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Beyond Accuracy and Speed: Task Demands and Mathematical Performance[1]

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It is an important issue to explore classroom environments which are conducive to developing students' mathematical performance. This study explores the effects of different classroom environments (solution-demand and corresponding-time setting) on mathematical performances. Fourteen and eighteen prospective teachers were required to prove a task under different conditions respectively:
a) Cognitive demand of multiple-solution corresponding time of three hours, and
b) Cognitive demand of a right solution corresponding time of 20 minutes.
We used SOLO as the assessment tool for mathematical performance from quality perspective. Significant differences were found in the quantity and quality of mathematical performance. The regular environment focusing on speed and accuracy were found to be directly linked to low levels of performance. The findings above provide implications to the cognitive benefits of multiple-solution demand and corresponding time setting.

Keywords: classroom environments; multiple-solution task; proving; trapezoid; cognitive demands
MESC Classification: B10, E50
MSC2010 Classification: 97B10, 97E50

PERSPECTIVE AND PURPOSE

A number of studies (*e.g.*, Silver & Stein, 1996) noted that the quality of mathematical opportunities, rather than the background or ability of students, is linked to low levels of performance and engagement in mathematics. Studying the classroom environmental factors that support and limit cognition development is important. How classrooms can be-

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come environments in which students have frequent opportunities to engage in higher-level cognition has been an important subject of research over the last decades (Doyle, 1983; Stein & Lane, 1996; Boston & Smith, 2009). Existing studies mainly focus on the following themes: scaffolding to inhibit / support engagement of students with the mathematical notion (Bayazit, 2006), processes to evaluate instructional decisions and the choice of materials to foster a challenging classroom climate by "mathematical task framework" (e.g. Stein, Smith, Henningsen, & Silver, 2000), and methods to select and implement task to sustain high-level engagement by a lesson protocol (e.g. Smith, Bill, & Hughes, 2008). However, these studies rarely link classroom environmental factors to student outcomes. This study focuses on how the different task demands may influence students` performance of proving a task.

TASK AND STUDENT LEARNING

Mathematical tasks are central to students' learning because "tasks convey messages about what mathematics is and what doing mathematics entails" (NCTM, 1991, p. 24). Different tasks may place different cognitive demands on students (Hiebert & Wearne, 1993). Different tasks may structure different ways students think and can serve to limit or to broaden their views of the subject matter and their actual experiences with mathematics (Schoenfeld, 1994). Selection and implementation of high-level tasks are one of the necessary conditions for high-level cognition in classroom learning (e.g. Silver, Ghousseini, Gosen, Charalambous, & Strawhun, 2005). As a kind of high-level task, multiple solutions for a problem have been recommended both as a critical way to make connectedness of one's mathematical knowledge (NCTM, 2000) and as a tool to develop flexible, transferable knowledge (Yakes & Star, 2011), one's problem-solving expertise (Silver, Ghousseini, Gosen, Charalambous, & Strawhun, 2005), and flexibility of mathematic thinking (Krutetskii, 1976).

Cai & Nie (2008) noted, "All too often students hold the misconception that there is only one 'right' way to approach and solve a problem and, therefore, they fail to develop flexibility in inventing and selecting appropriate strategies and finding solutions. This misconception might be largely due to their lack of experience in using multiple ways to approach a problem." The statement above indicates a rare emphasis on the multiple-solution approach to a problem in the classroom. For teacher education, multiple solutions for a problem are not only a key feature of teaching as suggested in the study (e.g. Ball, 1993), but also are stressed in the classroom of high performing countries (Stigler & Hiebert, 1999).

Waston (2007) confirmed the rationale of these notions from a theoretical perspective of the relationship between teaching and learning based on the SOLO taxonomy by Biggs & Collis (1982). “The links” between teaching and learning are as follows: if learners are only offered unistructural situations (single solution), they are less likely to develop multistructural performance (multiple solutions). Such statements highlight the importance of multiple solutions for a problem for preparation by a mathematics teacher.

