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Understanding of Teaching Strategies on Quadratic Functions in Chinese Mathematics Classrooms

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What strategies are used to help students understand quadratic functions in mathematics classroom? In specific, how does Chinese teacher highlight a connection between algebraic representation and graphic representation? From October to November 2009, an experienced teacher classroom was observed. It was found that when students started learning a new type of quadratic function in lessons, the teacher used two different teaching strategies for their learning:

(1) Eliciting students to plot the graphs of quadratic functions with pointwise approaches, and then construct the function image in their minds with global approaches; and

(2) Presenting a specific mathematical problem, or introducing conception to elicit students to conjecture, and then encouraging them to verify it with appoint approaches

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BACKGROUND

The concept of function plays a central and fundamental role in mathematics science. In China, the notion of a function and a quadratic function is greatly highlighted in the school mathematics curriculum. In specific, when students finished junior high school mathematics curriculum, they could find expression of a quadratic function in realistic situations, use a pointwise approach to draw parabola of a given expression, read traits from the graph, and use algebraic formula to determine vertex and symmetry axis of the quadratic function graph (Ministry of Education, 2001). Furthermore, teacher should employ quadratic functions to help senior-high school students understanding of maximum or minimum, monotonicity, and odevity of a function, and comprehend the relation between a quadratic function and a quadratic equation (Ministry of Education, 2003). However, studies have revealed that the learning quadratic function is more complex than what is typically assumed by designers of curriculum and instruction (Even, 1998; Huang, Tang, Gong, Yang & Tian, 2012; Pirie & Kieren, 1994; Zaslavsky, 1997; Zazkis, Liljedahl & Gadowsky, 2003). Therefore, how to effectively help students learn the difficult content is a question worthy of consideration. This paper focuses on a teacher strategy of teaching quadratic functions in Chinese mathematics classroom.

DIFFICULTIES IN LEARNING QUADRATIC FUNCTION

Limited Understanding of Function Concept

Pictorial entailments surrounding quadratic functions were reported in Zaslavsky's study (1997). She found that the students considered only the visible part of a parabola although it was defined on an infinite domain. For example, when the y-intercept of a parabola did not show on the graph, they believed that this point does not exist. Their visual reasoning also was reflected on determination whether a point is or not on a parabola, just depending on the graph in view, but ignoring the analytical properties and the definition of a quadratic function. These mentioned evidence indicated that students think of the graph of a function as being only fixed geometric curve in the plane. Oehrtman, Carlson & Thompson (2008) argued that students' such graph reading is caused by their understanding limited to an action view of function (Dubinsky, Breidenbach, Hawks & Nichols, 1992). With this action view, students do not view a function graph as defining a general mapping of a set of input values from the independent axis to a set of output values on the dependent axis (Carlson, Oehrtman & Engelke, 2010).

Curriculum Effect

In Chinese school mathematics curriculum, students start learning quadratic functions in Grade 9, after completing linear functions, and learning how to solve quadratic equations. Traditionally, linear functions and quadratic equations are considered prerequisites for quadratic functions. In fact, the learning sequence may impede the understanding of quadratic functions. For example, it was observed in a mathematics classroom that a student is asked to find the minimum of $y=3x^2+9x-3$, he obtained a simple expression of $y=x^2+3x-1$ by dividing by 3, and then use method of completing square to get the minimum of $y=x^2+3x-1$ as that of $y=3x^2+9x-3$. It showed that the student overgeneralized the property that the corresponding equations $3x^2+9x-3=0$ and $x^2+3x-1=0$ are equivalence, and applied it to the quadratic functions. Strong pulls toward traits of linearity also have been shown on students learning quadratic functions. For example, Zaslavsy (1987) asked students to find an expression for a graph of a parabola that contained three labeled points. They used them in a manner that suggested a linear mindset, which is they used them to calculate the slope value of the straight line through those two points, and then inserted the value into the parabola form of an expression as the leading coefficient.

Incomplete Variations and Examples

The expression form of quadratic function $y=ax^2+bx+c$ has different variations in which one of the coefficients of *b* and *c* is or not zero. If the number of examples and variations are too limited, students' misconceptions would appear. For example, student determined that a quadratic function with an expression of the form $y=ax^2+bx$ does not have a y-intercept, because they thought *c* does not exist, and *c* determines the y-intercept (Zaslavsky, 1997). The similar results were also found by Even's study (1998). For instance, a student checked quadratic functions of the form $y=ax^2+c$, and then concluded that *c* regulated the moving of the graph of $y=ax^2+bx+c$ up or down from $y=ax^2$.

