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COMPARISON RESULTS FOR THE PRECONDITIONED GAUSS-SEIDEL METHODS

Jae Heon Yun

Abstract. In this paper, we provide comparison results of several types of the preconditioned Gauss-Seidel methods for solving a linear system whose coefficient matrix is a Z-matrix. Lastly, numerical results are presented to illustrate the theoretical results.

1. Introduction

In this paper, we consider the following linear system

$$Ax = b, \quad x, b \in \mathbb{R}^n, \tag{1}$$

where $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is a nonsingular matrix. Throughout the paper, we always assume that $A = I - L - U$, where I is the identity matrix, L and U are strictly lower and strictly upper triangular matrices, respectively. The basic iterative method for solving the linear system (1) is

$$Mx_{k+1} = Nx_k + b, \; k = 0, 1, \ldots, \tag{2}$$

where x_0 is an initial vector, $A = M - N$ and M is nonsingular. Then (2) can be also written as

$$x_{k+1} = M^{-1}Nx_k + M^{-1}b, \; k = 0, 1, \ldots, \tag{3}$$

where $M^{-1}N$ is called an iteration matrix of the iterative method (3).

We now transform the original linear system (1) into the preconditioned linear system

$$PAx = Pb, \tag{4}$$

where P is called a preconditioner. If we apply the Gauss-Seidel method to the preconditioned linear systems (4), then we obtain the *preconditioned Gauss-Seidel method* for solving the linear system (1). The preconditioned Gauss-Seidel method has been studied by many authors [2, 3, 4, 5, 6, 8, 9, 11, 12].

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In 1991, Gunawardena et al. [3] proposed the preconditioner $P_s = I + S$, where

$$S = \begin{pmatrix} 0 & -a_{12} & 0 & \cdots & 0 \\ 0 & 0 & -a_{23} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -a_{n-1,n} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

In 2001, Evans et al. [2] proposed the preconditioner $P_1 = I + R_1$, where

$$R_1 = \begin{pmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ -a_{n1} & \cdots & 0 & 0 \end{pmatrix}.$$

In 2004, Niki et al. [8] proposed the preconditioner $P_r = I + R$, where

$$R = \begin{pmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ -a_{n1} & \cdots & -a_{n,n-1} & 0 \end{pmatrix}.$$

This paper is organized as follows. In Section 2, we present some notation, definitions and preliminary results which we refer to later. In Section 3, we provide comparison results of several types of the preconditioned Gauss-Seidel methods for solving the linear system (1) whose coefficient matrix is a Z-matrix. In Section 4, we provide numerical results to illustrate the theoretical results obtained in Section 3.

2. Preliminaries

For a vector $x \in \mathbb{R}^n$, $x \geq 0$ ($x > 0$) denotes that all components of x are nonnegative (positive). For two vectors $x, y \in \mathbb{R}^n$, $x \geq y$ ($x > y$) means that $x - y \geq 0$ ($x - y > 0$). For a vector $x \in \mathbb{R}^n$, $|x|$ denotes the vector whose components are the absolute values of the corresponding components of x. These definitions carry immediately over to matrices. A matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is called a *Z-matrix* if $a_{ij} \leq 0$ for $i \neq j$, and A is called an *M-matrix* if A is a Z-matrix and $A^{-1} \geq 0$. For a square matrix A, $\rho(A)$ denotes the spectral radius of A, and A is called *irreducible* if the directed graph of A is strongly connected [10].

A representation $A = M - N$ is called a *splitting* of A when M is nonsingular. A splitting $A = M - N$ is called *regular* if $M^{-1} \geq 0$ and $N \geq 0$, *weak regular* if $M^{-1} \geq 0$ and $M^{-1}N \geq 0$, and *an M-splitting* of A if M is an M-matrix and $N \geq 0$. A splitting $A = M - N$ is called the *Gauss-Seidel splitting* of A if M and $-N$ are lower triangular and strictly upper triangular parts of A, respectively. Some useful results which we refer to later are provided below.