Leikin & Levav-Waynberg (2007) further indicated that in-service teachers have limited conception, that is, they have less knowledge in terms of the cognitive benefits of this practice. Silver, Ghousseini, Gosen, Charalambous & Strawhun (2005) emphasized that this practice failed to be supported by our regular environment. Doyle (1988) also identified the classroom-based factors stressing speed and accuracy influence students to develop richness of strategy. Although numerous studies have been written about the advantages of multiple-solution task and the significance of multiple-solution tasks for a mathematics teacher, less experimental evidence exists for the cognitive benefits of multiple-solution demand and possible limitation of a regular classroom environment. Specially, discerning how different demands of solution types of tasks, namely, multiple-solution demand versus single-solution demand, can influence the performance of “mathematical understanding” is not enough.

Given the considerable amount of time required for the tasks of multiple solutions as compared with those of single solution, we found that three hours is the appropriate length of time for a multiple-solution-proof task of “the area formula of a trapezoid” in this current work. Conversely, in the most of Chinese classrooms, the longest time for a regular problem-solving activity in the naturalistic setting is 20 minutes, which is half of the length of a standard lesson.

SOLO FRAMEWORK

Biggs & Collis (1982) argued that the traditional quantitative assessment of learning based on aggregating units fails to chart longitudinal growth of conceptual knowledge and proposed the structure of observed learning outcomes (SOLO) taxonomy using quality methods that is, identifying the different levels of conceptual knowledge based on its use of available information and the complexity with which it is put together, namely, extended abstract, rational, multi-structural, uni-structural, and pre-structural levels. The SOLO framework is broadly applied to assess structures of learning results that occur within each Piagetian stage (Sensorimotor; Intuitive/Preoperational; Concrete Operational; Formal Operational) among diverse “learning-results”, such as mathematics, English, history, geography, economics, specially, reasoning, on school-related tasks and mathematical problem solving (Collis, Romberg & Jurdak, 1986). Can the SOLO framework be applied to examine proof space multiple proving results?

In the previous research, proof space was proposed the SOLO framework among the following different levels: extended abstract, rational, multi-structural, uni-structural, and pre-structural levels. The corresponding proof examples appear in (Sun, 2012).

PROOF TASK OF "THE AREA FORMULA OF A TRAPEZOID"

Tasks of area generally are widely addressed as core content of elementary mathematics by Chinese mathematicians. "Out-in Complementary Principle" (出入相补原理) applied to tasks of area and volume proposed by Liu Hui[2] is regarded as the foundation of Chinese mathematics (*cf.* Siu, 1993). For example, by using this principle, Zhao Shuang (赵爽), the famous ancient Chinese mathematician, proved Pythagorean Theorem. Zhang Jing Zhong (张景中), a famous modern Chinese mathematician not only proved all geometrical properties and theorems in secondary curriculum, but also proposed a creative curriculum reform plan, notion of educational mathematics, in mathematics education field (Zhang, 2005). On the other hand, multiple-solution-proof with area tasks reflects Chinese education philosophy of drawing inferences about other cases from one instance (ju yiyu er sanyufan "举一反三"). for example, Hua Hengfan（华蘅芳：1833–1902）produced more than twenty proofs of Pythagorean Theorem. These studies suggest that multiple-solution-proof area task would be an important in mathematical education field. However, it is interesting to note that multiple-solution-proof area task become a topic scarcely explored in the field of mathematic curriculum due to the fact that it should be categorized geometric section by its reasoning nature and it could not be grouped into traditional Euclid's geometry system by its content as a well-made algebra formula for area calculation in the beginning at primary period. Therefore, proof task of "the area formula" is new attempt for all prospective teachers.

The proof task of "the area formula of a trapezoid" in traditional curriculum materials generally takes three styles in terms of explanation in Figure 1 (Mathematics Textbook Developer Group for Elementary School, 2003, p. 88); namely, a trapezoid is divided into 2 triangles or a parallelogram and a triangle, or a trapezoid is reorganized into a parallelogram by copying the same trapezoid. By re-collecting these well-made explanations, multiple-proofs would be easy tasks, low cognitive tasks. On the other hand, constructing

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[2] Little is known about the life of Liu Hui (劉徽: AD 220–280?) merely because he lived so long ago, and many records were destroyed in book burnings throughout dynasties in ancient China. He was considered one of the most accomplished Chinese mathematicians of his time (*From* Math History Wiki).

creative proofs of “the area formula of a trapezoid” require to synthesise various geometry knowledge, which also place high cognitive demands on students (Sun, 2008). From this perspective, multiple proof task of a trapezoid would provide a flexible demand for prospective teachers of all levels.

