Journal of the Korea Society of Mathematical Education Series D:
Research in Mathematical Education
Vol. 16, No. 3, September 2012, 195–206

A Study on Cognitive Development of Scientifically Talented Students toward Definition and Theorem in the Course of Multivariable Calculus[1]

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(Received August 26, 2012; Revised September 24, 2012; Accepted September 29, 2012)

We adopt a spirit of Problem based learning to the class of Multivariable Calculus in a school of scientifically talented students and observed effects of our teaching-learning method in the Spring Semester of 2010. Twelve students who enrolled in this class participated in this research. We have proceeded with classroom experiment for the half of semester after midterm exam so that the students could compare our teaching-learning method with usual traditional one in the subject of multivariable calculus. Especially, we investigated changes in the learning attitude and cognitive development of the students toward definition and theorem of mathematics. Each group of 4 students worked on a sheet of our well-designed structured problems of several steps in each class and presented how they understood the way of constructing new definition and related theorems. Instructor's role in this research was to guide students' activities as questioner so that students could attain the clear meanings of definitions and theorems by themselves. We firstly analyzed students' process of mathematization of definition through observing their discussions and presentations as well as their achievements in the quizzes and final exams. Secondly, we analyzed students' class-diaries collected at the end of each class in addition to pre/post surveys.

Keywords: mathematization, learning, cognitive development, small group discussions, definition, theorems, problem based learning
MESC Classification: D45, C35
MSC2010 Classification: 97D40, 97C70, 97C30, 97I60

[1] The part of this article was presented at the 7th MCG International Conference at Korea Science Academy of KAIST, Busan, Korea; July 15–18, 2012 (*cf.* Kim & Choi, 2012).

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1. INTRODUCTION

In Korea, the most mathematics courses of undergraduate level have been given in the way of instructor-centered lecture. For a 3 credit math course, we usually provide two lectures of 75 minutes or three lectures of 50 minutes in a week. Additionally, one hour problem session is given for each course in some universities equipped with a graduate program and in the session, teaching assistants help students to solve problems related with theories and concepts treated in classes by instructors. Traditionally, the most math instructors of universities or colleges have been used to the instructor centered teaching in Korea. This instructor-centered traditional teaching-learning method needs to be changed into learner-centered method involving group discussion and presentation not only in the math class of graduate level but also in the math class of undergraduate level. For the math courses of university level, research on inquiry oriented instruction was started by Kwon (2005; 2007), classroom experiment on modified Moore method by Kim & Kim (2010) as well as research on learning environment at college mathematics education (Kim, Yang & Lee, 2004).

It is natural to suppose that mathematics class for the talented or gifted should adopt the learner-centered teaching-learning method if we intend to get students to be aware of their true potential, character or motives and to enjoy various concepts of mathematics by themselves. They also need strong background in mathematics to develop their talent in science and so we provided them with the opportunities to learn how to think mathematically through the learner-centered teaching-learning. In this research, we adopted a spirit of problem based learning (PBL) method in a school of talented/gifted students in science to the multivariable calculus class of college level, and investigated changes in the learning attitude and cognitive development of the students toward definitions and theorems (briefly, DT) in the multivariable calculus course. Before we started this classroom experiment, we had a question: Can we finish all material of the course within a semester by our teaching-learning method? In fact, we were pretty sure that our students could understand DT in depth in view of structure of knowledge through our method when we reviewed Chalice's research results. Chalice (1995) claimed through his research on classroom experiments by modified Moore method; there was no problem! He could finish all material of the course within a semester. Even he and his students experienced enlivening, enjoyable, and intellectually stimulating teaching-learning in the experiment. Modified Moore method and PBL have Socratic method (Nelson, 1949) as their same origin.

In this research, we expected students to understand the meaning of definitions and theorems and realize ways of construction of definition and theorems, to share their ideas through discussion in a small group, to clarify their ideas through presentations of their understanding, to think creatively by generalization and mathematization under instructor's guidance and questions, and eventually to transfer or apply their achieved ideas (knowledge) to other subjects in mathematics or sciences. Our research goal is to accomplish these through a classroom experiment equipped with the learner-centered teaching-learning method

2. RESEARCH BACKGROUND

We designed a classroom experiment for a multivariable calculus course offered at Korea Science Academy of KAIST (briefly, KSAK) in the Spring Semester of 2010. To enhance the potential of our talented/gifted students in the course of multivariable calculus, we looked for a suitable teaching-learning method. We focused on the spirit of PBL. KSAK is a school of scientifically talented/gifted students. Middle school or higher level students were selected through the entrance exams of 3 steps in 2010 academic year;

1. Evaluation through documents - academic achievements and prize in competition, etc.
2. Written tests in math and subjects of sciences
3. Oral test for a topic after 30 minutes-written test for the topic

