Commun. Korean Math. Soc. **27** (2012), No. 1, pp. 185–195
http://dx.doi.org/10.4134/CKMS.2012.27.1.185

GEOMETRY OF LIGHTLIKE HYPERSURFACES OF AN INDEFINITE COSYMPLECTIC MANIFOLD

Dae Ho Jin

Abstract. We study the geometry of lightlike hypersurfaces M of an indefinite cosymplectic manifold $\bar{M}$ such that either (1) the characteristic vector field ζ of $\bar{M}$ belongs to the screen distribution $S(TM)$ of M or (2) ζ belongs to the orthogonal complement $S(TM)^{\perp}$ of $S(TM)$ in $T\bar{M}$.

0. Introduction

The theory of lightlike submanifolds is one of the interesting topics of differential geometry. This theory is relatively new and in a developing stage. Many authors studied the geometry of lightlike submanifolds of indefinite Sasakian manifolds. Recently several authors have studied the geometry of lightlike submanifolds of an indefinite cosymplectic manifold [10].

The purpose of this paper is to study the geometry of lightlike hypersurfaces M of an indefinite cosymplectic manifold $\bar{M}$ subject to the conditions: (1) The characteristic vector field ζ of $\bar{M}$ belongs to the screen distribution $S(TM)$ of M, or (2) ζ belongs to the orthogonal complement $S(TM)^{\perp}$ of $S(TM)$ in $T\bar{M}$. We provide several new results on lightlike hypersurfaces M of this two types by using the structure tensors of M induced by the contact metric structure tensor J of $\bar{M}$.

1. Lightlike hypersurfaces

An odd dimensional smooth manifold $(\bar{M}, \bar{g})$ is called a contact metric manifold [1, 8] if there exist a $(1,1)$-type tensor field J, a vector field ζ, called the characteristic vector field, and its 1-form θ satisfying

$$\begin{aligned} &J^2X = -X + \theta(X)\zeta,\ J\zeta = 0,\ \theta\circ J = 0,\ \theta(\zeta) = 1,\\ &\bar{g}(\zeta,\zeta) = \epsilon, \quad \bar{g}(JX, JY) = \bar{g}(X,Y) - \epsilon\,\theta(X)\theta(Y),\\ &\theta(X) = \epsilon\bar{g}(\zeta, X), \quad d\theta(X,Y) = \bar{g}(JX, Y),\ \epsilon = \pm 1 \end{aligned} \tag{1.1}$$

Received August 31, 2010; Revised January 7, 2011.
2010 *Mathematics Subject Classification.* Primary 53C25, 53C40, 53C50.
Key words and phrases. totally umbilical, screen conformal, tangential and ascreen lightlike hypersurfaces, indefinite cosymplectic manifold.

for any vector fields X, Y on $\bar{M}$. Then the set $(J,\, \theta,\, \zeta,\, \bar{g})$ is called a contact metric structure on $\bar{M}$. We say that $\bar{M}$ has a normal contact structure [8] if $N_J + d\theta \otimes \zeta = 0$, where N_J is the Nijenhuis tensor field of J. A normal contact metric manifold is called a cosymplectic [1, 12] for which we have

$$\bar{\nabla}_X \theta = 0, \quad \bar{\nabla}_X J = 0 \tag{1.2}$$

for any vector field X on $\bar{M}$. A cosymplectic manifold $\bar{M} = (\bar{M},\, J,\, \zeta,\, \theta,\, \bar{g})$ is called an *indefinite cosymplectic manifold* [10] if $(\bar{M}, \bar{g})$ is a semi-Riemannian manifold of index $\mu(>0)$. For any indefinite cosymplectic manifold, apply the operator $\bar{\nabla}_X$ to $J\zeta = 0$ for any vector field X on $\bar{M}$ and use (1.2), we have $J(\bar{\nabla}_X \zeta) = 0$. Apply J to this and use (1.1) and $\theta(\bar{\nabla}_X \zeta) = 0$, we get

$$\bar{\nabla}_X \zeta = 0. \tag{1.3}$$

A hypersurface M of $\bar{M}$ is called a *lightlike hypersurface* if the normal bundle $TM^{\perp}$ of M is a vector subbundle of the tangent bundle TM of M, of rank 1. Then there exists a non-degenerate complementary vector bundle $S(TM)$ of $TM^{\perp}$ in TM, called a *screen distribution* on M, such that

$$TM = TM^{\perp} \oplus_{\text{orth}} S(TM), \tag{1.4}$$

where $\oplus_{\text{orth}}$ denotes the orthogonal direct sum. We denote such a lightlike hypersurface by $M = (M, g, S(TM))$. Denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $F(M)$ module of smooth sections of any vector bundle E over M. It is known [4] that, for any null section ξ of $TM^{\perp}$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a unique null section N of a unique vector bundle $\text{tr}(TM)$ of rank 1 in $S(TM)^{\perp}$ satisfying

$$\bar{g}\,(\xi, N) = 1, \quad \bar{g}(N, N) = \bar{g}(N, X) = 0,\ \forall\, X \in \Gamma(S(TM)).$$