Un-linking Different Representations to Solve Problems

Why students sometimes cannot solve problems on quadratic function, even if they understand function conception beyond action view, distinguish properties between a quadratic function and a quadratic equation, and hold enough variations and examples of the form of the expression of quadratic function. Even (1990) found that these difficulties are caused by a lack connectedness between representations. It's well known that same function can appear in different representations. In general, a function can be represented with a graph, a formula, an arrow diagram, a table, a set of ordered pairs, and words. In particular, to a quadratic function, symbolic (formula) and graphic representations are important. A lack of rich relationships and connectedness between the two representations seems to prevent students from relating the given expression to the graph of a quadratic function $y = ax^2 + bx + c$ (Even, 1990).

THEORETICAL FRAMEWORK

Duval (2006) argued that if one wishes to analyze difficulties in learning mathematics it is to the study of translations between different system representations (e.g. between symbol and graph) that one ought to give priority. Practically, students' knowledge and understanding of quadratic functions will intertwine with the flexibility in moving from one representation to another, for instance, different ways of approaching functions, context of the representations, and underlying notion (Even, 1998). The role of alternative ways of approaching functions will be focused on, and Even's framework will be made use of to explain teaching strategies on quadratic function tasks in Chinese mathematics classrooms in this study?

Pointwise Way

These are graphs of quadratic functions $y = ax^2$ (A), $y = bx^2$ (B), $y = cx^2$ (C), and $y = dx^2$ (D). Compare *a*, *b*, *c*, and *d*.

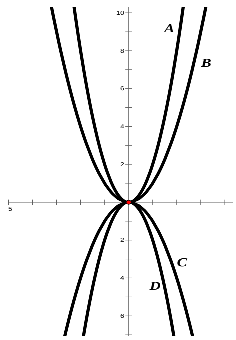

Figure 1. Comparison of coefficients of quadratic functions

A pointwise approach and a global approach were considered for linking different representations of function (Even, 1990; 1993; 1998). The pointwise way approaching quadratic function means to plot, read or deal with discrete points of a function either because we are interested in some specific points only. For example, if we want to find the *y*-intercept of $y = 2x^2 + x - 1$, we can choose $x = 0$, then calculate $y = -1$. Sometimes using a pointwise approach is successful in solving problems that involved symbolic and graphic presentations of quadratic functions. For example, presented with the problem in Figure 1, we just choose and draw the vertical line of $x = 1$, then order the coefficients simply through reading the intersection points (which ordinates are a, b, c, and d from the top down).

But a pointwise approach to graphing functions is, in some cases, less powerful. For example, graphing a quadratic function that has (–100, 78) as a vertex by plotting several points near (0, 0) will not produce a very informative graph (Even, 1990). Huang, Tang, Gong, Yang & Tian (2012) also found that when students were asked to use pointwise approach to draw translated parabolas, various patterns appeared, such as, parabolas' vertices coincided at the origin, intersected, and extended in different directions.

Global Way

Therefore, there are also times when we have to consider the function in a global way. For example, when we want to sketch the graph of a function given in symbolic form, or when we want to find an extremum of a function defined on the real numbers. Sometimes, in many specific problems, a global approach is more powerful than a pointwise approach, for instance, presented with problem in Figure 2. Use of a pointwise approach is complex for solving this problem. In details, we should find the value of a or $a^6 - 1$ in the expression with the coordinate of the given point, then put $x = 2$ into it to evaluate the value. However, we consider the parabola in a global way, which is symmetric graph by x-axis, so that (–2, –3) and (2, 3) are symmetric points on the parabola.

If a point (–2,–3) on the graph of $y = (a^6 - 1)x^2$, evaluate the value when $x = 2$.

Figure 2. Evaluation of a corresponding value based on a given expression

Maybe use of a global approach to function implies a better understanding. But sometimes this is not always the case. For example, Even (1998) found some students misused a global approach to judge c positive or negative according to the vertex location of a graph of $y = ax^2 + bx + c$, before they understood quadratic function deeply.