Theorem 2.1 ([1]). *Let $A \geq 0$ be a matrix. Then the following hold.*

(a) *If $Ax \geq \beta x$ for a vector $x \geq 0$ and $x \neq 0$, then $\rho(A) \geq \beta$.*

(b) *If $Ax \leq \gamma x$ for a vector $x > 0$, then $\rho(A) \leq \gamma$. Moreover, if A is irreducible and if $\beta x \leq Ax \leq \gamma x$, equality excluded, for a vector $x \geq 0$ and $x \neq 0$, then $\beta < \rho(A) < \gamma$ and $x > 0$.*

Lemma 2.2 ([5]). *Let $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ be an irreducible M-matrix with $a_{i,i+1} \neq 0$ for $1 \leq i \leq n-1$, and let $A_s = (I+S)A = M_s - N_s$ be the Gauss-Seidel splitting of A_s. Then $M_s^{-1}N_s$ has a positive Perron vector and $\rho(M_s^{-1}N_s) > 0$.*

Lemma 2.3 ([6]). *Let A be an M-matrix and let $A_s = (I+S)A = M_s - N_s$ be the Gauss-Seidel splitting of A_s. If $\rho(M_s^{-1}N_s) > 0$, then $Ax \geq 0$ for any nonnegative Perron vector of $M_s^{-1}N_s$.*

Lemma 2.4 ([7]). *Suppose that $A_1 = M_1 - N_1$ and $A_2 = M_2 - N_2$ are weak regular splittings of the monotone matrices A_1 and A_2, respectively, such that $M_2^{-1} \geq M_1^{-1}$. If there exists a positive vector x such that $0 \leq A_1 x \leq A_2 x$, then for the monotonic norm associated with x*

$$\|M_2^{-1}N_2\|_x \leq \|M_1^{-1}N_1\|_x.$$

In particular, if $M_1^{-1}N_1$ has a positive Perron vector, then

$$\rho(M_2^{-1}N_2) \leq \rho(M_1^{-1}N_1).$$

3. Comparison results for preconditioned Gauss-Seidel methods

In this section, we provide comparison results of several types of the preconditioned Gauss-Seidel methods for solving the linear system (1). We assume that $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ is a Z-matrix with $a_{n1} \neq 0$ and $a_{i,i+1} \neq 0$ for $1 \leq i \leq n-1$. For simplicity of exposition, let

$$\begin{array}{llll} P_s = I + S, & P_1 = I + R_1, & P_{s1} = I + S + R_1, & P_r = I + R, \\ A_s = P_s A, & A_1 = P_1 A, & A_{s1} = P_{s1} A, & A_r = P_r A. \end{array}$$

Let the Gauss-Seidel splittings of A, A_s, A_1, A_{s1} and A_r be defined by

$$A = M - N,\ A_s = M_s - N_s,\ A_1 = M_1 - N_1,\ A_{s1} = M_{s1} - N_{s1},\ A_r = M_r - N_r.$$

Let $SL = \Lambda_0 + E_0$, $R_1 U = \Lambda_1 + E_1$ and $RU = \Lambda_2 + E_2$, where Λ_0, Λ_1 and Λ_2 are diagonal matrices, and E_0, E_1 and E_2 are strictly lower triangular matrices. By simple calculation, one obtains

$$\begin{array}{ll} M = I - L, & N = U, \\ M_s = (I - \Lambda_0) - (L + E_0), & N_s = U - S + SU, \\ M_1 = (I - \Lambda_1) - (L - R_1 + E_1), & N_1 = U, \\ M_{s1} = (I - \Lambda_0 - \Lambda_1) - (L - R_1 + E_0 + E_1), & N_{s1} = U - S + SU, \\ M_r = (I - \Lambda_2) - (L - R + RL + E_2), & N_r = U. \end{array}$$

Notice that $N = N_1 = N_r = U$ and $N_s = N_{s1} = U - S + SU$. Let

$$T = M^{-1}N, \quad T_s = M_s^{-1}N_s, \quad T_1 = M_1^{-1}N_1, \quad T_{s1} = M_{s1}^{-1}N_{s1}, \quad T_r = M_r^{-1}N_r.$$

Then T is an iteration matrix of Gauss-Seidel method, and T_s, T_1, T_{s1} and T_r are iteration matrices of several types of preconditioned Gauss-Seidel methods.

