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REMARKS ON CS-STARCOMPACT SPACES

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Abstract. A space X is *cs-starcompact* if for every open cover $\mathcal{U}$ of X, there exists a convergent sequence S of X such that $St(S,\mathcal{U}) = X$, where $St(S,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap S \neq \emptyset\}$. In this paper, we prove the following statements:

(1) There exists a Tychonoff cs-starcompact space having a regular-closed subset which is not cs-starcompact;

(2) There exists a Hausdorff cs-starcompact space with arbitrary large extent;

(3) Every Hausdorff centered-Lindelöf space can be embedded in a Hausdorff cs-starcompact space as a closed subspace.

1. Introduction

By a space, we mean a topological space. Let us recall that a space X is *countably compact* if every countable open cover of X has a finite subcover. Fleischman [5] defined a space X to be *starcompact* if for every open cover $\mathcal{U}$ of X, there exists a finite subset F of X such that $St(F,\mathcal{U}) = X$, where $St(F,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap F \neq \emptyset\}$, and he proved that every countably compact space is starcompact. Conversely, van Douwen-Reed-Roscoe-Tree [3] proved that every Hausdorff starcompact space is countably compact, but this does not hold for T_1-space (see [10, Example 2.5]). As generalizations of starcompactness, the following classes of spaces are given:

Definition 1.1. A space X is *cs-starcompact* if for every open cover $\mathcal{U}$ of X, there exists a convergent sequence S of X such that $St(S,\mathcal{U}) = X$.

Definition 1.2 ([9])**.** A space X is *$\mathcal{K}$-starcompact* if for every open cover $\mathcal{U}$ of X, there exists a compact subset K of X such that $St(K,\mathcal{U}) = X$.

Definition 1.3 ([6])**.** A space X is *star-Lindelöf* if for every open cover $\mathcal{U}$ of X, there exists a countable subset F of X such that $St(F,\mathcal{U}) = X$.

In [8], a cs-starcompact space is called star determined by convergent sequence and a $\mathcal{K}$-starcompact space is called star-compact; in [3], a star-Lindelöf

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space is called strongly star-Lindelöf. From the above definitions, it is clear that every star-compact space is cs-starcompact and every cs-starcompact space is both $\mathcal{K}$-starcompact and star-Lindelöf.

Thorough this paper, the cardinality of a set A is denoted by $|A|$. For a space X, the extent $e(X)$ of X is defined as the smallest cardinal number κ such that the cardinality of every discrete closed subset of X is not greater than κ. Let ω denote the first infinite cardinal and $\mathfrak{c}$ the cardinality of the set of all real numbers, For a cardinal κ, let κ^+ be the smallest cardinal greater than κ. As usual, a cardinal is the initial ordinal and an ordinal is the set of smaller ordinals. When viewed as a space, every cardinal has the usual order topology. For each pair of ordinals α, β with $\alpha < \beta$, we write $(\alpha, \beta) = \{\gamma : \alpha < \gamma < \beta\}$. Other terms and symbols that we do not define will be used as in [4].

2. Closed subsets of cs-starcompact spaces

In this section, we construct the example stated in the abstract by using the Example from [8, Theorem 3.5], but we include its construction here for the sake of completeness in the example. For a Tychonoff space X, let βX denote the Čech-Stone compactification of X.

Example 2.1. There exists a Tychonoff cs-starcompact space X having a regular-closed subset Y which is not cs-starcompact.

Proof. Let $\omega^* = \beta\omega \setminus \omega$. Then, every infinite closed set in ω^* has cardinality $2^{\mathfrak{c}}$ and there are $2^{\mathfrak{c}}$-many infinite closed subsets of ω^*. Thus, by a standard transfinite recursion, we can construct a subset G of ω^* such that every compact subset of G is finite and every compact subset of $H = \omega^* \setminus G$ is finite as well, and G and H are countably compact. Pick a countably infinite discrete subspace D of G and let $S_1 = G \setminus (\overline{D} \setminus D)$(where closures are taken in ω^*). Then, D is a closed and discrete subspace of S_1. So, S_1 is not countably compact. However, $S_1 \setminus D$ is countably compact. It was showed that S_1 is not $\mathcal{K}$-starcomopact in [8]. Then, S_1 is not cs-starcompact, since every cs-starcompact space is $\mathcal{K}$-starcomapact. Since D is a infinite discrete closed subset of S_1, we can enumerate D as $\{d_n : n \in \omega\}$.