Teacher can require students to plot a graph of $y = x^2 + 4x + 3$, so that they will find $c = 3$ positive though the vertex location is in the third quadrant they can find $c = 3$ via the expression $y = x^2 + 4x + 3$, how can they find $c = 3$ from graphic?. Therefore, sometimes a pointwise approach can be used to improve students' understanding of conception.

If $a \neq 0, y = ax + b, y = ax^2$, then which graph is correct as following.

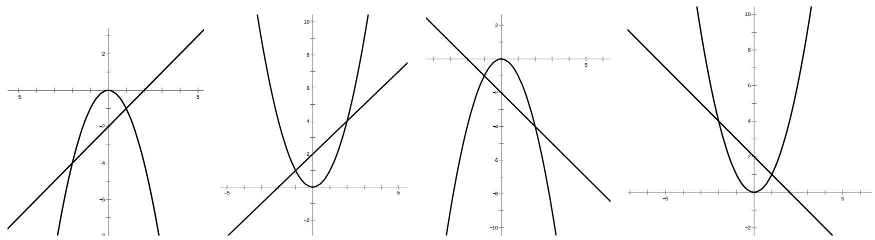

Figure 3. Selection of a corresponding graph based on given expressions

The Two Ways Demanded Together

Sometimes, in order to solve mathematics problem, the two approaches are demanded together. For example, the best way for student to solve the problem presented in Figure 3, is to determine the y-intercept of $y = ax + b$ in a pointwise way firstly, and refined the linear graph in two possibilities, then use a global approach to choose which maintains the consistence of the linear direction and the parabola open-direction controlled by a.

From examples mentioned above, it is indicated that each one of the alternative ways of approaching functions is different from the others and neither one of them is appropriate for all situations. In fact, a pointwise approach emphasizes a function as a process, and a global approach highlights it as an object (Sfard, 1991; Sfard & Linchevski, 1994). The process conception is the first step in understanding of function. Transition from the process to the abstract object is long and inherently difficult. These two aspects, although ostensibly incompatible, are in fact complementary as different sides of the same coin. Therefore, a pointwise approach and a global approach can be seen as two powerful strategies for students understanding of the duality, and connecting symbolic and graphic representations.

It is well known, that the term “understanding” has been freely used in mathematics education literature. However mathematical understanding was a difficult conception to define and stated prior to Skemp’s distinction between knowledge and understanding. In particular, Skemp (1976) classified relational understanding as knowing what to do and why it should be done and instructional understanding as having rules without the reasons. Other more general views proposed that understanding is the development of connections between ideas, facts, or procedurals (Davis, 1984; Hiebert, 1986). Recent constructivist conceptualizations of understanding have proposed that understanding is built by forming mental objects and making connections among them in addition to Pirie and Kieren’s model of the growth of mathematical understanding (Kaput, 1985; Sfard, 1991; Sierpinska, 1990; Tall & Vinner, 1981). Pirie and Kieren's model (Pirie & Kieren, 1994) of mathematical understanding may be borrowed here for better understand of the strategies employed by the teacher. The model indicates that understanding is a nonlinear dynamic hierarchical process, in which mathematical understanding could be divided into eight levels, from the inside followed by a primitive knowing, image making, image having, property noticing, formalizing, observing, structuring, and inventing.

In this study the research question is how Chinese teacher use strategies to help students learning quadratic function with the two approaches?

METHOD

The Teacher' Information

Mr. Shen graduated from the Department of Mathematics of a normal university in 1999. He has 10 years teaching experience in Qinchuan Middle School, where students' academic performance is excellent among local schools. He is a part-time student for his Master's Degree in Education, and now he is preparing his dissertation. Qinchuan Junior School covers about 66 acres, located in the west in Yucheng, which is a small city in the southern of Jiangsu. This school has about 2,500 students divided into three grades from seven to nine. Each grade has 16 classes, and each class has 45~ 50 students mostly from common families.

Classroom Observation

Quadratic functions should be taught at the sixth semester in Grade 9 according to the mathematics curriculum. However, all the contents in the mathematics curriculum are generally completed by students till the fifth semesters in local schools. Preparation for entrance examination is made at the last semester. Therefore, when Mr. Shen taught the lessons of quadratic function, he as well as his students had no textbook. He criticized the reformed series textbooks used in local school are too much easy for students to pass entrance examination; therefore he planned his lessons based more than one recourses.