In this case, the tangent bundle $T\bar{M}$ of $\bar{M}$ is decomposed as follow:

$$T\bar{M} = TM \oplus \text{tr}(TM) = \{TM^{\perp} \oplus \text{tr}(TM)\} \oplus_{\text{orth}} S(TM). \tag{1.5}$$

We call $\text{tr}(TM)$ and N the *transversal vector bundle* and the *null transversal vector field* of M with respect to the screen $S(TM)$ respectively.

Let $\bar{\nabla}$ be the Levi-Civita connection of $\bar{M}$ and P the projection morphism of $\Gamma(TM)$ on $\Gamma(S(TM))$ with respect to the decomposition (1.4). Then the local Gauss-Weingartan formulas of M and $S(TM)$ are given by

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y)N\,, \tag{1.6}$$

$$\bar{\nabla}_X N = -A_N X + \tau(X)N\,, \tag{1.7}$$

$$\nabla_X PY = \nabla^*_X PY + C(X, PY)\xi, \tag{1.8}$$

$$\nabla_X \xi = -A^*_\xi X - \tau(X)\xi \tag{1.9}$$

for all X, $Y \in \Gamma(TM)$ respectively, where ∇ and ∇^* are the liner connections on TM and $S(TM)$ respectively, B and C are the local second fundamental forms on TM and $S(TM)$ respectively, A_N and A^*_ξ are the shape operators on TM and $S(TM)$ respectively and τ is a 1-form on M. Since $\bar{\nabla}$ is torsion-free, ∇ is also torsion-free and B is symmetric on M. From the fact that $B(X,Y) = \bar{g}(\bar{\nabla}_X Y, \xi)$ for all X, $Y \in \Gamma(TM)$, we show that B is independent of the choice of a screen distribution $S(TM)$ and satisfies

$$B(X,\xi) = 0, \quad \forall\, X \in \Gamma(TM). \tag{1.10}$$

The induced connection ∇ of M is not metric and satisfies

$$(\nabla_X g)(Y,Z) = B(X,Y)\,\eta(Z) + B(X,Z)\,\eta(Y) \tag{1.11}$$

for any X, Y, $Z \in \Gamma(TM)$, where η is a 1-form such that

$$\eta(X) = \bar{g}(X,N), \quad \forall\, X \in \Gamma(TM). \tag{1.12}$$

But the connection ∇^* on $S(TM)$ is metric. Above two local second fundamental forms B and C are related to their shape operators by

$$B(X,Y) = g(A^*_\xi X, Y), \qquad \bar{g}(A^*_\xi X, N) = 0, \tag{1.13}$$

$$C(X,PY) = g(A_N X, PY), \quad \bar{g}(A_N X, N) = 0. \tag{1.14}$$

From (1.13), the operator A^*_ξ is $S(TM)$-valued self-adjoint on TM such that

$$A^*_\xi \xi = 0. \tag{1.15}$$

From the equations (1.6), (1.9) and (1.10), we show that

$$\bar{\nabla}_X \xi = -A^*_\xi X - \tau(X)\xi, \ \forall\, X \in \Gamma(TM). \tag{1.16}$$

2. Tangential lightlike hypersurfaces

In general, the characteristic vector field ζ belongs to $T\bar{M}$. Thus, from the decomposition (1.5) of $T\bar{M}$, ζ is decomposed by

$$\zeta = P\zeta + a\xi + bN, \tag{2.1}$$

where a and b are smooth functions defined by $a = \epsilon\theta(N)$ and $b = \epsilon\theta(\xi)$.

Proposition 2.1 ([8]). *Let M be a lightlike hypersurface of an indefinite almost contact manifold $\bar{M}$. Then there exists a screen $S(TM)$ such that*

$$J(S(TM)^\perp) \subset S(TM).$$

Note 1. Although $S(TM)$ is not unique, it is canonically isomorphic to the factor vector bundle $TM^* = TM/Rad(TM)$ considered by Kupeli [11]. Thus all screen distributions $S(TM)$ are mutually isomorphic. For this reason, we consider only lightlike hypersurface M of $\bar{M}$ equipped with a screen distribution $S(TM)$ such that $J(S(TM)^\perp) \subset S(TM)$.