Theorem 3.1. *Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ be a Z-matrix. If $a_{1n}a_{n1} < 1$ and $a_{i,i+1}a_{i+1,i} < 1$ for $1 \le i \le n-1$, then*
(a) $\rho(T_{s1}) < \rho(T)$ if $\rho(T) < 1$,
(b) $\rho(T_{s1}) = \rho(T)$ if $\rho(T) = 1$,
(c) $\rho(T_{s1}) > \rho(T)$ if $\rho(T) > 1$.

Proof. Notice that A_{s1} is also a Z-matrix. Since $a_{1n}a_{n1} < 1$ and $a_{i,i+1}a_{i+1,i} < 1$ for $1 \le i \le n-1$, $A_{s1} = M_{s1} - N_{s1}$ is an M-splitting of A_{s1}. Since $a_{n1} \neq 0$ and $a_{i,i+1} \neq 0$ for $1 \le i \le n-1$, A is irreducible. Since $A = M - N$ is an M-splitting of A and $N \neq 0$, there exists a positive eigenvector x such that $Tx = \lambda x$, where $\lambda = \rho(T) > 0$. From $Tx = \lambda x$ and $R_1L = 0$, one easily obtains

$$\begin{aligned} Ux &= \lambda(I - L)x, \\ SUx &= \lambda(S - \Lambda_0 - E_0)x, \\ R_1Ux &= \lambda R_1 x. \end{aligned} \tag{5}$$

Using (5) and $R_1U = \Lambda_1 + E_1$,

$$\begin{aligned} T_{s1}x - \lambda x &= M_{s1}^{-1}\left(U - S + SU - \lambda(I - \Lambda_0 - \Lambda_1) + \lambda(L - R_1 + E_0 + E_1)\right)x \\ &= M_{s1}^{-1}\left((\lambda-1)S + \lambda(\Lambda_1 + E_1) - \lambda R_1\right)x \\ &= M_{s1}^{-1}\left((\lambda-1)S + \lambda R_1U - \lambda R_1\right)x \\ &= (\lambda-1)M_{s1}^{-1}(S + \lambda R_1)x. \end{aligned} \tag{6}$$

If $\lambda < 1$, then from (6) $T_{s1}x < \lambda x$. Since $x > 0$, Theorem 2.1 implies that $\rho(T_{s1}) < \lambda$. For the cases of $\lambda = 1$ and $\lambda > 1$, $T_{s1}x = \lambda x$ and $T_{s1}x > \lambda x$ are obtained from (6), respectively. Hence, the theorem follows from Theorem 2.1. □

Theorem 3.2. *If $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is an M-matrix, then*

$$\rho(T_{s1}) \le \rho(T_s) < 1.$$

Proof. Since A is an irreducible M-matrix with $a_{i,i+1} \neq 0$, by Lemma 2.2 there exists a positive eigenvector x such that $T_sx = \rho(T_s)x$ and $\rho(T_s) > 0$. Since $N_s = N_{s1}$, $A_{s1} - A_s = R_1A = M_{s1} - M_s$ and thus