Let $S_2 = (\omega_1 + 1) \times (\omega + 1) \setminus \{\langle \omega_1, \omega \rangle\}$ be the Tychonoff plank and $D' = \{\langle \omega_1, n \rangle : n \in \omega\}$. Then, D' is an infinite discrete closed set in S_2. Now, we show that S_2 is cs-starcompact. For this end, let $\mathcal{U}$ be an open cover of S_2. For each $n \in \omega$, there exists an $U_n \in \mathcal{U}$ such that $\langle \omega_1, n \rangle \in U_n$, hence there exists an $\alpha_n < \omega_1$ such that $(\alpha_n, \omega_1] \times \{n\} \subseteq U_n$. Let $\alpha' = \sup\{\alpha_n : n \in \omega\}$. Then, $\alpha' < \omega_1$. If we pick $S' = \{\alpha' + 1\} \times (\omega + 1)$, then S' is a convergent sequence of S_1 such that $D' \subseteq St(S', \mathcal{U})$. On the other hand, since $\omega_1 \times (\omega + 1)$ is countably compact, then there exists a finite subset F of $\omega_1 \times (\omega + 1)$ such that $\omega_1 \times (\omega + 1) \subseteq St(F, \mathcal{U})$. If we put $S = S' \cup F$, then S is a convergent sequence of S_2 such that $S_2 = St(S, \mathcal{U})$, which shows that S_2 is cs-starcompact.

Assume $S_1 \cap S_2 = \emptyset$. Let $f : D \to D'$ be a map defined by $f(d_n) = \langle \omega_1, n \rangle$ for each $n \in \omega$. Let X be the quotient space obtained from the discrete sum $S_1 \oplus S_2$

by identifying d_n of D with $\langle \omega_1, n\rangle$ of D' for each $n \in \omega$. Let $\pi : S_1 \oplus S_2 \to X$ be the quotient map.

Let $Y = \pi(S_1)$. It is easy to check that Y is a regular closed in X, however, it is not cs-starcompact, since it is homeomorphic to S_1.

Next, we show that X is cs-starcompact. For this end, let $\mathcal{U}$ be an open cover of X. Since $\pi(S_2)$ is cs-starcompact, then there exists a convergent sequence F_1 of $\pi(S_2)$ such that

$$\pi(S_2) \subseteq St(F_1, \mathcal{U}).$$

On the other hand, since $\pi(S_1 \setminus D)$ is homeomorphic to $S_1 \setminus D$ and $S_1 \setminus D$ is countably compact, then there exists a finite subset F_2 of $\pi(S_1 \setminus D)$ such that $\pi(S_1 \setminus D) \subseteq St(F_2, \mathcal{U})$. Let $S = F_1 \cup F_2$. Then, S is a convergent sequence of X and $X = St(S, \mathcal{U})$. Hence, X is cs-starcompact, which completes the proof. □

Remark 1. The author does not know if there exists a Tychonoff cs-starcompact space having a G_δ-regular closed subset which is not cs-starcompact.

For a space X, recall that the Alexandroff duplicate $A(X)$ of a space X, denoted by $A(X)$, is constructed in the following way: The underlying set of $A(X)$ is $X \times \{0, 1\}$ and each point of $X \times \{1\}$ is isolated; a basic neighbourhood of a point $\langle x, 0\rangle \in X \times \{0\}$ is a set of the form $(U \times \{0\}) \cup ((U \times \{1\}) \setminus \{\langle x, 1\rangle\})$, where U is a neighborhood of x of X. It is well-known that $A(X)$ is compact (countably compact, Lindelöf) if and only if so is X and $A(X)$ is Hausdorff (regular, Tychonoff, normal) if and only if so is X.

Example 2.2. There exists a Tychonoff cs-starcompact space X such that $A(X)$ is not cs-starcompact.

Proof. Let X be the same space as the Tychonoff plank $S_2 = (\omega_1 + 1) \times (\omega + 1) \setminus \{\langle \omega_1, \omega\rangle\}$. Then, X is cs-starcompact.

We show that $A(X)$ is not cs-starcompact. Let $A = \{\{\omega_1\} \times \omega\} \times \{1\}$ and let us consider the open cover $\mathcal{U} = \{A(X) \setminus A\} \cup \{\langle\langle \omega, n\rangle, 1\rangle : n \in \omega\}$ of $A(X)$. Let C be a convergent sequence of $A(X)$. Then, $C \cap A$ is finite, since A is discrete closed in $A(X)$ and C is a convergent sequence. Hence, there exists a $n_0 \in \omega$ such that $\langle\langle \omega, n_0\rangle, 1\rangle \notin C$. Thus $\langle\langle \omega, n_0\rangle, 1\rangle \notin St(C, \mathcal{U})$, since $\{\langle\langle \omega, n_0\rangle, 1\rangle\}$ is the only element of $\mathcal{U}$ containing $\langle\langle \omega, n_0\rangle, 1\rangle$, which completes the proof. □

Next, we give a machine that produces a Hausdorff cs-starcompact space by a star-Lindelöf space. Let X be a star-Lindelöf space with $|X| = \kappa$. Let T be X with the discrete topology and let $Y = T \cup \{\infty\}$, where $\infty \notin T$, be the one-point compactification of T. We define

$$S(X) = X \cup (Y \times \kappa^+)$$

and we topologize $S(X)$ as follows: $Y \times \kappa^+$ has the usual product topology and is an open subspace of $S(X)$, and a basic neighborhood of a point x of X takes the form $G(U, \alpha) = U \cup (U \times (\alpha, \kappa^+))$, where U is a neighborhood of x in X and $\alpha < \kappa^+$.