Proposition 2.2. *Let M be a lightlike hypersurface of an indefinite cosymplectic manifold $\bar{M}$. Then ζ does not belong to $TM^\perp$ and* $\mathrm{tr}(TM)$.

Proof. Assume that the vector field ζ belongs to $TM^{\perp}$ [or $\mathrm{tr}(TM)$]. Then we have $\zeta = a\xi$ and $a \neq 0$ [or $\zeta = bN$ and $b \neq 0$]. From this we have

$$\epsilon = \bar{g}(\zeta, \zeta) = a^2 \bar{g}(\xi, \xi) = 0 \text{ [or } \epsilon = \bar{g}(\zeta, \zeta) = b^2 \bar{g}(N, N) = 0].$$

It is a contradiction to $\epsilon = \pm 1$. From this result we deduce our assertion. □

Note 2. Călin [2] has proved that if the characteristic vector field ζ is tangent to M, then it belongs to $S(TM)$ which we assume in this paper.

Definition 1. A lightlike hypersurface M of an indefinite cosymplectic manifold $\bar{M}$ is said to be a *tangential lightlike hypersurface* [9] of $\bar{M}$ if the characteristic vector field ζ of $\bar{M}$ is tangent to M.

For any tangential M, by Note 2, we show that ζ belongs to $S(TM)$, i.e., $a = b = 0$. In this case, there exists a non-degenerate almost complex distribution D_o on M with respect to J, i.e., $J(D_o) = D_o$, such that

$$S(TM) = \{J(TM^{\perp}) \oplus J(\mathrm{tr}(TM))\} \oplus_{\mathrm{orth}} D_o.$$

Now consider the 2-lightlike almost complex distribution D such that

$$TM = D \oplus J(\mathrm{tr}(TM)), \quad D = \{TM^{\perp} \oplus_{\mathrm{orth}} J(TM^{\perp})\} \oplus_{\mathrm{orth}} D_o \tag{2.2}$$

and two null vector fields U and V and their 1-forms u and v such that

$$U = -JN, \; V = -J\xi, \; u(X) = g(X, V), \; v(X) = g(X, U). \tag{2.3}$$

Denote by S the projection morphism of TM on D. By the first equation of (2.2)[denote (2.2)-1], any vector field X on M is expressed as follows

$$X = SX + u(X)U, \quad JX = FX + u(X)N, \tag{2.4}$$

where F is a tensor field of type $(1, 1)$ defined on M by

$$FX = JSX, \quad \forall\, X \in \Gamma(TM).$$

Apply J to (1.6), (1.7) and (1.16) and use (1.6), (1.7), (2.3) and the second equation of (2.4), for all $X, Y \in \Gamma(TM)$, we have

$$B(X, U) = C(X, V), \tag{2.5}$$

$$\nabla_X U = F(A_N X) + \tau(X)U, \tag{2.6}$$

$$\nabla_X V = F(A^*_\xi X) - \tau(X)V, \tag{2.7}$$

$$(\nabla_X F)(Y) = u(Y)A_N X - B(X, Y)U. \tag{2.8}$$

Theorem 2.3. *Let M be a tangential lightlike hypersurface of an indefinite cosymplectic manifold $\bar{M}$. Then ζ is parallel on M and $S(TM)$. Moreover ζ is conjugate to any vector field on M with respect to B and C.*

Proof. Replace Y by ζ to (1.6) and use (1.3) and $\zeta \in \Gamma(TM)$, we get

$$\nabla_X \zeta + B(X, \zeta)N = 0, \; \forall\, X \in \Gamma(TM).$$

Taking the scalar product with ξ in this equation, we have

$$\nabla_X \zeta = 0, \quad B(X, \zeta) = 0, \; \forall\, X \in \Gamma(TM). \tag{2.9}$$

Thus ζ is parallel on M and conjugate to any vector field on M with respect to B. Replace PY by ζ to (1.8) and use (2.9) and $\zeta \in \Gamma(S(TM))$, we have

$$\nabla^*_X \zeta + C(X, \zeta)\xi = 0, \ \forall\, X \in \Gamma(TM).$$

Taking the scalar product with N to this equation we have

$$\nabla^*_X \zeta = 0, \quad C(X, \zeta) = 0, \ \forall\, X \in \Gamma(TM). \tag{2.10}$$

Thus ζ is also parallel on $S(TM)$ and conjugate to any vector field on M with respect to C. Thus we have our assertions. □