Figure 1. The example introducing three explanations of trapezoid area formula in Chinese textbook (Mathematics Textbook Developer Group for Elementary School, 2003, p.88)

THE STUDY

To explore a different cognitive demand and its effect on mathematical performance, two classroom environments were developed; one, henceforth, referred to as a regular environment (RE) with a right solution demand in accordance with the naturalistic time setting for a regular problem-solving activity of 20 minutes, and the other was the experimental environment (EE) with multiple-solutions demand corresponding time setting of three hours.

RESEARCH QUESTION

What are the differences of “mathematical performance” under different conditions, namely, regular environment versus experimental environment? “Mathematical performance” refers the quality (the structure extent of multiple proving) and quantities of proving (numbers of multiple proofs).

DATA SOURCE

Our study is set in the context of a four-year program to obtain the degree in mathematics education. In particular, our experiment was conducted within a one-semester course of mathematics teaching methods, addressed to 11/18 third-year prospective secondary teachers respectively whose entrance performance with same examination are not significant difference. In the pilot study, we found that two groups of prospective teachers held limited conception of problem solving and bounded knowledge of classroom environments for high cognitive development, and none of them had tried to take two hours to solve a single problem. Due to tight time arrangement, there is one 3-hours-task in the course, which aims to aware their misconception of problem solving with single solution alone by providing experience of multiple solutions and enhance awareness of classroom time conditions for implementation multiple solution tasks[3].

ANALYSES

As part of a larger project (Sun & Chan, 2009), the research employed both quantitative and qualitative methods. To examine the quantity and quality of mathematical performance under different environments, we analyzed proof space, the collections of proofs of a statement produced by the individuals or groups (Leikin, 2009) by the SOLO assessment tool. The SOLO taxonomy developed by Biggs & Collis (1982) describes the different quality in terms of the structure of observed learning outcomes and are claimed to be applicable to any subject areas of increasing complexity in performance by a student. Practical data strongly support the validity of the tool. Specially, this tool was suggested to be applied to the area of mathematical problem solving (Collis, Romberg & Jurdak, 1986).

The following quality standards of proof performance of "the area formula of a trapezoid" were established according to the SOLO framework: A *pre-structural* proof might outline the proofs without valid methods. A *unistructural* proof might outline the proofs with separating or compensating methods. A *multistructural* proof might outline the proofs with both separating and compensating methods, but two methods are never brought together. A *relational* proof will prove the formula by an integrated separating and compensating methods. An *extended abstract* proof would cover the ground of the

[3] A following discussion on how a teacher could develop learning opportunity for multiple solutions and proving in classrooms was arranged at next lesson.

relational proof, but then, it might theorize the various methods mentioned above. The corresponding proof examples appear in the followings.

0. Pre--structural proof

A *pre-structural* proof might outline the proofs without valid methods. The *pre-structural* proof in Figure 2 refers to the isosceles trapezoid proving only.

$$h \cdot \left(\frac{b-a}{4}\right) + h \cdot \left(\frac{b-a}{4}\right) + ah$$
$$= h \cdot \frac{(2b-2a)}{4} + \frac{4ah}{4}$$
$$= \frac{(a+b)}{2} h$$

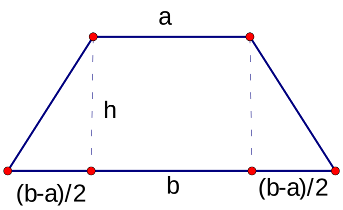
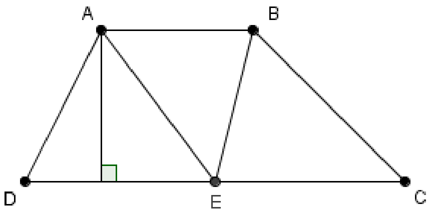


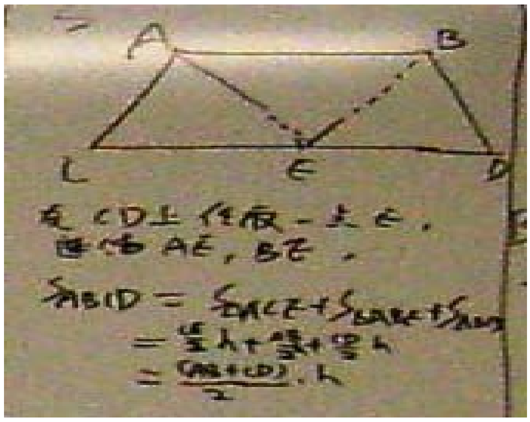
Figure 2. A *pre-structural* proof might outline the proofs without valid methods.