$$M_s^{-1} - M_{s1}^{-1} = M_{s1}^{-1}R_1AM_s^{-1}. \tag{7}$$

From (7), one obtains

$$T_s - T_{s1} = M_{s1}^{-1}R_1AT_s. \tag{8}$$

Multiplying by x on both sides of (8) gives

$$\rho(T_s)x - T_{s1}x = \rho(T_s)M_{s1}^{-1}R_1Ax. \tag{9}$$

Since $\rho(T_s) > 0$, from Lemma 2.3

$$Ax \geq 0. \tag{10}$$

From (9) and (10),

$$T_{s1}x \leq \rho(T_s)x. \tag{11}$$

From Theorem 2.1 and (11), it follows that $\rho(T_{s1}) \leq \rho(T_s) < 1$. □

Theorem 3.3. *If $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is an M-matrix, then*

$$\rho(T_{s1}) \leq \rho(T_1) < 1.$$

Proof. We first consider the case where A_1 is an irreducible matrix. Since A is an irreducible M-matrix and $A = M - N$ is a regular splitting of A, there exists a positive eigenvector $x > 0$ such that $Tx = \rho(T)x$ and $\rho(T) > 0$. Since $0 < \rho(T) < 1$, $Ax \geq 0$ and hence

$$A_{s1}x = (I + S + R_1)Ax \geq (I + R_1)Ax = A_1x \geq 0.$$

It is easy to show that A_1 and A_{s1} are M-matrices and $M_{s1}^{-1} \geq M_1^{-1}$. Hence, from Lemma 2.4 $\|T_{s1}\|_x \leq \|T_1\|_x$. Since A_1 is an irreducible M-matrix, T_1 has a positive Perron vector. From Lemma 2.4, it also follows that $\rho(T_{s1}) \leq \rho(T_1)$.

We next consider the case where A_1 is a reducible matrix. Let $A(\epsilon) = (a_{ij}(\epsilon))$ be defined by

$$a_{ij}(\epsilon) = \begin{cases} a_{n1} - \epsilon & \text{if } i = n \text{ and } j = 1 \\ a_{ij} & \text{otherwise,} \end{cases}$$

where $\epsilon > 0$. Let $A_1 = (I + R_1)A = (\bar{a}_{ij})$, $A_1(\epsilon) = (I + R_1)A(\epsilon) = (\bar{a}_{ij}(\epsilon))$, $A_{s1} = (I + S + R_1)A = (\hat{a}_{ij})$ and $A_{s1}(\epsilon) = (I + S + R_1)A(\epsilon) = (\hat{a}_{ij}(\epsilon))$. Then, it can be shown that $\bar{a}_{n1} = \hat{a}_{n1} = 0$,

$$\bar{a}_{ij}(\epsilon) = \begin{cases} -\epsilon & \text{if } i = n \text{ and } j = 1 \\ \bar{a}_{ij} & \text{otherwise} \end{cases}$$

and

$$\hat{a}_{ij}(\epsilon) = \begin{cases} -\epsilon & \text{if } i = n \text{ and } j = 1 \\ \hat{a}_{n-1,1} + \epsilon a_{n-1,n} & \text{if } i = n-1 \text{ and } j = 1 \\ \hat{a}_{ij} & \text{otherwise.} \end{cases}$$

Since A, A_1 and A_{s1} are M-matrices, it can be easily shown that $A(\epsilon)$, $A_1(\epsilon)$ and $A_{s1}(\epsilon)$ are also M-matrices for any sufficiently small $\epsilon > 0$. Since A is irreducible, $A(\epsilon)$ and $A_1(\epsilon)$ are irreducible matrices for any $\epsilon > 0$. Let $M(\epsilon) = A(\epsilon) + N$, $M_1(\epsilon) = A_1(\epsilon) + N_1$ and $M_{s1}(\epsilon) = A_{s1}(\epsilon) + N_{s1}$. Then $A(\epsilon) = M(\epsilon) - N$, $A_1(\epsilon) = M_1(\epsilon) - N_1$ and $A_{s1}(\epsilon) = M_{s1}(\epsilon) - N_{s1}$ are the Gauss-Seidel splittings of $A(\epsilon)$, $A_1(\epsilon)$ and $A_{s1}(\epsilon)$, respectively. Let $T(\epsilon) = M(\epsilon)^{-1}N$, $T_1(\epsilon) = M_1(\epsilon)^{-1}N_1$ and $T_{s1}(\epsilon) = M_{s1}(\epsilon)^{-1}N_{s1}$. Since $A(\epsilon)$ is irreducible and