Definition 2. We say that M is *totally umbilical* [4] if, on any coordinate neighborhood $\mathcal{U}$, there is a smooth function β such that

$$B(X, Y) = \beta\, g(X, Y), \ \forall\, X,\, Y \in \Gamma(TM). \tag{2.11}$$

In case $\beta = 0$ on $\mathcal{U}$, we say that M is *totally geodesic.*

Theorem 2.4. *Let M be a totally umbilical tangential lightlike hypersurface of an indefinite cosymplectic manifold $\bar{M}$. Then M is totally geodesic.*

Proof. As M is totally umbilical, from (2.9) and (2.11), we have

$$\beta\, g(X, \zeta) = 0, \quad \forall\, X \in \Gamma(TM).$$

Replace X by ζ in this equation and use $g(\zeta, \zeta) = \epsilon$, we have $\beta = 0$. □

Definition 3. A screen $S(TM)$ is called *totally umbilical* [4] in M if there exists a smooth function γ on a neighborhood $\mathcal{U}$ in M such that

$$C(X, PY) = \gamma\, g(X, Y), \ \forall\, X,\, Y \in \Gamma(TM). \tag{2.12}$$

In case $\gamma = 0$ on $\mathcal{U}$, we say that $S(TM)$ is *totally geodesic* in M.

Theorem 2.5. *Let M be a tangential lightlike hypersurface of an indefinite cosymplectic manifold $\bar{M}$ such that $S(TM)$ is totally umbilical in M. Then $S(TM)$ is totally geodesic in M.*

Proof. Assume that $S(TM)$ is totally umbilical in M. Replace Y by ζ to (2.12) and use (2.10), we have

$$\gamma\, g(X, \zeta) = 0, \quad \forall\, X \in \Gamma(TM).$$

Replace X by ζ to this equation and use $g(\zeta, \zeta) = \epsilon$, we obtain $\gamma = 0$. □

Theorem 2.6. *Let M be a tangential lightlike hypersurface of an indefinite cosymplectic manifold $\bar{M}$. D is integrable on M if and only if*

$$B(X, FY) = B(FX, Y), \quad \forall\, X,\, Y \in \Gamma(D).$$

Moreover, if M is totally umbilical, then D is autoparallel with respect to ∇.

Proof. Take $X, Y \in \Gamma(D)$. Then we have $FY = JY \in \Gamma(D)$ due to (2.4). Apply $\bar{\nabla}_X$ to $FY = JY$ and use (1.2), (1.6), (2.3) and (2.4), we get

$$B(X, FY) = g(\nabla_X Y, V),\ (\nabla_X F)Y = -B(X, Y)U. \tag{2.13}$$

By straightforward calculations from (2.13), we have

$$B(X, FY) - B(FX, Y) = g([X, Y], V).$$

If D is integrable on M, then $[X, Y] \in \Gamma(D)$ for any $X, Y \in \Gamma(D)$. Thus we get $g([X, Y], V) = 0$. This implies $B(X, FY) = B(FX, Y)$ for all $X, Y \in \Gamma(D)$. Conversely if $B(X, FY) = B(FX, Y)$ for all $X, Y \in \Gamma(D)$, then we have $g([X, Y], V) = 0$. Thus we get $[X, Y] \in \Gamma(D)$ for all $X, Y \in \Gamma(D)$. Therefore D is integrable on M.

Moreover, if M is totally umbilical, from (2.13)-1 and Theorem 2.4, we get $g(\nabla_X Y, V) = 0$ for all $X, Y \in \Gamma(D)$. This imply $\nabla_X Y \in \Gamma(D)$ for all $X, Y \in \Gamma(D)$. Thus D is autoparallel with respect to ∇. □

Theorem 2.7. *Let M be a tangential lightlike hypersurface of an indefinite cosymplectic manifold $\bar{M}$. Then F is parallel on D with respect to ∇ if and only if D is autoparallel with respect to ∇.*

Proof. If F is parallel on D with respect to ∇, i.e., $(\nabla_X F)Y = 0$ for any $X, Y \in \Gamma(D)$, taking the scalar product with V to (2.13)-2 with $(\nabla_X F)Y = 0$, we have $B(X, Y) = 0$ for all $X, Y \in \Gamma(D)$. From (2.13)-1, we have $g(\nabla_X Y, V) = 0$. This imply $\nabla_X Y \in \Gamma(D)$ for all $X, Y \in \Gamma(D)$. Thus D is autoparallel with respect to ∇.