Theorem 2.3. *Let κ be a cardinal. Let X be a star-Lindelöf space with $|X| = \kappa$. Then, $S(X)$ is cs-starcompact. Moreover, if X is a Hausdorff space, so is $S(X)$.*

Proof. Put $S = S(X)$. Then, it is easy to see that X is a closed subset of S and S is Hausdorff if X is Hausdorff.

We show that S is cs-starcompact. To this end, let $\mathcal{U}$ be an open cover of S. Without loss of generality, we assume that $\mathcal{U}$ consists of basic open sets of S. For each $x \in X$, there exists a $U_x \in \mathcal{U}$ such that $x \in U_x$. Hence, there exist $\alpha_x < \kappa^+$ and an open neighborhood V_x of x in X such that $G(V_x, \alpha_x) \subseteq U_x$. Let $\mathcal{V} = \{V_x : x \in X\}$. Then, $\mathcal{V}$ is an open cover of X. Hence, there exists a countable subset F of X such that $St(F, \mathcal{V}) = X$, since X is star-Lindelöf.

If we put $\alpha_0 = \sup\{\alpha_x : x \in X\}$, then $\alpha_0 < \kappa^+$, since $|X| = \kappa$. Let $F_1 = (F \times \{\alpha_0\}) \cup \{\langle \infty, \alpha_0 \rangle\}$. Then, F_1 is a convergent sequence of $S(X)$ by the construction of the topology of $S(X)$. Since $U_x \cap F_1 \neq \emptyset$ for every $x \in X$, then $X \subseteq St(F_1, \mathcal{U})$.

On the other hand, since Y is compact and κ^+ is countable compact, then $Y \times \kappa^+$ is countable compact, hence there exists a finite subset F_2 of $Y \times \kappa^+$ such that $Y \times \kappa^+ \subseteq St(F_2, \mathcal{U})$. If we put $S' = F_1 \cup F_2$, then S' is a convergent sequence of S and $S = St(S', \mathcal{U})$, which shows that S is cs-starcompact. This completes the proof. □

A family of subsets is *centered* (*linked*) provided every finite subfamily (every two elements, respectively) has nonempty intersection and a family is called *σ-centered* (*σ-linked*) if it is the union of countably many centered subfamilies (linked subfamilies, respectively). A space X is *centered-Lindelöf* (*linked-Lindelöf*) (see [1, 2]) if for every open cover $\mathcal{U}$ of X has σ-centered (σ-linked) subcover. Clearly, every centered-Lindelöf space is linkered-Lindelöf. Bonanzinga and Matveev [1] proved that every Hausdorff (regular, Tychonoff) linked-Lindelöf space can be embedded in Hausdorff (regular, Tychonoff, respectively) star-Lindelöf space as closed subspace. Thus, we get the following corollary by Theorem 2.4.

Corollary 2.4. *Every Hausdorff centered-Lindelöf space can be embedded in a Hausdorff cs-starcompact space as a closed subset.*

Matveev [7] constructed an example of a Tychonoff star-Lindelöf space X with arbitrary large extent. In the following, we construct an example of a Hausdorff cs-starcompact space X with arbitrary large extent by using Matveev's example.

Example 2.5. For every cardinal τ, there exists a Hausdorff cs-starcompact space Y such that $e(Y) \geq \tau$.

Proof. Let $X = (D^\tau \times \omega) \cup (Z \times \{\omega\})$ be the subspace of the product of $D^\tau \times (\omega + 1)$ (see [7]). Then, X is Tychonoff and $e(X) \geq \tau$, since $Z \times \{\omega\}$ is discrete closed in X. Matveev showed that X is star-Lindelöf. Let $Y = S(X)$.

Then, $Z \times \{\omega\}$ is discrete closed in Y by the construction of the topology of Y and X is closed in Y. Hence, $e(Y) \geq \tau$. Clearly, Y is Hausdorff and cs-starcompact by Theorem 2.4, which completes the proof. $\square$

Remark 2. In [8], Mill-Tkachuk-Wilson showed that the Tychonoff plank is cs-starcompact, which shows that the extent of a Tychonoff cs-starcompact space may contain a countably infinite closed discrete subspace. But, the author does not know if there exists a Tychonoff cs-starcompact space X such that $e(X) \geq \mathfrak{c}$.

Remark 3. The author does not know if every Tychonoff star-Lindelöf space can be represented in a Tychonoff cs-starcompact space as a closed subspace.

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