From October to November 2009, Mr. Shen's Grade 9 algebra class was observed. The unit of quadratic functions ha three sections: definition of quadratic function, graph and attribute of quadratic function, and application of quadratic function. The second section highlights a connection between algebraic representation and graphic representation. This content was divided into five lessons by the teacher, each of which had about 40 minutes. Table 1 presents these lessons in details. During observation, one researcher videotaped, and another wrote field notes for each lesson. The lesson plan was collected, and the teacher was semi-structure interviewed by the researcher who wrote notes after each lesson. At the same time, the researcher who videotaped the lesson records the whole process of interview. Videos of the lessons and interviews were translated into transcripts by four assistants.

Table 1. The Content of Each Lesson in Second Section of the Unit of Quadratic Functions

Content	Lesson sequence
Graph and property of $y = ax^2$	Lesson 2 (L2)
Graph and property of $y = ax^2$ & Exercise	Lesson 3 (L3)
Graph and property of $y = ax^2 + k$	Lesson 4 (L4)
Graph and property of $y = a(x+h)^2$	Lesson 5 (L5)
Graph and property of $y = a(x+h)^2 + k$ & Exercise	Lesson 6 (L6)

Teacher Interviews

The authors interviewed the teacher after each lesson, and audio recorded each interview conversation. The goal of the interviews was to obtain the teacher's perspective on students' learning difficulties, as well as to learn about why the teacher used strategies in particular ways to improve students' understanding during the implementation of mathematics tasks. In the interviews, the three questions were mainly focused on:

(1) What do you think the emphasis and difficulty in this class?
(2) Where do you think students will encounter difficulties, and why?
(3) What strategies you used to help students understand the content, and why?

Data Analysis

Data analysis relied primarily on full transcripts and video recordings themselves, and secondary on interviews. The pointwise approaches and globe approaches were coded, which were employed by students to solving problems in lessons. Actions of pointwise approaches were coded into three categories:

(a) Plotting function graphs,
(b) Explaining traits of moved graphs, and
(c) Finding algebraic expressions of translated functions.

At the same time, actions of globe approaches were coded into two categories:

(a) Generating conjectures, and
(b) Sketching graphs according to algebraic expressions.

RESULTS

Roles of the Two Ways in Different Lessons

Table 2 shows that only in the L2, the two approaches were used at same level, but in the following lessons, pointwise approaches were used significantly lower than global approaches. It does not mean that pointwise approaches were used decreasing gradually in the sequence of the lessons. It also does not mean that this change is arbitrary or accidental. L2 is the first lesson of graphs and properties of $= ax^2$, in which pointwise approaches accounted for 50%, and L3 is the second lesson of the same topic, in which pointwise approaches were reduced to 23.53%. When students learned another new topic (graphs and properties of $y = ax^2 + k$) in L4, the frequency of pointwise approaches was increased to 30%. In L5, graphs and properties of $y = a(x + h)^2$ were learned, which are too difficult to understand for students (Zazkis, Liljedahl & Gadowsky, 2003), therefore, pointwise approaches were used increased continually. Although graphs and properties of $y = a(x + h)^2 + k$ in L6 is a new topic for students, the teacher did not think so (Huang, Tang, Gong, Yang & Tian 2010), thus, pointwise approaches were reduced from 36.37% to 28.57%, and at the same time global approaches were increased. It indicated that, on the one hand, when students started learning a new topic in the lessons, pointwise approaches would be increased in the frequency, and at the same time, global approaches would be corresponding reduced. On the other hand, when the classroom activities were to consolidate the contents which was learned in prior lessons, global approaches would be increasable used. It was inferred that the teacher had formed his own understanding of the two approaches--a pointwise approach can help students understand the properties of quadratic functions, and a global approach can consolidate and deepen their understanding.

Table 2. The Number of Time of Two Ways Used by Students in Different Lessons

	No. (percentage) of pointwise approaches	No. (percentage) of global approaches
L2	6 (50%)	6 (50%)
L3	4 (23.53%)	13 (76.47%)
L4	6 (30%)	14 (70%)
L5	8 (36.37%)	14 (63.63%)
L6	6 (28.57%)	15 (71.43) %
Total	30 (32.61%)	62 (67.39%)

Pointwise Ways Used in Lessons

Plotting function graphs

Pointwise approaches were used by the teacher and his students to plot function graphs. For example, Mr. Shen requested his students to complete the mathematical task in L4: Depicting points to draw parabolas of $y=1/2x^2$, $y=1/2x^2+1$, and $y=1/2\,x^2-1$ at the same coordinate system. The teacher listed a table on the blackboard, took $x = -3, -2, -1, 0, 1, 2, 3$ symmetrically, and found the corresponding value of $y=1/2x^2$ in order.