Conversely if D is autoparallel with respect to ∇, from (2.13)-1, we have

$$B(X, FY) = 0,\ \forall\, X, Y \in \Gamma(D).$$

For $Y \in \Gamma(D)$, we show that $F^2 Y = -Y + \theta(Y)\zeta$. Replace Y by FY to $B(X, FY) = 0$ for all $X \in \Gamma(D)$ and use (2.9)-2, we have $B(X, Y) = 0$ for any $X, Y \in \Gamma(D)$. Thus F is parallel on D with respect to ∇ by (2.13). □

Theorem 2.8. *Let M be a tangential lightlike hypersurface of an indefinite cosymplectic manifold $\bar{M}$. If F is parallel on M with respect to ∇, then D is parallel on M and M is locally a product manifold $L_u \times M^\sharp$, where L_u is a null curve tangent to $J(tr(TM))$ and $M^\sharp$ is a leaf of D.*

Proof. Assume that F is parallel on M with respect to ∇. Then F is parallel on D with respect to ∇. By Theorem 2.7, D is autoparallel with respect to ∇. Let $X, Y \in \Gamma(TM)$. Apply F to (2.8) with $(\nabla_X F)Y = 0$, we have $u(Y)F(A_N X) = 0$ due to $FU = 0$. Replace Y by U to this and use (2.3), we have $F(A_N X) = 0$. From this and (2.6), we get $\nabla_X U = \tau(X)U$ for all $X \in \Gamma(TM)$. Thus $J(\mathrm{tr}(TM))$ is also autoparallel with respect to ∇. By the decomposition theorem of de Rham [3], we have $M = L_u \times M^\sharp$, where L_u is a null curve tangent to $J(\mathrm{tr}(TM))$ and $M^\sharp$ is a leaf of D. □

Corollary 1. *Let M be a totally umbilical tangential lightlike hypersurface of an indefinite cosymplectic manifold $\bar{M}$ such that $S(TM)$ is totally umbilical in M. Then M is locally a product manifold $L_u \times M^\natural$, where L_u is a null curve tangent to $J(tr(TM))$ and $M^\natural$ is a leaf of D.*

Proof. From Theorems 2.4 and 2.5, we have $B = 0$ and $A_N = 0$. Thus, from (2.8), we show that $(\nabla_X F)Y = 0$ for all $X,\ Y \in \Gamma(TM)$, i.e., F is parallel on M with respect to ∇. By Theorem 2.8, we have our theorem. □

Theorem 2.9. *Let M be a totally umbilical tangential lightlike hypersurface of an indefinite cosymplectic manifold $\bar{M}$ such that $S(TM)$ is totally umbilical. Then M is locally a product manifold $L_\xi \times L_u \times L_v \times M^\sharp$, where L_ξ, L_u and L_v are null curves tangent to $TM^\perp, J(tr(TM))$ and $J(TM^\perp)$ respectively and $M^\sharp$ is a leaf of the integrable distribution D_o.*

Proof. By Theorem 2.6, D is autoparallel with respect to ∇. Thus, for all $X,\ Y \in \Gamma(D_o)$, we have $\nabla_X Y \in \Gamma(D)$. From (1.8) and (2.13)-2, we have

$$\begin{aligned} C(X, FY) &= g(\nabla_X FY,\ N) = g((\nabla_X F)Y + F(\nabla_X Y),\ N) \\ &= g(F(\nabla_X Y),\ N) = -g(\nabla_X Y, JN) = g(\nabla_X Y,\ U), r \end{aligned} \tag{2.14}$$

due to $FY \in \Gamma(D_o)$. If $S(TM)$ is totally umbilical in M, then we have $C = 0$ due to Theorem 2.5. By (1.8) and (2.14), we get

$$g(\nabla_X Y, N) = 0, \quad g(\nabla_X Y, U) = 0,\ \forall\, X \in \Gamma(TM),\ \forall\, Y \in \Gamma(D_o).$$

These imply $\nabla_X Y \in \Gamma(D_o)$ for all $X,\ Y \in \Gamma(D_o)$. Thus D_o is autoparallel with respect to ∇ such that $TM = TM^\perp \oplus J(\mathrm{tr}(TM)) \oplus J(TM^\perp) \oplus_{\mathrm{orth}} D_o$. Since M and $S(TM)$ are totally umbilical, by Theorems 2.4 and 2.5, we have $A^*_\xi = A_N = 0$. Thus (1.9), (2.6) and (2.7) deduce respectively

$$\nabla_X \xi = -\tau(X)\xi,\ \nabla_X U = \tau(X)U,\ \nabla_X V = -\tau(X)V,\ \forall\, X \in \Gamma(TM).$$