Calculus-2 offered in KSAK is a college level one semester course for the first and second year students. One semester consists of 15 weeks. Three lectures of 1 hour are given for Calculus-2 per week. The contents of the course cover second part of multivariable calculus which consists of vector analysis and multiple integrations and their applications. Multivariable Calculus contains the generalized theories for functions of 2–3 variables (up to n variables) which are analogous to those of functions of one variable. Accordingly, it is a good subject providing students with the opportunity to enhance abilities of generalization. Moreover, it drives students to understand a way of construction of definition and theorem and even to construct new definitions and theorems. Students experience ‘(vertical) mathematization’ through this mathematical activity. Freudenthal (1991) said that

> “The essence of mathematics is mathematical activities”.

He clarified that mathematics was not the body of mathematical knowledge, but the activity of solving problems and looking for problems, and, more generally, the activity of organizing matter from reality or mathematical matter - which is called ‘mathematization’ (Freudenthal, 1973, p. 134). In very clear terms, Freudenthal clarified what mathematics is about:

> “There is no mathematics without mathematizing”.

Meanwhile, theorem is a kind of logical connections of defined terms which can be proved to be true. More theorems can be constructed by known theorems. Theorem is a true proposition, true mathematical statement. It consists of 'hypothesis and conclusion'. Let's see an example: If a function *f* is differentiable at a point P, then *f* is continuous at the point P. This theorem is stated through such definitions as 'function', differentiability', 'continuity'.

Whitehead (1929) elaborated rhythmic three cycles such as romance, precision, generalization in the article, and the 'rhythm of education' which is included in the 'aims of education'. The education proceeds through the rhythmic cycles on the basis of his metaphysical philosophy and educational philosophy. The 'rhythm of education' means that the intellectual levels of learners are elevated through the rhythmic cycles of stages of romance, precision, and generalization over and over again. As a result of these cyclic repetitions, the learners become truly free of inner prejudice against knowledge. It is very important for learners to stay and enjoy in the cycle of romance through group discussions when they face with new subjects such as definitions or theorems in mathematics. They need time to be familiar with the subjects in the cycle. Doing so, they can achieve strength to go far with the subjects. But many instructors do not let them stay enough in the cycle of romance. In this research, we provided group discussion and presentation for students to deal with new subjects in the stage. We also expected students could experience a spirit of PBL as well as these rhythmic three cycles.

Characteristics of PBL are originally from the next ideas;

- Learning is driven by challenging, open-ended, ill-defined and ill-structured problems.
- Students generally work in collaborative groups.
- Teachers take on the role as 'facilitators' of learning.

MacDonald and Isaacs (2001, p.317) offer this distinguishing characteristic of problem-based learning:

> "The characteristic that distinguishes PBL from other learning methods centering on what students do, rather than what teaching staff do (student-centered methods) is that the problem comes before the knowledge (in the broadest sense) needed to solve or resolve it".

3. RESEARCH METHOD

3.1. Participants

Two authors who majored in both math and math education participated in this research. One of them teaches scientifically talented/gifted students in KASK, the other teaches mathematics course including multivariable calculus, theory of mathematics education and its related topics in a university. Also, 12 students (age 16-18) who registered in one of authors, Prof. Choi's class for Calculus-2 participated in the classroom experiment of this research from April 7 to May 20 of 2010 (2nd half of Spring semester). One assistant videotaped classroom activities.

3.2. Overview

We had pre/post surveys and presented structured or ill-structured problems for main concepts in each lecture to students. 12 students were divided into 3 small groups of 4 students for group discussions according to their academic achievements in Calculus 1 and their mid-term scores and personalities. Students handed in reflective journal (class diary) at the end of each class. The final test, quizzes and homework were given.

3.3. Pre/Post Survey

We investigated students' background and activities of discussion, presentation, learning attitude to DT and Understanding of DT. Reflective Journal reflects items of Definition, Theorem, Discussion, Presentation and Others.

3.4. Classroom Experiment

Each class proceeded in the following order:

- Hand out problems in the beginning of class
- comments on previous reflective journals by instructor,
- group activity: discussion on the problems,
- presentation of what students found and learned
- teacher's guide for students' unresolved problem through questions so that students found out the concepts and meaning of DT (*cf.* Polya, 1985),
- introduce sketch of main definitions for next lesson by instructor
- students write reflective journals

3.5. Structured / Ill structured problems

Figures 1 and 2 show the third week structured or ill-structured problems that we gave to students. Students tried to find or construct definitions or theorems through the problems which we produced based on our evaluation of students' activities of mathematization in every week.