Mr. Shen: Do you think the value of $y=1/2x^2+1$ must be calculated in this way? ... Yes, don't calculate any longer, as long as the latter's corresponding value of the $y=1/2x^2$ plus one."

When Students were to find the corresponding value of function $y=1/2\,x^2-1$, he also encouraged them to focus attention on this method. Then, the teacher required each student to plot the graphs of $y=1/2x^2$, $y=1/2x^2+1$, and $y=1/2\,x^2-1$ on the cross-section paper.

Graph is an important representation of function. Especially, in school mathematics curriculum, the function traits are always introduced and developed ground on function graphs. Therefore, using table-list and point-plotting to draw function graph is the most common strategy in school teaching function. For example, in the other two lessons of introduction new topics (L2 and L5), the teacher employed the same teaching strategy as that in L4. In fact, it is believed that the strategy can help students understand and grasp function traits based on function graphs, and establish a link between function graph and algebraic expression. However, when the graphs of $y = 1/2\,x^2$, $y = 1/2\,x^2 + 1$, and $y=1/2\,x^2-1$ need to draw in the same coordinate system, Students' misconception was exposed in the classroom:

(1) The three parabolas' vertices coincided at the origin;
(2) Parabolas intersected;
(3) Parabolas extended in different directions, but did not intersect (Huang, Tang, Gong, Yang & Tian, 2012).

In other words, although the plotting method can help students draw a function graph, but cannot enhance students understanding of the global trait of a function, and the position relation between transformed graphs.

Explaining traits of moved graphs

The teachers often requested students to used pointwise approach to explain graph traits. The following excerpt from L4 illustrates the teacher elicited students to explain the position between graphs of $y=1/2\,x^2+1$ and $y = 1/2\,x^2$ with pointwise approach.

Mr Shen: Do the graphs of $y=1/2\,x^2+1$ and $y = 1/2\,x^2$ intersect? Liping.

Liping: No. If x is equal to zero, then the first value is equal to one, the second is equal to zero. The same way, x is equal to one, with three over two, and one over two, then x is equal to two, with three and two. The first value is always larger than the second one, so the first curve is always above the second one.

Mr Shen: Great! Let x equal to any a, then get two points on the graphs. The first point of the vertical coordinate is always larger one than the second one, so the first point is always above the second point.

Note that after questions raised by Mr. Shen, Liping picked up some specific numbers, and found out the corresponding values, then got the general conclusion by inductive reasoning—"The first value is always larger than the second, so the first curve is always above the second one". Mr. Shen did not satisfy the explanation given by Liping. He took a general number a, and found the two values under the corresponding expressions, then obtained the true sentence $1/2a^2 + 1 > 1/2a^2$. At last, Mr. Shen concluded that the first point is above the second point. Because the point is arbitrary, the first graph is above the second one. In this process, the teacher enhanced students to understand traits of function graph with pointwise approach.

The next excerpt from L5 illustrates the teacher help students understand the horizontal translation on quadratic function through focusing on the vertex coordinates.

Mr. Shen: When a graph moves, each point on it is moved under the same law. … Which point is chosen better? … Vertex! The vertex of $y = ax^2$ is (0, 0). What is the vertex of $y = a(x+1)^2$? … It is (h, 0). Do you know if the h is a positive number, where does the graph move? … Yes, the right side. If the h is a negative number, the graph moves toward the left side.

In fact, it would be hard for students to understand horizontal translation on quadratic function, because horizontal translation is against intuition (Eisenberg & Dreyfus, 1994). If $y = x^2$ is supposed to be translated to $y = (x+1)^2$, x in the former expression is added into a positive one, however the graph moved left toward in the negative direction along the axis x. In order to students' understanding of the correspondence between algebraic expression and graph in translation process, Mr. Shen stressed that a special point should be focused on, and seize its coordinates change in the translation process, then speculated translation direction of the graph according to variation of the point location.

Further evidence from a post-lesson interview with Mr. Shen confirmed that he believed pointwise approach to be effective for students' understanding of translations on quadratic function.

Mr. Shen: Focus on points! Why did we pay attention to the vertex of a parabola? In fact, any one point is Ok. But the vertex is special. Using its coordinates change in the translation process can help them understanding, rather than memorize the law.