$A(\epsilon) = M(\epsilon) - N$ is an M-splitting of $A(\epsilon)$, there exists a positive eigenvector x such that $T(\epsilon)x = \rho(T(\epsilon))x$ and $\rho(T(\epsilon)) > 0$. Hence, $A(\epsilon)x \geq 0$, which implies that

$$A_{s1}(\epsilon)x \geq A_1(\epsilon)x \geq 0.$$

It is easy to show that $M_{s1}(\epsilon)^{-1} \geq M_1(\epsilon)^{-1}$. Hence, from Lemma 2.4

$$\|T_{s1}(\epsilon)\|_x \leq \|T_1(\epsilon)\|_x.$$

Since $A_1(\epsilon)$ is irreducible and $A_1(\epsilon) = M_1(\epsilon) - N_1$ is an M-splitting of $A_1(\epsilon)$, $T_1(\epsilon)$ has a positive Perron vector. From Lemma 2.4, it also follows that

$$\rho(T_{s1}(\epsilon)) \leq \rho(T_1(\epsilon)). \tag{12}$$

If $\epsilon \to 0$, then (12) implies that $\rho(T_{s1}) \leq \rho(T_1)$. Hence, the proof is complete. □

Theorem 3.4. *If $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ is an M-matrix, then*

$$\rho(T_1) \leq \rho(T) < 1.$$

Proof. Since A is an irreducible M-matrix and $A = M - N$ is a regular splitting of A, there exists a positive eigenvector $x > 0$ such that $Tx = \rho(T)x$ and $\rho(T) > 0$. Since $N_1 = N$, $M_1 - M = R_1 A$ and thus

$$M^{-1} - M_1^{-1} = M_1^{-1} R_1 A M^{-1}. \tag{13}$$

From (13), one obtains

$$T - T_1 = M_1^{-1} R_1 A T. \tag{14}$$

Multiplying by x on both sides of (14) gives

$$\rho(T)x - T_1 x = \rho(T) M_1^{-1} R_1 A x. \tag{15}$$

Since $0 < \rho(T) < 1$, $Ax \geq 0$. From (15),

$$T_1 x \leq \rho(T)x. \tag{16}$$

From Theorem 2.1 and (16), it follows that $\rho(T_1) \leq \rho(T) < 1$. □

Theorem 3.5. *If $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ is an M-matrix, then*

$$\rho(T_r) \leq \rho(T_1) < 1.$$

Proof. We first consider the case where A_1 is an irreducible matrix. Since A_1 is an M-matrix and $A_1 = M_1 - N_1$ is a regular splitting of A_1, there exists a positive eigenvector $x > 0$ such that $T_1 x = \rho(T_1)x$ and $\rho(T_1) > 0$. Since $N_r = N_1$, $M_r - M_1 = (R - R_1)A$ and thus

$$M_1^{-1} - M_r^{-1} = M_r^{-1}(R - R_1) A M_1^{-1}. \tag{17}$$

From (17), one obtains

$$T_1 - T_r = M_r^{-1}(R - R_1) A T_1. \tag{18}$$

Multiplying by x on both sides of (18) gives

$$\rho(T_1)x - T_r x = \rho(T_1) M_r^{-1}(R - R_1) A x. \tag{19}$$

Since $(R - R_1)(I + R_1)^{-1} = (R - R_1)$, (19) can be transformed into

$$\rho(T_1)x - T_r x = \rho(T_1)M_r^{-1}(R - R_1)A_1 x. \tag{20}$$

Since $0 < \rho(T_1) < 1$, $A_1 x \geq 0$. Since $(R - R_1) \geq 0$, from (20)

$$T_r x \leq \rho(T_1)x. \tag{21}$$

From Theorem 2.1 and (21), it follows that $\rho(T_r) \leq \rho(T_1) < 1$.