3.6. Assessment

We evaluated students through 2 exams (20% for each), home works (briefly HW, 30%), quizzes (20%) and group discussion & presentation (10%). During 1st–7th week, Midterm exam, HW, Quiz are given and during 8–15th week, Final exam, HW, Quiz are given.

15.7 Maximum and Minimum Values

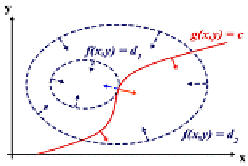
Problem 1 : 1변수 함수에서 극대 극소를 갖는 점의 성질을 되새기자.
1) 이 성질들을 2변수 함수로 어떻게 확장할 수 있을까?
2) 1계, 2계 편도함수가 만족해야 하는 조건들은?
3) 토론의 결과를 정리로 만들어 보자.

Problem 2 : 1변수 함수가 폐구간에서 연속이면 최댓값과 최솟값을 가졌다. 이를 2변수 함수로 어떻게 확장할 수 있을까? 즉 2변수 연속함수의 정의역이 어떤 조건을 만족하면 최댓값과 최솟값을 보장할 수 있을까? 추측해보자.

Figure 1. Problem for maximum and minimum

15.8 Lagrange Multipliers

Problem 1 : 어떤 함수가 주어진 조건(제한) 아래에서 최댓값과 최솟값을 갖는다고 하자.
이때에 최댓값과 최솟값을 어떻게 찾을까?
Lagrange는 이런 상황을 이변수 함수의 경우에 다음과 같이 설명한다. 주어진 조건이 $g(x, y) = k$이고, 이 조건 아래에서 최댓값과 최솟값을 구하고자 하는 함수를 $f(x, y)$라 할 때에, $f(x, y)$의 최댓값과 최솟값은 이 함수의 level curve가 $g(x, y) = k$의 그래프와 접할 때에 발생한다는 것을 알았다.
이 사실을 이용하여 $f(x, y)$의 최댓값과 최솟값을 가질 수 있는 후보점들을 찾을 수 있었는데 어떻게 찾았을까?



Problem 2 : 이 아이디어를 3변수 함수에 어떻게 적용할 수 있을까?

Figure 2. Problem for Lagrange Multipliers

4. RESEARCH RESULT & ANALYSIS

We analyzed the results of pre/post surveys and students' reflective journals in this section. In particular, we selected students' responses related with DT from survey results. Those responses include students' effort for new definitions, role of definitions, how to study theorem before/after our classroom experiment and the results are given in this section. In the tables, 'before' and 'after' mean 'before experiment' and 'after experiment'.

Question 1 (Group discussion). Why do you think you need group discussion in the class?

Students are allowed to choose multiple items for the question and they are asked to prioritize the list of their choice. We counted the first in each student's list.

Table 1. Group discussion

Items	Before	After
To listen colleague’s ideas	25%	27.3%
To show my ideas to colleague	8.3%	
To understand math concepts in depth	8.3%	27.3%
Opportunity to improve my ability to express what I know		18.2%
To take active part in the class	58.3%	27.3%
Opportunity to gain confidence		

More students realized better that they needed group discussion to understand math concepts in depth and through group discussion they could improve their abilities to express what they know after our experiment.

Question 2 (Definition). How much effort do you make to understand a new definition?

Table 2. Effort for new definition

Items	Before	After
Just learn them in the class	8.7 %	
Memorize them to keep in mind	8.7 %	
Try to understand in depth after the class	25 %	18.2 %
Examine them closely again through solving problem after the class	58.3 %	81.8 %
Analyze them in detail to understand meanings of them		

More students became to examine new theorem again through solving problem after the class more closely than before.

Question 3 (Definition). What do you think is the role of the definitions in mathematics?

It was a surprise to us that some students thought definition as a decoration for the first page of chapter and considered it not so useful before experimental class. After experiment, most of them realized the essence of definition and its role.

Table 3. Role of definition

Before	After
- convey the mathematical concepts. express a new concept in terms of mathematical language - most fundamental of various mathematical principles - decoration for the first page of chapter - The most basic elements for solving problems. - rule to explain something new in mathematics - Never give a serious thought about it. - Something to memorize. Not so useful	- theorems are derived from the definitions and very useful for solving problems - starting point for developing mathematical thought, and important for deriving theorems - arranged to be used most easily and conveniently and essential for deriving theorems

Question 4 (Theorem). How much effort do you make to understand a new theorem?