Thus $TM^\perp, J(\mathrm{tr}(TM))$ and $J(TM^\perp)$ are autoparallel with respect to ∇. Thus we have $M = L_\xi \times L_u \times L_v \times M^\sharp$, where L_ξ, L_u and L_v are null curves tangent to $TM^\perp, J(\mathrm{tr}(TM))$ and $J(TM^\perp)$ respectively and $M^\sharp$ is a leaf of the integrable distribution D_o. □

3. Ascreen lightlike hypersurfaces

Definition 4. A lightlike hypersurface M of an indefinite cosymplectic manifold $\bar{M}$ is said to be an *ascreen lightlike hypersurface* [9] of $\bar{M}$ if the vector field ζ on $\bar{M}$ belongs to $S(TM)^\perp = TM^\perp \oplus \mathrm{tr}(TM)$.

For any ascreen M, the characteristic vector field ζ is decomposed by

$$\zeta = a\xi + bN. \tag{3.1}$$

Then, by Proposition 2.2, we show that $a \neq 0$ and $b \neq 0$.

Definition 5. A lightlike hypersurface M is called *screen conformal* [5, 6, 7] if there exists a non-vanishing smooth function φ on a neighborhood $\mathcal{U}$ in M such that $A_N = \varphi\, A^*_\xi$, or equivalently,

$$C(X, PY) = \varphi\, B(X, Y),\ \forall\, X,\, Y \in \Gamma(TM). \tag{3.2}$$

Note 3. For a screen conformal M, since C is symmetric on $S(TM)$, $S(TM)$ is integrable and M is locally a product manifold $L_\xi \times M^*$ where L_ξ is a lightlike curve tangent to $TM^\perp$ and M^* is a leaf of $S(TM)$ [4].

Theorem 3.1. *Let M be an ascreen lightlike hypersurface of an indefinite cosymplectic manifold $\bar{M}$. Then M is screen conformal with $\varphi = -\frac{a}{b}$.*

Proof. Apply $\bar{\nabla}_X$ to (3.1) and use (1.3), (1.7) and (1.16), we have

$$aA^*_\xi X + bA_N X = \{Xa - a\tau(X)\}\xi + \{Xb + b\tau(X)\}N,\ \forall\, X \in \Gamma(TM).$$

Taking the scalar product with ξ and N by turns we have

$$A_N X = \varphi A^*_\xi X,\ Xa = a\tau(X),\ Xb = -b\tau(X),\ \forall\, X \in \Gamma(TM), \tag{3.3}$$

where $\varphi = -\frac{a}{b}$. Thus M is screen conformal with $\varphi = -\frac{a}{b}$. □

From Theorem 3.1 and Note 3, we have the following result:

Theorem 3.2. *Let M be an ascreen lightlike hypersurface of an indefinite cosymplectic manifold $\bar{M}$. Then $S(TM)$ is integrable and M is locally a product manifold $L_\xi \times M^*$ where L_ξ is a lightlike curve tangent to the normal bundle $TM^\perp$ and M^* is a leaf of $S(TM)$.*

The induced Ricci type tensor $R^{(0,2)}$ of M is defined by

$$R^{(0,2)}(X, Y) = \mathrm{trace}\{Z \to R(Z, X)Y\},\ \forall\, X,\, Y \in \Gamma(TM).$$

In general, the tensor field $R^{(0,2)}$ is not symmetric [4, 5, 7]. A tensor field $R^{(0,2)}$ of lightlike hypersurfaces M is called its *induced Ricci tensor* [5] of M if it is symmetric. A symmetric $R^{(0,2)}$ tensor will be denoted by *Ric*.

Definition 6. We define the connection $\nabla^\perp$ on the transversal bundle $\mathrm{tr}(TM)$ by $\nabla^\perp_X N = \tau(X)N$ for all $X \in \Gamma(TM)$. We say that $\nabla^\perp$ is the *transversal connection* of M. Define the curvature tensor $R^\perp$ of $tr(TM)$ by

$$R^\perp(X, Y)N = \nabla^\perp_X \nabla^\perp_Y N - \nabla^\perp_Y \nabla^\perp_X N - \nabla^\perp_{[X,Y]} N$$

for all $X, Y \in \Gamma(TM)$. If $R^\perp$ vanishes identically, then the transversal connection $\nabla^\perp$ of M is said to be *flat* (or *trivial*) [8].

