472 ERIC EJ334993

_____ (1990). The duality of qualitative and the quantitative knowing in school geometry. *Int. J. Math. Educ. Sci. Technol*. **21(2)**, 303–308. ME **1990l.**02060

Rahim, M. H. & Siddo, R. A. (2010). Students approaches in constructing convincing arguments in geometry using technology: A case study. *J. Korea Soc. Math. Educ. Ser. D* 14(3), 219–231.

_____ (2011). High School Student-Teachers Attempts to Justify Mathematical Propositions Utilizing Spatial Structuring on Shape Transform. In: J. Cho, S. Lee & Y. Choe (Eds.), *Proceedings of the 47th Korean National Meeting of Mathematics Education held at Kyungsung University, Nam-gu, Busan 608-736, Korea; November 4–5, 2011* (pp. 277–294).

Steffe, L. P. (1988). Modifications of the counting scheme. In: L. P. Steffe, P. Cobb. & E. von Glasersfeld (Eds.), *Construction of Arithmetical Meanings and Strategies*. New York: Springer-Verlang.

Steffe, L. P.; Cobb, P. & von Glasersfeld E. (Eds.) (1988). Construction of arithmetical meanings and strategies. New York: Springer-Verlang.

von Glasersfeld, E. (1974). Piaget and the radical constructivist epistemology. In: C. D. Smock & E. von Glasersfeld (Eds.), *Epistemology and education* (pp. 1–24). Athens, GA: Follow Through Publications. Avaiable from: http://www.vonglasersfeld.com/034

_____ (1984). An introduction to radical constructivism. In: P. Watzlawick (Ed.), *The invented reality* (pp. 17–40). New York: Norton. [Originally published in German: von Glasersfeld, E. (1981). Einführung in den Radikalen Konstruktivismus.P. Watzlawick (Ed.), *Die erfundene Wirklichkeit*. München: Piper.] Avaiable from: http://www.univie.ac.at/constructivism/EvG/papers/070.1.pdf

_____ (1995). *Radical constructivism: A way of knowing and learning*. Studies in Mathematics Education Series: 6. Washington, DC: Falmer Press. ERIC ED381352 Avaiable from: http://www.eric.ed.gov/PDFS/ED381352.pdf

Wirszup, I. (1976). Breakthroughs in the psychology of learning and teaching geometry. In: J. L. Martin & D. A. Bradbard (Eds.), *Space and geometry: Papers from research workshop* (pp. 75–97). Athens, GA: University of Georgia, Georgia Center for the Study of Learning and Teaching Mathematics. ERIC ED132033 Avaiable from: http://www.eric.ed.gov/PDFS/ED132033.pdf