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A NOTE ON COMPACT ALMOST KÄHLER LOCALLY SYMMETRIC SPACES

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Abstract. Concerning the integrability of almost Kähler manifolds, we prove that a compact almost Kähler locally symmetric space is Kähler if the length of the star Ricci tensor is not smaller than that of the Ricci tensor.

1. Introduction

An almost Hermitian manifold $M = (M, J, g)$ is called an almost Kähler manifold if the Kähler form of M is closed. Further, an almost Hermitian manifold $M = (M, J, g)$ is a Kähler manifold if J is integrable and its Kähler form is closed, or equivalently if J is parallel with respect to the Levi-Civita connection ∇ of g. Hence a Kähler manifold is necessarily an almost Kähler manifold. However, in general the converse is not true. During the last few decades, some curvature conditions for the integrability of almost Kähler manifolds were investigated. For instance, T. Oguro proved that an almost Kähler manifold of constant curvature is a flat Kähler manifold [3]. Further, in case of dimension 4, N. Murakoshi, T. Oguro and K. Sekigawa showed that a compact almost Kähler locally symmetric space is a Kähler manifold [2]. In case of dimension $2n(> 4)$, T. Oguro proved that if a compact almost Kähler locally symmetric space M is a weakly $*$-Einstein manifold with non-negative $*$-scalar curvature, then M is a Kähler manifold [4]. Therefore, it is natural to raise a question whether a compact almost Kähler locally symmetric space of dimension $2n(> 4)$ is a Kähler manifold or not. In this note, we shall prove the following:

Theorem 1.1. *Let $M = (M, J, g)$ be a compact almost Kähler locally symmetric space. If the inequality $||\mathrm{Ric}^*|| \geq ||\mathrm{Ric}||$ holds on M, then M is a Kähler manifold.*

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Here Ric* and Ric denote the $*$-Ricci tensor and the Ricci tensor, respectively.

2. Preliminaries

Let $M = (M, J, g)$ be an almost Hermitian manifold (that is, a manifold with almost complex structure J and compatible Riemannian metric g, i.e., $g(JX, JY) = g(X, Y)$ for vector fields X, Y. The Riemannian curvature tensor R, the Ricci tensor Ric and the scalar curvature s are given by

$$R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z,$$
$$\mathrm{Ric}(X,Y) = \mathrm{Trace}\{Z \to R(Z,X)Y\},$$
$$s = \mathrm{Trace}_g \mathrm{Ric},$$

where ∇ is the Levi-Civita connection of g and X, Y, Z are smooth vector fields on M. Furthermore, the $*$-Ricci tensor Ric* and $*$-scalar curvature s^* of (J, g) are given by

$$\mathrm{Ric}^*(X,Y) = \mathrm{Trace}\{Z \to -JR(Z,X)JY\},$$
$$s^* = \mathrm{Trace}_g \mathrm{Ric}^*.$$

By using the first Bianchi identity, we have

$$\mathrm{Ric}^*(X,Y) = \frac{1}{2}\mathrm{Trace}\{Z \to R(X,JY)JZ\}. \tag{2.1}$$

Therefore, the $*$-Ricci tensor Ric* is usually neither symmetric nor skew symmetric and satisfies the identity

$$\mathrm{Ric}^*(JX,JY) = \mathrm{Ric}^*(Y,X).$$

We also regard the curvature tensor R as a (0,4)-tensor as follows:

$$R(X,Y,Z,V) = g(R(X,Y)Z,V).$$

With respect to an orthonormal frame $\{e_i\}_{i=1,\ldots,2n}$, we shall adopt the following notational convention:

$$\begin{aligned}
R_{ijkl} &= R(e_i, e_j, e_k, e_l),\\
\mathrm{Ric}_{ij} &= \mathrm{Ric}(e_i, e_j),\\
\mathrm{Ric}^*_{ij} &= \mathrm{Ric}^*(e_i, e_j),\\
J_{ij} &= g(Je_i, e_j),\\
\nabla_i J_{jk} &= g((\nabla_{e_i} J)e_j, e_k),\\
R_{ijk\bar{l}} &= R(e_i, e_j, e_k, Je_l),\\
R_{\bar{i}\bar{j}\bar{k}\bar{l}} &= R(Je_i, Je_j, Je_k, Je_l),\\
\mathrm{Ric}_{\bar{i}\bar{j}} &= \mathrm{Ric}(Je_i, Je_j),\\
\mathrm{Ric}^*_{\bar{i}\bar{j}} &= \mathrm{Ric}^*(Je_i, Je_j).
\end{aligned}$$

We adopt the summation convention of Einstein, but, as we work with orthonormal frames, there is no need to raise and lower the indices. For instance,

$$\mathrm{Ric}_{jk} = R_{ijki}.$$

Since J is g-orthogonal, we get

$$\begin{aligned} J_{ij} &= -J_{ji} = J_{\bar{i}\bar{j}}, \\ \nabla_i J_{jk} &= -\nabla_i J_{\bar{j}\bar{k}}. \end{aligned} \tag{2.2}$$