1. Uni-structural proof

A *uni-structural* proof might outline the proofs with separating or compensating methods. The proofs with separating methods require re-organizing a quadrangle or triangle within a trapezoid. The proofs with compensating methods require re-organizing a polygon outside of the trapezoid.

Proofs with separating methods:





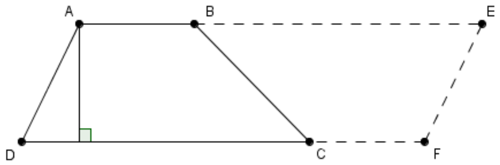
E is the midpoint of CD.
Connect AE and BE. So,

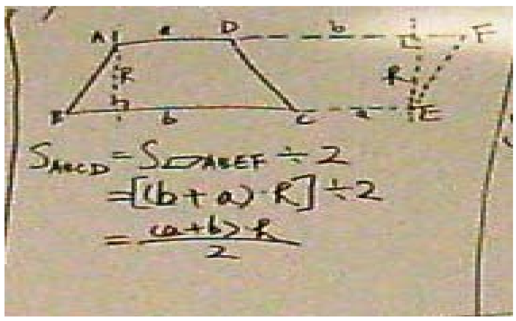
$$S_{ABCD} = S_{\Delta ADE} + S_{\Delta ABE} + S_{\Delta BCE}$$
$$= \frac{1}{2}\cdot\frac{b}{2}\cdot h + \frac{ah}{2} + \frac{1}{2}\cdot\frac{b}{2}\cdot h = \frac{(a+b)h}{2}$$

Figure 3. The trapezoid is divided into 3 triangles.

COMMENT: The trapezoid is divided into 3 triangles. The key point of the method is finding of midpoint, which make proving simple .Of course, any a point on the line DC is an available too.

Proofs with compensating methods:





Extend AB to E, so as to $BE = CD$.
Extend DC to F, so as to $CF = AB$.
Then $AE = FD$ and $AE \parallel FD$.
So AEFD is a parallelogram.

$$S_{ABCD} = \frac{1}{2}S_{AEFD} = \frac{(a+b)h}{2}$$

Figure 4. A trapezoid is reorganized into a parallelogram by copying the same trapezoid.

2. Multi-structural proof

A multi-structural proof might outline the proofs with both separating and compensating methods above, but never bring two methods together.

3. Relational proof

A relational proof will prove the formula by combining two methods above (separating methods, compensating methods) (*cf.* Figure 7).

E is midpoint of BC. Connect AE. F is the intersection of extended line DC and extended line AE.

$$\begin{cases} \angle ABE = \angle FCE \\ BE = CE \\ \angle BEA = \angle CEF \end{cases} \Rightarrow \Delta ABE \cong \Delta FCE \Rightarrow \begin{cases} AB = FC \\ S_{\Delta ABE} = S_{\Delta FCE} \end{cases}$$

Then

$$S_{ABCD} = S_{\Delta ADF} = \frac{(a+b)h}{2}$$

COMMENT: A trapezoid is skillfully transformed into a triangle with same area by replacing ΔABE by ΔFCE. It is a creative proving.

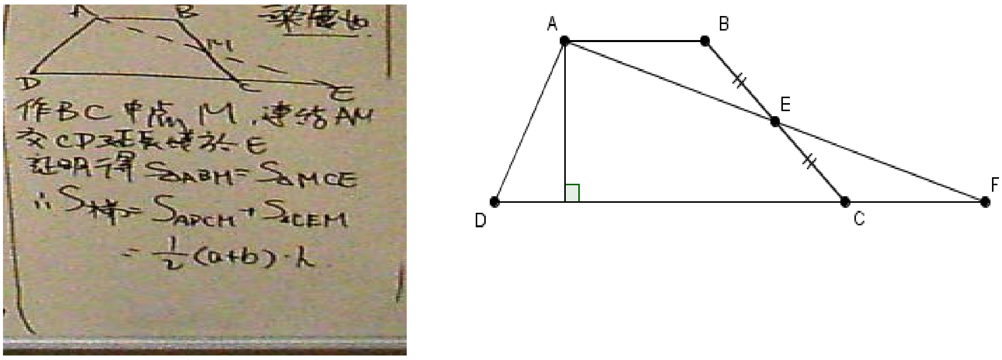
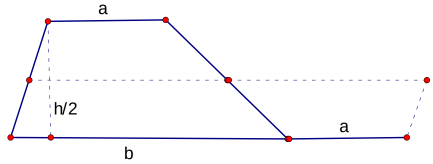