Theorem 3.3 ([8]). *Let M be a lightlike hypersurface of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. The following assertions are equivalent*:

(i) *Each 1-form τ is closed, i.e., $d\tau = 0$, on any $\mathcal{U} \subset M$.*
(ii) *The Ricci type tensor $R^{(0,2)}$ is an induced Ricci tensor of M.*
(iii) *The transversal connection of M is flat, i.e., $R^\perp = 0$.*

Theorem 3.4. *Let M be an ascreen lightlike hypersurface of an indefinite cosymplectic manifold $\bar{M}$. Then $R^{(0,2)}$ is an induced symmetric Ricci tensor of M and the transversal connection $\nabla^{\perp}$ of M is flat.*

Proof. Apply the operator ∇_X to $Ya = a\tau(Y)$ and use (3.3), we have

$$XYa = aX(\tau(Y)) + a\tau(X)\tau(Y), \quad \forall\, X,\, Y \in \Gamma(TM).$$

From this equation we have the following result:

$$2a\, d\tau(X,Y) = \{XY - YX - [X,Y]\}a = 0, \quad \forall\, X,\, Y \in \Gamma(TM).$$

Taking the product with $b \neq 0$ to this equation and using $2ab = \epsilon$, we have $d\tau(X,Y) = 0$. Thus, by Theorem 3.3, we have our assertion. □

From now on we may assume that $\epsilon = 1$ without loss of generality. In this case, substituting (3.1) into $g(\zeta,\zeta) = 1$, we have $2ab = 1$. Consider the local unit timelike vector field V^* on M and its 1-form v^* defined by

$$V^* = -\,b^{-1}J\xi, \quad v^*(X) = -\,g(X, V^*), \quad \forall\, X \in \Gamma(TM). \tag{3.4}$$

Let $U^* = -\,a^{-1}JN$. Then U^* is a unit timelike vector field on $S(TM)$ such that $g(V^*, U^*) = 1$. Apply J to (3.1) and use (1.1) and $2ab = 1$, we have

$$0 = aJ\xi + bJN = -\,(V^* + U^*)/2, \quad \text{i.e.,} \quad U^* = -\,V^*.$$

From this equation we deduce the result: $J(TM^{\perp}) = J(\mathrm{tr}(TM))$. From this fact, the tangent bundle TM of M is decomposed as follow:

$$TM = TM^{\perp} \oplus_{\mathrm{orth}} S(TM) = TM^{\perp} \oplus_{\mathrm{orth}} \{J(TM^{\perp}) \oplus_{\mathrm{orth}} D^*\}, \tag{3.5}$$

where D^* is a non-degenerate and almost complex distribution on M with respect to J, otherwise $S(TM)$ is degenerate.

Denote by Q the projection morphism of TM on D^*. Then, using (3.5) and $JV^* = a\xi - bN$, any vector field X on M is expressed as follows

$$X = QX + v^*(X)V^* + \eta(X)\xi, \tag{3.6}$$

$$JX = fX + av^*(X)\xi - b\eta(X)V^* - bv^*(X)N, \tag{3.7}$$

where f is a tensor field of type $(1,1)$ defined on M by

$$fX = JQX, \quad \forall\, X \in \Gamma(TM).$$

Apply J to (1.16) and use (1.2), (1.6), (1.13), (3.3), (3.4) and (3.7), we get

$$\nabla_X V^* = 2a\{f(A^*_{\xi}X) - aB(X, V^*)\xi\}, \ \forall\, X \in \Gamma(TM). \tag{3.8}$$

Theorem 3.5. *Let M be an ascreen lightlike hypersurface of an indefinite cosymplectic manifold $\bar{M}$. Then the following assertions are equivalent:*

(i) *V^* is parallel with respect to the induced connection ∇ on M.*
(ii) *M is totally geodesic.*
(iii) *$S(TM)$ is totally geodesic on M.*

Proof. (i) $\Leftrightarrow$ (ii). If V^* is parallel with respect to ∇, then, taking the scalar product with N to (3.8), we have $B(X, V^*) = 0$. Thus we have $f(A^*_\xi X) = 0$ for all $X \in \Gamma(TM)$. From this result and (3.7), we obtain $J(A^*_\xi X) = 0$ for any $X \in \Gamma(TM)$. Apply J in this equation and use (1.1) and the fact $\theta(A^*_\xi X) = 0$, we have $A^*_\xi X = 0$ for all $X \in \Gamma(TM)$. Thus M is totally geodesic. Conversely if M is totally geodesic, then, by (3.8), we have $\nabla_X V^* = 0$ for all $X \in \Gamma(TM)$.