Table 4. Effort for new theorem

Items	Before	After
Just learn them in the class	8.3 %	9.1%
Memorize them to keep in mind		
Try to understand in depth after the class	8.3 %	9.1%
Examine them closely again through solving problem after the class	75 %	72.7 %
Analyze them in detail to understand meanings of them	8.3 %	9.1%

Question 5 (Theorems). How do you study theorem? (The multiple choice possible)

Table 5. How to study theorem

items	Before	After
review each step of proof for theorem referring to text book	81.8%	63.6%
try to prove the theorem by myself without referring to text book	9.1%	45.5%
examine whether the converse of theorem is true and try to find counterexample if the converse is not true	9.1 %	9.1 %
examine why the given conditions in the hypothesis of theorem are necessary	9.1%	45.5%
examine how the conclusion are changed when the hypothesis of theorem are weakened or strengthened	0%	9.1%

More students became to try to prove theorem by them self without referring to text book and examine why the given conditions in the hypothesis of theorem are necessary and also how the conclusion are changed when the hypothesis of theorem are weakened or strengthened after our experiment.

Next, we categorized and summarized students' responses, opinions and feelings for DT and group discussion expressed in the reflective journals in Tables 6–8.

Table 6. Reflective Journal: Definition

Students' responses, opinions and feelings
- I can understand the multivariable functions fundamentally by comparing those with the functions of single variable.
- To understand the definition by constructing it directly is more effective for remembering the concepts of limit and continuity.
- It is impressive to recognize the essential meaning of definitions instead of learning them by repetition.
- It is amazing to get generalizations by extending previous cases similarly and refining further.
- The procedure of generalizations seems to be adding to the previous cases in a complicated way. But, it is not straightforward, and makes one think up an effective way.

Table 7. Reflective Journal: Theorems

Students' responses, opinions and feelings
- I can understand the procedure of derivations better than before
- It was enjoyable to understand theorems
- I have more focus on the meaning and structure of proof of theorem.
- I understand how to approach to theorems in various way
- I acquire the deep meaning of them
- I got more comfortable with them

Looking at Tables 6 and 7, we would contend that students were elevated through the rhythmic cycles of stages of romance, precision, and generalization and also they became truly free of inner prejudice against knowledge of DT as Whitehead (1929) said. Accordingly, we could assure that our teaching-learning method brought out students' cognitive development.

In the next table, students confessed that they realized merits providing by group discussions. That is what we expected as a result of this research.

Table 8. Reflective Journal: Discussion

Students' responses, opinions and feelings
- It allows the longer memory than the conventional lecture - It allows deep understanding of the subject to express to others explicitly what I know - It allows for learning others' diverse ways of thinking to listen to others' thought in the discussion - It was enjoyable to understand theorems through group discussion

We divided our experimental class into 2 groups of high achievement and lower achievement according to their mid-term scores. We found that students in the lower achievement show their improvement in the average scores of exams through this classroom experiment

Table 9. Academic achievements

Experimental class	High achievement	Lower achievement	Standard deviation	
			High	Lower
Mid-term	98.67	76	4.41	15.05
Final	97	89.67	20.86	15.15

We also compared academic achievement of our experimental class with the average scores of exams in Calculus-2 classes which other professors lectured through the instructor centered teaching-learning in the same school. Same exams were given to all students in Calculus-2 classes and students were graded altogether using the same rubric.

Table 10. Comparison with other classes

Group	Midterm exam	Final exam
Experimental class	87.33	93.33
Comparison class	86.06	92.29

5. CONCLUSION

The students had experience of constructing and (re-)discovering definitions, deriving theorems by using definitions, and applying the acquired knowledge to solving problems effectively in this research. This was made possible through collaborative group discussions of the well-designed structured/ill-structured problems and the presentations of the results in the class with the instructor's questioning and guidance. Students confessed that they realized merits providing by group discussions.

The students' cognition of the definitions in mathematics has changed; the definitions are the fundamental ground from which theorems are derived and useful sources of problem solving, not mere rules or assumptions. Through this research, most of them realized the essence of definition and its role. The students' learning attitude toward theorems in mathematics has changed. More attention is paid to understanding of the structure of theorems, and possible generalizations of theorems, which goes beyond applying theorems for problem solving. Most students experienced generalization and deformation of definitions and theorems which yield improvement of students' holistic/analytic thoughts. Therefore, students were elevated through the rhythmic cycles of stages of romance, precision, and generalization and also they became truly free of inner prejudice against knowledge of DT as Whitehead (1929) insisted. Accordingly, we could assure that our teaching-learning method brought out students' cognitive development. Especially, the change of learning attitude has led to the meaningful improvement of the grades, especially for students of lower achievement. We also could finish all material of the course within a semester by our teaching-learning method. Also, our students could understand DT in depth in view of structure of knowledge through our teaching-learning method as well as without the loss of their average scores in the exams.

Through this research, we suggest instructors to let students stay enough in the first stage of romance when they get into new definition or theorem that they have never known before. Instructors can do so by presenting students with concrete examples relevant to the knowledge. This would give them driving force so that they can go through the rhythmic cycles of stages of romance, precision, and generalization over and over again.

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