Figure 5. A relational proof combining separating methods with compensating methods

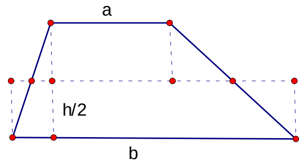
4. Extended abstract proof

An extended abstract proof would cover the ground of the relational proof, but then might synthesize various methods (either separating, or compensating). The following extended abstract proofs are developed by reverse thinking in a more advanced way, namely, constructing a figure from algebraic formula, 1/2*h* (*a*+*b*).

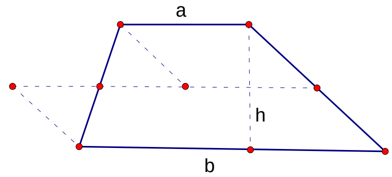
$$S = \frac{1}{2}(a+b)h$$



$$S = a \cdot \frac{h}{2} + b \cdot \frac{h}{2} = \frac{1}{2}(a+b)h$$



$$S = a \cdot \frac{h}{2} + b \cdot \frac{h}{2} = \frac{1}{2}(a+b)h$$



$$S = \frac{1}{2}(a+b)h$$

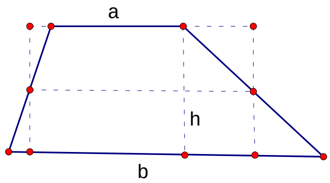


Figure 6. Extended abstract proofs

THE TASK

We chose "the area formula of a trapezoid" in the study as the proof task for the reason that there is rich proof from simple ones to complex ones, which easily detect diversity of understanding performance. Task 1 presented to the participants of 14 third-year prospective secondary teachers was as follows:

> Generate a right proving for "the area formula of a trapezoid" and then write down your proving in the worksheet in twenty minutes.

We collected the worksheets after 20 minutes following their distribution. Participants of 18 third-year prospective secondary teachers were invited to prove task 2 in three hours. Task 2 presented to the participants of the study was as follows:

> *Go on to generate your multiple proving for "the area formula of a trapezoid" as well as you can, and then write down multiple solutions in the worksheet.*

Again, the worksheets were collected after 3 hours.

PRELIMINARY RESULTS AND DISCUSSION

Experimental environment is unusual requirement compared with their habit of problem solving. However, it is amazing students totally involved in the proof activity .None quitted in the whole process of three-hours-proving (We found it actually took at least 1 hour to engage in this kind of task).

Performance differences of “mathematical performance” under different conditions are impressive.

1. Prospective teachers in an experimental environment generated much more solutions.

Overall, average number of solutions of prospective teachers in an experimental environment and control environment is 7.6 and 2.1 respectively. The corresponding multiple proof examples appear in Appendix. Maximum / Minimum number of solutions of prospective teachers in an experimental environment and control environment is 22 /4 and 4 /0 respectively (Table 1). Clearly, prospective teachers in an experimental environment generated much more solutions.

Table 1. Distribution of number of solutions in different classroom environments

Different classroom environments (Cognitive demand and time requirement)	Maximum number of solutions	Average number of solutions	Minimum number of solutions
Control environment (One right solution in 20 minutes)	4	2.1	0
Experimental environment (Multiple solutions in three hours)	22	7.6	4

2. Prospective teachers in the experimental environment made more comprehensive performance.

It is impressive that prospective teacher in experimental environment generated 4 relational structure proof and 2 extended abstract proof .None of these examples appear in control environment (Table 2). Prospective teacher in the experimental environment made more comprehensive performance.

Table 2. Distribution of SOLO level of proof space in different classroom environments

Different classroom environments (cognitive demand and time requirement)	Pre-structural level of proof space	Uni-structural level of proof space	Multi-structural of proof space	Relational structure level of proof space	Extended abstract structure level of proof space
Control environment (One right solution demand in 20 minutes)	3	5	2	0	0
Experimental environment (Multiple-solutions demand in three hours)	4	9	7	4	2

3. Prospective teachers in experimental environment produce much more complex structure of proof space performance than those in regular environment.