Finding algebraic expressions of translated functions

A pointwise approach is also an effective method used to find algebraic expressions of translated functions. Mr. Shen believed that function translation is not hard, but the difficulty comes from how to set up the connection between a function expression and its graph in the translation process. In order to promoting students' understanding of this connection, at the beginning of L5 the teacher implemented a task on probing the expression of a graph horizontal translated from the parabola of $y = x^2$. The classroom episode is excerpted as following.

Mr. Shen: The graph of $y=x^2$ can move up and down. Of course, it also can move left and right. If the graph move left one unit, can you find the expression?

Students: Yes.

Mr. Shen: Who can answer it in details? Ming.

Ming: Because the shape of the graph is same in the translation, the coefficient a is same, and equals to 1. But, I don't know b and c.

Mr. Shen: Good. Now, we let the expression $y=x^2+bx+c$. here, b and c are two unknown. Can you set up a system of equations? [No response] We have found a vertex point (–1, 0) on the moved graph. Well, the following question comes. Please listen to me carefully. When the graph moves left one unit, does each point on it move left one unit?

Students: Yes.

Mr. Shen: If I know the coordinates of a point, after it moves left one unit. Can we know the new coordinates of the point?

Students: Yes.

Mr. Shen: In the same way, we can take any point on $y = x^2$, for example point (1, 1). It moves left one unit, what are the new coordinates?

Students: The coordinates is (0, 1).

[Then students set up a system of two equations with the two points' coordinates, and found out the expression $y = (x–1)^2$.]

Although the teacher controlled the classroom in the process of finding a moved graph expression, his teaching behavior demonstrated high-level performance. It was found by Henningsen and Stein (1997) that a teacher modeling high-level performance support high-level student engagement.

Global Ways Used in Lessons

Generating conjectures

Mr. Shen encouraged students to use global approaches to generate conjectures. The following excerpt from L4 illustrates the teacher elicited students made an analogy translation on linear function to translation on quadratic one.

Mr. Shen: We have learned the function of $y = ax^2$. Today we will explore the features of the function $y = ax^2+c$. These look quite like. As to how obtain $y = ax^2+c$, just add c to $y = ax^2$. What changes will the graph?

Students: Translate it upward or downward.

Mr. Shen: Yun, can you explain?

Yun: Just like the translation on linear function.

Mr. Shen: Can you give us an example?

Yun: $y = 2x$, $y = 2x+1$

In this excerpt, the teacher attempted students to employ analogy reasoning to generate a global impression on the translation on quadratic function. He expected that students would preliminary perceive the translation on a parabola before use of pointwise approaches. This strategy, which is consistent with the view of Gestalt, could facilitate students to establish a gestalt firstly, which can assimilate specific knowledge. Further evidence from a postobservation interview with Mr. Shen confirmed his pedagogical intention.

Mr. Shen: I already elicited students to conjecture. The purpose is to promote them to generate an overall impression on the translation.

In L5, where core concept is the horizontal translation on quadratic function, the teacher also used the strategy to deal it with. Firstly, Mr. Shen intended to give students a global impression on the horizontal translation. He said: "the graph of $y=x^2$ can move up and down. Of course, it also can move left and right. If the graph move left one unit, can you conjecture its expression?"

In the post-lesson interview, Mr. Shen explained the rule of global approaches.

Mr. Shen: They have the graph in their mind before they draw the moved graph, so I think they will not make errors in depicting points to draw it. I think this is a discovery process in which students have a hypothesis firstly, and then verify it.

In other words, the teacher believed that students should be elicited to use a global approach in learning mathematics because the approach could promote students to conjecture audaciously, and generate a preliminary understanding, then make justification in details, on the other hand, could predetermine an orientation in the process of problem solving, but prevent the students into the details, where disappearing the forest for the trees.

Sketching graphs according to algebraic expressions.

Mr. Shen always requested students to use global approaches to sketch parabolas when they had drawn them with pointwise approaches. In the L2, after students plotted the graph of $y=x^2$, and explored its traits, the teacher required students to sketch the graphs of $y = x^2$, and $y = \pm 2x^2$ directly. It also occurred in the L4 to L6 that students were required to sketch graphs of specific examples of $y = ax^2 + k$ or $y = a(x+h)^2$, when they have drawn them with pointwise approaches. Mr. Shen explained in the post-lesson interview.