An almost Hermitian manifold $M = (M, J, g)$ is called an almost Kähler manifold if the Kähler form Ω defined by $\Omega(X, Y) = g(JX, Y)$ is closed, or equivalently if

$$g((\nabla_X J)Y, Z) + g((\nabla_Y J)Z, X) + g((\nabla_Z J)X, Y) = 0.$$

In terms of components, the above identity is expressed as

$$\nabla_i J_{jk} + \nabla_j J_{ki} + \nabla_k J_{ij} = 0. \tag{2.3}$$

If M is an almost Kähler manifold, then M is a quasi-Kähler manifold, that is,

$$(\nabla_{JX} J)JY + (\nabla_X J)Y = 0$$

holds on M [5] or in terms of components we get

$$\nabla_{\bar{i}} J_{\bar{j}k} + \nabla_i J_{jk} = 0. \tag{2.4}$$

It is well known that the curvature tensor and the almost complex structure of an almost Kähler manifold satisfy the following identity [1]:

$$\begin{aligned} &R(X,Y,Z,W) - R(X,Y,JZ,JW) - R(JX,JY,Z,W) \\ &\quad + R(JX,JY,JZ,JW) + R(JX,Y,JZ,W) + R(JX,Y,Z,JW) \\ &\quad + R(X,JY,JZ,W) + R(X,JY,Z,JW) \\ &= 2g((\nabla_q J)X,Y)g((\nabla_q J)Z,W). \end{aligned}$$

In terms of components, the above identity is expressed as

$$R_{ijkl} - R_{ij\bar{k}\bar{l}} - R_{\bar{i}\bar{j}kl} + R_{\bar{i}\bar{j}\bar{k}\bar{l}} + R_{\bar{i}j\bar{k}l} + R_{\bar{i}jk\bar{l}} + R_{i\bar{j}\bar{k}l} + R_{i\bar{j}k\bar{l}} = 2\nabla_q J_{ij} \nabla_q J_{kl}. \tag{2.5}$$

3. Proof of Theorem 1.1

Let $M = (M, J, g)$ be a compact almost Kähler locally symmetric space satisfying $||\mathrm{Ric}^*|| \geq ||\mathrm{Ric}||$. From the definition of the $*$-scalar curvature s^*, we have

$$\begin{aligned} -2\triangle s^* &= \nabla^2_{pp}(R_{ijkl} J_{ij} J_{kl}) \\ &= (\nabla^2_{pp} R_{ijkl}) J_{ij} J_{kl} + 4(\nabla_p R_{ijkl})(\nabla_p J_{ij}) J_{kl} \\ &\quad + 2R_{ijkl} \nabla_p J_{ij} \nabla_p J_{kl} + 2R_{ijkl}(\nabla^2_{pp} J_{ij}) J_{kl} \\ &= 2R_{ijkl} \nabla_p J_{ij} \nabla_p J_{kl} + 2R_{ijkl}(\nabla^2_{pp} J_{ij}) J_{kl}, \end{aligned}$$

since $\nabla R = 0$. Taking account of the Ricci identity, (2.1), (2.2), (2.3), (2.4) and (2.5), we get from the above identity

$$\begin{aligned}\triangle s^* =& -R_{ijkl}\nabla_p J_{ij}\nabla_p J_{kl} - R_{ijkl}(\nabla^2_{pp}J_{ij})J_{kl}\\ =& -\frac{1}{4}\nabla_p J_{ij}\nabla_p J_{kl}\nabla_q J_{ij}\nabla_q J_{kl} + R_{ijkl}(\nabla^2_{pi}J_{jp} + \nabla^2_{pj}J_{pi})J_{kl}\\ =& -\frac{1}{4}\nabla_p J_{ij}\nabla_q J_{ij}\nabla_p J_{kl}\nabla_q J_{kl} + 2R_{ijkl}(\nabla^2_{pi}J_{jp})J_{kl}\\ =& -\frac{1}{4}\nabla_p J_{ij}\nabla_q J_{ij}\nabla_p J_{kl}\nabla_q J_{kl} - 2R_{ijkl}(R_{pijs}J_{sp} + R_{pips}J_{js})J_{kl}\\ =& -\frac{1}{4}\nabla_p J_{ij}\nabla_q J_{ij}\nabla_p J_{kl}\nabla_q J_{kl} - 4(\mathrm{Ric}^*_{i\bar{j}}\mathrm{Ric}^*_{i\bar{j}} - \mathrm{Ric}^*_{i\bar{j}}\mathrm{Ric}_{i\bar{j}})\\ =& -\langle\nabla_p\Omega,\nabla_q\Omega\rangle\langle\nabla_p\Omega,\nabla_q\Omega\rangle - 4(||\mathrm{Ric}^*||^2 - \langle\mathrm{Ric}^*,\mathrm{Ric}\rangle)\\ \leq& -\langle\nabla_p\Omega,\nabla_q\Omega\rangle\langle\nabla_p\Omega,\nabla_q\Omega\rangle - 4||\mathrm{Ric}^*||(||\mathrm{Ric}^*|| - ||\mathrm{Ric}||) \leq 0,\end{aligned}$$

since $||\mathrm{Ric}^*|| \geq ||\mathrm{Ric}||$. Integrating the above inequality over M, we have

$$\begin{aligned}0 = \int_M \triangle s^* dM \leq& -\int_M \langle\nabla_p\Omega,\nabla_q\Omega\rangle\langle\nabla_p\Omega,\nabla_q\Omega\rangle dM\\ &- 4\int_M ||\mathrm{Ric}^*||(||\mathrm{Ric}^*|| - ||\mathrm{Ric}||)dM \leq 0,\end{aligned}$$

which yields $\int_M \langle\nabla_p\Omega,\nabla_q\Omega\rangle\langle\nabla_p\Omega,\nabla_q\Omega\rangle dM = 0$, and hence $\nabla J = 0$. This completes the proof of Theorem 1.1.

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