The most complex structure level of proof space in group level in experimental environment and control environment is E (prove by combing separating methods with compensating) / M (prove by either separating or compensating methods above, but never bring two methods together) respectively. The mode structure level of proof space in individual level in experimental environment and control environment is M/ U respectively (Table 3).

Prospective teachers in experimental environment had better performance is obvious.

Table 3. Distribution of SOLO structural level of proof space in group/ individual level in different classroom environments

Different classroom environments (Cognitive demand and time requirements)	The most complex structure level of proof space in group level	The mode structure level of proof space in individual level
Control environment (One right solution demand in 20 minutes)	M	U
Experimental environment (Multiple-solutions demand in three hours)	E	M

The results above indicated that different cognitive demands shape different efforts and predetermine different cognitive opportunities under environment set. More important, traditional classroom environment relying totally on conceptions of proving solving—in one-way afford limited and fragmented conditions for multiple solution tasks is identified.

SIGNIFICANCE OF THE STUDY

This study simply compares the effects of different solution demands and corresponding time settings on understanding performances of a proof task. Despite effects of the pervious mathematical background between two groups, the findings indicate that, consistent with the existing literature (Silver & Stein, 1996), there are significant differences were found in the quantity and quality of mathematical performance between two classroom environment group. The experimental environment (multiple-solution corresponding time of three hours) are much more conducive to deep understanding than those of the regular environment (20 minutes with a right solution demand). This is to say that our regular classroom environment is, in fact, less conducive to deep understanding than the experimental environment. The results are useful in providing policy of environmental conditions of curriculum and assessment for development of high level cognition. Findings of this study suggest task requirements are linked to levels of performance and engagement in mathematics, which conclude that results of prior study on multiple-solution problems as a basic task selection and implementation mechanism (e.g. Stein, Smith, Henningsen & Silver, 2000) may be generalized to cover new domains (proof) and a new age group (prospective teachers).

Current study is first step in providing experimental evidence to the cognitive benefits of multiple-solution demand and corresponding time setting. In light of our discussion, it is not surprising that the experimental environment shows an advantage absent from the regular environment. This present paper may provide an indication to expose a specific environment, multiple-solution demand, and corresponding time setting in classroom practice, which might enable us to see which parts of the educational environment, can be improved. We often take for grant to develop our classroom environment by our intuitions, experiences and common senses (*e.g.*, Regular mathematical class time in mainland China, Hong Kong, and Macao is within 1 hour. Regular mathematical requirement is, rather than multiple solutions, a single solution). This study reminds us how our policies of setting task requirement and time could or could not develop understanding performance in classrooms. Future work will be required to further identify the potentials of affective benefits (*e.g.*, persistence, flexibility). The future investigation how our time requirement and solution requirement would shape student thinking habit in a long run are needed.

Similar results had been found with “Mid-Point theorem of triangles” (Sun & Chan, 2009), “area formula of a triangle”, “the intercept theorem” or Thales’ theorem. Future work will be required to further identify the extent of the conclusions. The more rigor experiment studies with control group (either using secondary students or using other multiple solutions tasks) as sample are also recommended.

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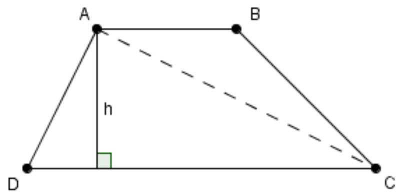
APPENDIX
MULTIPLE PROOF EXAMPLES

1. Method of Chan: Creating two triangles:

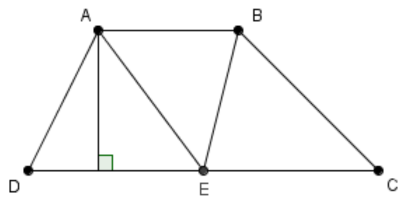


Connect AC. The triangle ΔABC and ΔACD have the same height h. So

$$S_{ABCD} = S_{\Delta ABC} + S_{\Delta ACD}$$
$$= \frac{ah}{2} + \frac{bh}{2} = \frac{(a+b)h}{2}$$

COMMENT: This is a simplest proving method among all methods presented by the text-books of different countries.