We next consider the case where A_1 is a reducible matrix. Let $A(\epsilon) = (a_{ij}(\epsilon))$ be defined by

$$a_{ij}(\epsilon) = \begin{cases} a_{n1} - \epsilon & \text{if } i = n \text{ and } j = 1 \\ a_{ij} & \text{otherwise,} \end{cases}$$

where $\epsilon > 0$. Let $A_1 = (I + R_1)A = (\bar{a}_{ij})$, $A_1(\epsilon) = (I + R_1)A(\epsilon) = (\bar{a}_{ij}(\epsilon))$, $A_r = (I + R)A = (\tilde{a}_{ij})$ and $A_r(\epsilon) = (I + R)A(\epsilon) = (\tilde{a}_{ij}(\epsilon))$. Then $\bar{a}_{ij}(\epsilon)$ is defined the same as in the proof of Theorem 3.3, and

$$\tilde{a}_{ij}(\epsilon) = \begin{cases} \tilde{a}_{n1} - \epsilon & \text{if } i = n \text{ and } j = 1 \\ \tilde{a}_{ij} & \text{otherwise.} \end{cases}$$

Since A and A_1 are M-matrices, $A(\epsilon)$ and $A_1(\epsilon)$ are also M-matrices for any sufficiently small $\epsilon > 0$. Since A is irreducible, $A(\epsilon)$ and $A_1(\epsilon)$ are also irreducible for any $\epsilon > 0$. Let $M_1(\epsilon) = A_1(\epsilon) + N_1$ and $M_r(\epsilon) = A_r(\epsilon) + N_r$. Then $A_1(\epsilon) = M_1(\epsilon) - N_1$ and $A_r(\epsilon) = M_r(\epsilon) - N_r$ are the Gauss-Seidel M-splittings of $A_1(\epsilon)$ and $A_r(\epsilon)$, respectively. Let $T_1(\epsilon) = M_1(\epsilon)^{-1}N_1$ and $T_r(\epsilon) = M_r(\epsilon)^{-1}N_r$. In a similar manner as was done in the first case, one can obtain

$$\rho(T_r(\epsilon)) \leq \rho(T_1(\epsilon)). \tag{22}$$

If $\epsilon \to 0$, then (22) implies that $\rho(T_r) \leq \rho(T_1)$. Hence, the proof is complete. □

Combining Theorems 3.2 to 3.5, the following corollary is obtained.

Corollary 3.6. *If $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is an M-matrix, then*

(a) $\rho(T_r) \leq \rho(T_1) \leq \rho(T) < 1$,
(b) $\rho(T_{s1}) \leq \rho(T_1) \leq \rho(T) < 1$,
(c) $\rho(T_{s1}) \leq \rho(T_s) \leq \rho(T) < 1$.

4. Numerical results

In this section, we provide numerical results for the preconditioned Gauss-Seidel methods to illustrate the theoretical results obtained in Section 3. All test matrices $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ satisfy the assumptions given in Section 3, that is, $a_{n1} \neq 0$, $a_{i,i+1} \neq 0$ $(1 \leq i \leq n-1)$ and A is an M-matrix. All spectral radii for iteration matrices of preconditioned Gauss-Seidel methods are computed using MATLAB. All notations are defined the same as in Section 3.