(ii) $\Leftrightarrow$ (iii). From (3.2), we show that $A^*_\xi X = 0 \iff A_N X = 0$ for all $X \in \Gamma(TM)$ due to $\varphi \neq 0$. Thus we have our assertions. $\square$

Take $Y \in \Gamma(D^*)$. Then we have $fY = JY \in \Gamma(D^*)$ due to (3.7). Apply J to (1.6) and use this, (1.2), (1.6), (3.2), (3.4) and (3.7), we have

$$(\nabla_X f)Y = -ag(\nabla_X Y, V^*)\xi + 2aB(X, Y)V^*, \tag{3.9}$$

$$B(X, fY) = bg(\nabla_X Y, V^*), \ \forall\, X \in \Gamma(TM) \tag{3.10}$$

for all $X \in \Gamma(TM)$. By the procedure same as for Theorem 2.6 and Theorem 2.7 and by using (3.9) and (3.10), instead of (2.13)-1 and (2.13)-2, and $S(TM)$ is integrable due to (3.2), the following two theorems hold:

Theorem 3.6. *Let M be an ascreen lightlike hypersurface of an indefinite cosymplectic manifold $\bar{M}$. D^* is integrable if and only if we have*

$$B(X, fY) = B(fX, Y), \quad \forall\, X, Y \in \Gamma(D^*).$$

Moreover, if M is totally geodesic, then D^* is autoparallel with respect to ∇.

Theorem 3.7. *Let M be an ascreen lightlike hypersurface of an indefinite cosymplectic manifold $\bar{M}$. Then f is parallel on D^* with respect to ∇ if and only if D^* is autoparallel with respect to ∇.*

Theorem 3.8. *Let M be an ascreen lightlike hypersurface of an indefinite cosymplectic manifold $\bar{M}$. If M is totally geodesic, then M is locally a product manifold $L_\xi \times L_{V^*} \times M^\natural$, where L_ξ and L_{V^*} are null and timelike curves tangent to $TM^\perp$ and $J(TM^\perp)$ respectively and $M^\natural$ is a leaf of D^*.*

Proof. Assume that M is totally geodesic. Then, from Theorem 3.6, we show that D^* is autoparallel with respect to ∇. From (1.9) and (3.8), we have $\nabla_X \xi = -\tau(X)\xi$ and $\nabla_X V^* = 0$. Thus $TM^\perp$ and $J(TM^\perp)$ are also autoparallel with respect to ∇. Thus we have $M = L_\xi \times L_{V^*} \times M^\natural$, where L_ξ and L_{V^*} are lightlike and timelike curves tangent to $TM^\perp$ and $J(TM^\perp)$ respectively and $M^\natural$ is a leaf of the integrable distribution D^*. $\square$

References

[1] D. E. Blair, *Riemannian Geometry of Contact and Symplectic Manifolds*, Birkhäuser, 2002.

[2] C. Călin, *Contributions to geometry of CR-submanifold*, Thesis, University of Iasi, Romania, 1998.

[3] G. de Rham, *Sur la réductibilité d'un espace de Riemannian*, Comm. Math. Helv. **26** (1952), 328–344.

[4] K. L. Duggal and A. Bejancu, *Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications*, Kluwer Acad. Publishers, Dordrecht, 1996.

[5] K. L. Duggal and D. H. Jin, *Null curves and Hypersurfaces of Semi-Riemannian Manifolds*, World Scientific, 2007.

[6] ______, *A classification of Einstein lightlike hypersurfaces of a Lorentzian space form*, J. Geom. Phys. **60** (2010), no. 12, 1881–1889.

[7] D. H. Jin, *Screen conformal lightlike real hypersurfaces of an indefinite complex space form*, Bull. Korean Math. Soc. **47** (2010), no. 2, 341–353.

[8] ______, *Geometry of lightlike hypersurfaces of an indefinite Sasakian manifold*, Indian J. Pure Appl. Math. **41** (2010), no. 4, 569–581.

[9] ______, *Special half lightlike submanifolds of an indefinite Sasakian manifold*, to appear in Bull. Korean Math. Soc.

[10] S. K. Kim, *Lightlike submanifolds of indefinite cosymplectic manifolds*, Ph. D. Thesis, Ulsan University, Korea, 2007.

[11] D. N. Kupeli, *Singular Semi-Riemannian Geometry*, Kluwer Acad. Publishers, Dordrecht, 1996.

[12] G. D. Ludden, *Submanifolds of cosymplectic manifolds*, J. Differential Geometry **4** (1970), 237–244.

Department of Mathematics
Dongguk University
Gyeongju 780-714, Korea
E-mail address: jindh@dongguk.ac.kr