2. Method of Bin: Creating three triangles:



E is the midpoint of CD. Connect AE and BE. So,

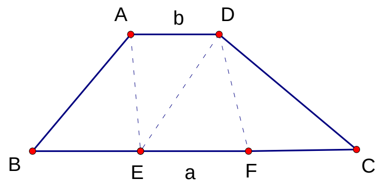
$$S_{ABCD} = S_{\Delta ADE} + S_{\Delta ABE} + S_{\Delta BCE}$$
$$= \frac{1}{2} \cdot \frac{b}{2} \cdot h + \frac{ah}{2} + \frac{1}{2} \cdot \frac{b}{2} \cdot h = \frac{(a+b)h}{2}$$

COMMENT: The trapezoid is divided into 3 triangles. The key point of the method is finding of midpoint, which make proving simple .Of course, any a point on the line DC is an available too.

3. Method of Hau: Creating multiple triangles:

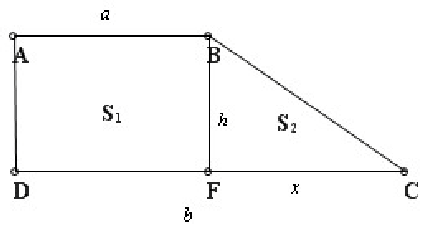
E is the 1/m point of CD (divide CD into m equal sections). Connect AE and BE. So,

$$S_{ABCD} = 1/2\ 1/m\ a\ h\ m + 1/2bh = \frac{(a+b)h}{2}$$



COMMENT: This is an extended method from 2 sections to m sections above.

4. Method of Lin: Creating into a triangle and a rectangle:



If it is a right-angle one, the trapezoid can be divided into a triangle and a rectangle by its height. Suppose

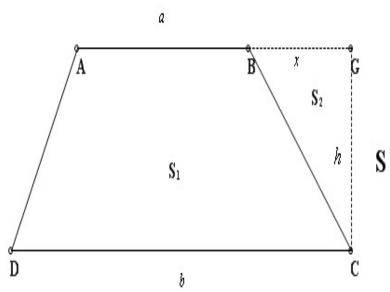
$$AB = a, \quad CD = b, \quad AD = h$$

$$S_2 = \frac{(b-a)h}{2}$$

$$S = S_1 + S_2 = ah + \frac{(b-a)h}{2} = \frac{2ah + bh - ah}{2} = \frac{(a+b)}{2}h$$

COMMENT: A right-angle trapezoid can be divided into a triangle and a rectangle. The formula of a right-angle trapezoid is proved.

If it is not a right-angle one, the trapezoid can be made a right-angle one by drawing its height. Suppose BG = x, then



$$S_{right-angle-trapzoid} = \frac{(a+x+b)}{2}h$$

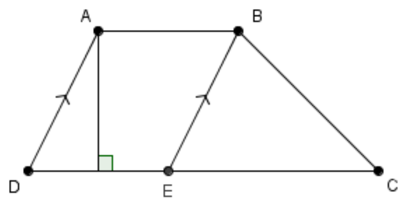
$$S_{\Delta} = \frac{x}{2}h$$

$$S_{trapzoid} = S_{right-angle-trapzoid} - S_{\Delta}$$

$$= \frac{(a+x+b)}{2}h - \frac{x}{2}h - = \frac{(a+b)}{2}h$$

COMMENT: A trapezoid can be divided into a triangle and a right-angle trapezoid. Based on the conclusion on the right-angle trapezoid above, the formula of a trapezoid is proved. This solution includes two steps from particular one to general one.

5. Method of Xiang: Creating into a triangle and a parallelogram:

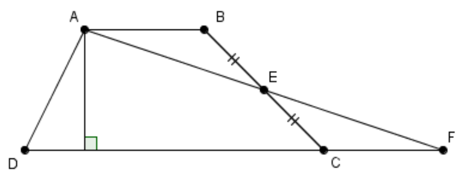


Draw BE//AD. E is the intersection of CD and BE.
Then ABED is a parallelogram because $DE = AB = a$,

$$S_{ABCD} = S_{ABED} + S_{\Delta BCE}$$
$$= ah + \frac{(b-a)h}{2} = \frac{(a+b)h}{2}$$

COMMENT: A trapezoid is divided into a triangle and a parallelogram.