Example 4.1. Consider a 5×5 matrix A of the form

$$A = \begin{pmatrix} 1 & -0.1 & -0.2 & 0 & -0.1 \\ 0 & 1 & -0.2 & -0.1 & 0 \\ -0.2 & 0 & 1 & -0.1 & -0.2 \\ -0.1 & -0.2 & 0 & 1 & -0.1 \\ -0.2 & 0 & -0.1 & -0.2 & 1 \end{pmatrix}.$$

Spectral radii for iteration matrices of preconditioned Gauss-Seidel methods are listed in Table 1. From Table 1, it can be seen that all comparison results in Section 3 are satisfied. For this matrix A, the following holds:

$$\rho(T) > \rho(T_1) > \rho(T_r) > \rho(T_s) > \rho(T_{s1}). \tag{23}$$

Table 1. Spectral radii for iteration matrices of preconditioned Gauss-Seidel methods for Example 4.1

$\rho(T)$	$\rho(T_1)$	$\rho(T_r)$	$\rho(T_s)$	$\rho(T_{s1})$
0.2367	0.1932	0.1809	0.1625	0.1055

Example 4.2. Consider a 5×5 matrix A of the form

$$A = \begin{pmatrix} 1 & -0.2 & -0.3 & -0.2 & -0.2 \\ -0.1 & 1 & -0.2 & -0.3 & -0.1 \\ 0 & 0 & 1 & -0.1 & -0.2 \\ -0.1 & 0 & 0 & 1 & -0.3 \\ -0.3 & 0 & -0.1 & 0 & 1 \end{pmatrix}.$$

Spectral radii for iteration matrices of preconditioned Gauss-Seidel methods are listed in Table 2. From Table 2, it can be seen that all comparison results in Section 3 are satisfied. For this matrix A, the following holds:

$$\rho(T) > \rho(T_s) > \rho(T_1) > \rho(T_r) > \rho(T_{s1}). \tag{24}$$

Table 2. Spectral radii for iteration matrices of preconditioned Gauss-Seidel methods for Example 4.2.

$\rho(T)$	$\rho(T_1)$	$\rho(T_r)$	$\rho(T_s)$	$\rho(T_{s1})$
0.3594	0.2863	0.2591	0.2987	0.2251

Example 4.3. Consider a 5×5 matrix A of the form

$$A = \begin{pmatrix} 1 & -0.1 & -0.4 & -0.2 & -0.2 \\ 0 & 1 & -0.1 & -0.4 & -0.2 \\ -0.2 & 0 & 1 & -0.1 & -0.6 \\ 0 & -0.1 & 0 & 1 & -0.8 \\ -0.3 & -0.2 & -0.1 & -0.3 & 1 \end{pmatrix}.$$

Spectral radii for iteration matrices of preconditioned Gauss-Seidel methods are listed in Table 3. From Table 3, it can be seen that all comparison results in Section 3 are satisfied. For this matrix A, the following holds:

$$\rho(T) > \rho(T_s) > \rho(T_1) > \rho(T_{s1}) > \rho(T_r). \tag{25}$$

TABLE 3. Spectral radii for iteration matrices of preconditioned Gauss-Seidel methods for Example 4.3.

$\rho(T)$	$\rho(T_1)$	$\rho(T_r)$	$\rho(T_s)$	$\rho(T_{s1})$
0.7949	0.7661	0.7209	0.7691	0.7366

Notice that $\rho(T_s) < \rho(T_1)$ and $\rho(T_s) < \rho(T_r)$ for Example 4.1, but $\rho(T_s) > \rho(T_1)$ and $\rho(T_s) > \rho(T_r)$ for Examples 4.2 and 4.3. Also notice that $\rho(T_{s1}) < \rho(T_r)$ for Examples 4.1 and 4.2, but $\rho(T_{s1}) > \rho(T_r)$ for Example 4.3. Hence, it can be concluded from Examples 4.1 to 4.3 that there exist no comparison results between $\rho(T_s)$ and $\rho(T_1)$, between $\rho(T_s)$ and $\rho(T_r)$, and between $\rho(T_{s1})$ and $\rho(T_r)$ under the same assumptions used in Section 3.

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DEPARTMENT OF MATHEMATICS
COLLEGE OF NATURAL SCIENCES
CHUNGBUK NATIONAL UNIVERSITY
CHEONGJU 361-763, KOREA
E-mail address: `gmjae@chungbuk.ac.kr`