Mr. Shen: Plotting graph by points is the basic skill students should master. However, this is not enough to just stop here. Students should be able to directly sketch it. If a student can draw a parabola directly according to the expression, it is indicated that his or her mind has been an image of the function. In other words, he or she can link graph and expression of function. Linking the two representations is a difficulty of learning function. Another reason is the need for problem-solving. Many exercises need students to draw function graph directly, otherwise cannot be solved.

It is indicated that Mr. Shen thought understanding of quadratic function is a process. Plotting graphs of quadratic functions with pointwise approaches means the function algebraic representation was internalized into graphic representation. This is just the first step in understanding process. Next, student should translate the graphic representation of function into personal mental image. Mr. Shen believed that this step is vitally important because it is the precondition of linking algebraic and graphic representations in mental action. In other words, student having parabola image in mind could be facilitated by global way, which is an important stage in resolving difficulties of learning quadratic function.

Mr. Shen believed that directly drawing function graph is often demanded to solve mathematics problems. That is, in many cases, global way is necessary to solve problems. Sometimes global approach and pointwise approach are demanded together in a mathematics task. A task the teacher presented as an exercise at the end of L5 is a typical example (Figure 4). To solve this task, students should firstly draw the function graph by global way, based on the algebraic expression and the symmetry axis equation of the function. Then, they could use the linear equations and the distance between the two intersection points to find the points coordinates by pointwise way, so that the value of a could be found.

A parabola of $y = a(x - m)^2$, which symmetry axis is $x = 2$, is intersect by $y = -2$. The distance between the two intersection points is 2. Find the value of a.

Figure 4. A task the two approaches demanded

In the L2 to L6, a total of 16 tasks as exercises were presented by the teacher. There are 11 tasks (68.75%) demand global approaches or the two approaches together. Furthermore, in the uniform unit test of quadratic function, there are seven tasks demanded global approaches or the two approaches together, accounting for 36.84%. It is indicated that, as Mr. Shen said, many mathematical tasks related to quadratic functions demand global approaches.

CONCLUSION AND DISCUSSION

In this study, the teacher employed different teaching strategies to elicit students in different situations to coordinate pointwise approaches and global approaches to understand quadratic functions. In particular, when students start learning a new type of quadratic function in lessons, teachers have used two different teaching strategies for their learning. The first one is that the teacher asked students to plot the graphs of quadratic functions with pointwise approaches, and then constructed the function image in their minds with global approaches. On the contrary, the second one is that the teacher, firstly presented a specific mathematical problem, or introduced conception by analogy to elicit students to conjecture, and then encourage them to verify the it with appoint approaches.

In the classroom, we also found that the teacher elicited students with appoint approaches, focusing on a general or a special point of a parabola for promoting student understanding of the quadratic function. In fact, the teacher strategy reflected his view on the teaching of mathematics. It can be referred that ultimate goal of the teacher is to improve students understanding of mathematics.

Moreover, the teacher believed that use of a global approach in context is the prerequisite of solving many mathematical problems, but also an evaluation indicator for understanding of conception deeply. It seems that the teacher encouraged students to solve problems using more abstract and generalized approaches. This finding is same as what was concluded by Cai in another study (*cf.* Cai, 2005; Cai & Lester, 2005).

In the perspective of P-K model, the use of appoint approach to plotting graph could be seen as the process of image making, and the use of global approach to constructing image in the mind is on the level of image having. Therefore, the first strategy employed by the teacher is completely necessary and reasonable, conforming to the understanding growth from the inside out in the P-K model. However, understanding growth is nonlinear. When the outer layer of understanding students can not establish, fold-back is when necessary. The inner layer is still weak and should be further constructed for meet the needs of the outer layer. When the teacher encouraged students to conjecture by global way, where students' understanding is on the image having level. However, the teacher did not encourage students to develop their understanding further, and then fold back. Students were required to use the appoint approach to construct understanding in the inner level, ensuring solid support for further growth.

At the same time, we note that use of appoint approach or global approach to quadratic function, is confined to the inner level of understanding in the P-K model. In other words, the teacher strategy is just emphasized the foundation of mathematical understanding, which perhaps reflects the tradition of Chinese mathematics teaching (Zhang, Li, & Tang, 2004). In order to improve students understanding, teacher should take more effective teaching strategies.

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