6. Method of Zhu: Re-shaping into a triangle by cutting :



E is midpoint of BC. Connect AE. F is the intersection of extended line DC and extended line AE.

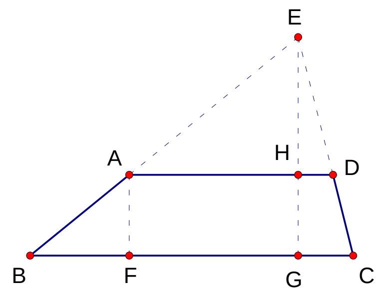
$$\begin{cases}\angle ABE = \angle FCE \\ BE = CE \\ \angle BEA = \angle CEF\end{cases} \Rightarrow \Delta ABE \cong \Delta FCE \Rightarrow \begin{cases} AB = FC \\ S_{\Delta ABE} = S_{\Delta FCE}\end{cases}$$

Then

$$S_{ABCD} = S_{\Delta ADF} = \frac{(a+b)h}{2}$$

COMMENT: A trapezoid is skilfully transformed into a triangle with same area by replacing ΔABE by ΔFCE. It is a creative proving.

7. Method of Chang: Re-shaping into a triangle by extending:



Extend BA and DC. E is intersection of BA and CD. Draw height EG and height AF. G is the intersection of EG and BC. F is the intersection of AF and BC.

Because $AD // BC$, the triangle EAD is similar to the triangle EBC,

$$\frac{EH}{EG} = \frac{EH}{h + EH} = \frac{AD}{BC} = \frac{a}{b}$$

Then

$$EH = \frac{ah}{b - a}$$

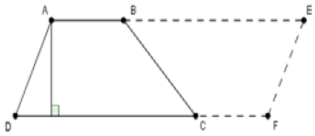
$$\begin{aligned} S_{ABCD} &= S_{\Delta EBC} - S_{\Delta EAD} \\ &= \frac{a(h + EH)}{2} - \frac{bEH}{2} = \frac{(a+b)h}{2}\end{aligned}$$

COMMENT: A trapezoid is extended into a triangle by extending its two sides. The EH was eliminated according to the property of the similar triangle.

8. Method of Xian: Re-shaping into a triangle by extending:

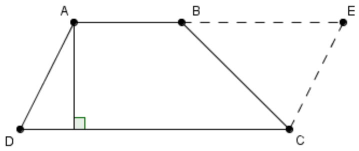
Extend AB to E, so as to $BE = CD$. Extend DC to F, so as to $CF = AB$. Then $AE = FD$ and $AE // FD$. So AEFD is a parallelogram.

$$S_{ABCD}=\frac{1}{2}S_{AEFD}=\frac{(a+b)h}{2}$$



COMMENT: A trapezoid is reorganized into a parallelogram by copying the same trapezoid.

9. Method of Feng: Re-shaping into a parallelogram by extending :

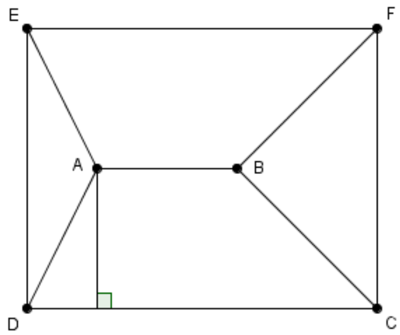


Draw $CE // DA$ such that line CE passes through point E. Then we have AECD is a parallelogram.

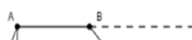
$$S_{ABCD}=S_{AECD}-S_{\Delta BEC}$$
$$=bh-\frac{(b-a)h}{2}=\frac{(a+b)h}{2}$$

COMMENT: A trapezoid is reorganized into a parallelogram by making a parallel line of a side.

10. Method of Wei rectangles -re-shaping into two rectangles by connecting:



Draw the symmetry points E and F of D and C based on symmetry axis AB.
CDEF is a rectangle, and $CD=EF=b$, $CF=DE=2h$, So



$$\begin{aligned} S_{ABCD} &= \frac{1}{2}\left(S_{CDEF} - S_{\Delta ADE} - S_{\Delta BCF}\right) \\ &= \frac{1}{2}\left[2bh - \frac{(b-a)\cdot 2h}{2}\right] = \frac{(a+b)h}{2} \end{aligned}$$

COMMENT: The two same trapezoids are reorganized into a rectangle by making a symmetry figure.

The End