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A NEW APPROXIMATION SCHEME FOR FIXED POINTS OF ASYMPTOTICALLY ϕ-HEMICONTRACTIVE MAPPINGS

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Abstract. In this paper, we introduce an asymptotically ϕ-hemicontractive mapping with a ϕ-normalized duality mapping and obtain some strongly convergent result of a kind of multi-step iteration schemes for asymptotically ϕ-hemicontractive mappings.

1. Introduction and preliminaries

Approximation schemes including Mann and Ishikawa iterative schemes for fixed points were studied by many authors [1, 2, 4, 5, 6, 8, 9, 10]. Noor [5] introduced a three-step iterative scheme and studied the approximate solution of variational inclusion in Hilbert spaces. Glowinski and Le Tallec [2] used three-step iterative schemes to find the approximate solutions of the elastoviscoplasticity problem, liquid crystal theory, and eigenvalue computation. Moreover, they showed that the three-step iterative scheme gives better numerical results than two-step and one-step approximate iterations. Rhoades and Soltuz [8] introduced a multi-step iterative scheme and showed that the convergences of Mann and Ishikawa iterative schemes are equivalent to the convergence of a multi-step iterative scheme for continuous and strongly pseudocontractive mappings.

The asymptotically nonexpansive mapping was introduced by Goebel and Kirk [3]. Cho et al. [1] established weak and strong convergences result of the three-step iterative scheme with errors for asymptotically nonexpansive mappings. Nearly twenty years ago, Schu [9] introduced asymptotically pseudocontractive mappings and established some strong convergence theorems of the one-step iteration for completely continuous, uniformly L-Lipschitzian and asymptotically pseudocontractive mappings in Hilbert spaces. Isac and Li

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[4] studied two-step iteration schemes for completely continuous nonexpansive mappings in Hilbert spaces. Recently, Ofoedu [6] studied one-step iteration schemes for L-Lipschitzian mappings and asymptotically pseudocontractive mappings in Banach spaces.

Motivated and inspired by these facts, we introduce an asymptotically ϕ-hemicontractive mapping with a ϕ-normalized duality mapping and obtain some strongly convergent result of a kind of multi-step iteration schemes for asymptotically ϕ-hemicontractive mappings in Banach spaces.

Let E be a real Banach space and $\phi : \mathbb{R}^+ = [0, \infty) \to \mathbb{R}^+$ be a continuous strictly increasing function such that $\phi(0) = 0$ and $\lim_{t\to\infty} \phi(t) = \infty$. To the function ϕ, we associate a ϕ-normalized duality mapping $J_\phi : E \to 2^{E^*}$ defined by

$$J_\phi(x) = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|\phi(\|x\|) \text{ and } \|f^*\| = \phi(\|x\|)\},$$

where E^* denotes the dual space of E and $\langle \cdot, \cdot \rangle$ denotes the duality pairing. We shall denote a single-valued duality mapping by j_ϕ.

If $\phi(t) = t$, then J_ϕ is the usual normalized duality mapping J.

We have the following relation between J_ϕ and J, which can be easily shown.

Remark 1.1. For such J_ϕ and J,

$$J_\phi(x) = \frac{\phi(\|x\|)}{\|x\|} J(x) \quad \text{for} \quad x \neq 0.$$

Let $T : D(T) \subset E \to E$ be a mapping with domain $D(T)$ and $F(T)$ be the nonempty set of fixed points of T.

Definition 1.1. T is said to be ϕ-nonexpansive if for all $x, y \in D(T)$, the following inequality holds:

$$\|Tx - Ty\| \leq \phi(\|x - y\|).$$

Definition 1.2. T is said to be ϕ-uniformly L-Lipschitzian if there exists $L > 0$ such that for all $x, y \in D(T)$

$$\|T^n x - T^n y\| \leq L \cdot \phi(\|x - y\|).$$

Definition 1.3. T is said to be asymptotically ϕ-nonexpansive, if there exists a sequence $\{k_n\}_{n\geq 0} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n \phi(\|x - y\|) \text{ for all } x, y \in D(T), \ n \geq 1.$$

Definition 1.4. T is said to be asymptotically ϕ-pseudocontractive, if there exist a sequence $\{k_n\}_{n\geq 0} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ and $j_\phi(x - y) \in J_\phi(x - y)$ such that

$$\langle T^n x - T^n y, j_\phi(x - y) \rangle \leq k_n (\phi(\|x - y\|))^2 \text{ for all } x, y \in D(T), \ n \geq 1.$$

Example 1.1. Let $E = \mathbb{R}$ have the usual norm and $K = [0, 2\pi]$. Define $T : K \to \mathbb{R}$ by

$$Tx = \frac{2x \cos x}{3}$$

for each $x \in K$. Define a function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\phi(x) = \ln(x + 1)$ for each $x \in \mathbb{R}^+$ and take $j_\phi(x - y) = \ln(|x - y| + 1)$. By induction, for x, $y \in K$ with $x > y$,

$$\begin{aligned} \langle T^n x - T^n y, j_\phi(x - y)\rangle &\leq \left(\frac{2}{3}\right)^n |x - y|\phi(|x - y|) \\ &\leq \left\{\left(\frac{2}{3}\right)^n + 1\right\} |x - y| \ln(|x - y| + 1) \\ &\leq k_n|x - y|^2, \end{aligned}$$

where $k_n = (\frac{2}{3})^n + 1$. Hence, T is asymptotically ϕ-pseudocontractive.

Definition 1.5. *T is said to be asymptotically ϕ-hemicontractive, if there exist a sequence $\{k_n\}_{n\geq 0} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ and $j_\phi(x - y) \in J_\phi(x - y)$ such that for some $n_0 \in \mathbb{N}$*

$$\langle T^n x - y, j_\phi(x - y)\rangle \leq k_n(\phi(\|x - y\|))^2 \text{ for all } x \in D(T),\ y \in F(T)\ n \geq n_0.$$

Remark 1.2. We have the following relations;

(i) Every ϕ-nonexpansive mapping is asymptotically ϕ-nonexpansive.

(ii) Every asymptotically ϕ-nonexpansive mapping is ϕ-uniformly L-Lipschitzian.

(iii) Every asymptotically ϕ-nonexpansive mapping is asymptotically ϕ-pseudocontractive.

Proof. (iii) If T is asymptotically ϕ-nonexpansive, then there exists a sequence $\{k_n\}_{n\geq 0} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n \cdot \phi(\|x - y\|) \text{ for all } x, y \in D(T), \quad n \geq 1.$$

Hence,

$$\begin{aligned} \langle T^n x - T^n y, j_\phi(x - y)\rangle &\leq \|T^n x - T^n y\| \|j_\phi(x - y)\| \\ &= \|T^n x - T^n y\| \phi(\|x - y\|) \\ &\leq k_n \cdot (\phi(\|x - y\|))^2, \end{aligned}$$

which shows that T is asymptotically ϕ-pseudocontractive. □

Remark 1.3. There exists an asymptotically ϕ-pseudocontractive mapping, which is not asymptotically ϕ-nonexpansive. In fact, Rhoades [7] showed that the class of asymptotically pseudocontractive mappings properly contains the class of asymptotically nonexpansive mappings.

The following inequality for a ϕ-normalized duality mapping is needed for our main results.

Lemma 1.1. *Let $J_\phi : E \to 2^{E^*}$ be a ϕ-normalized duality mapping. Then for any x, $y \in E$, we have*

$$\|x+y\|^2 \le \|x\|^2 + 2\frac{\|x+y\|}{\phi(\|x+y\|)}\langle y, j_\phi(x+y)\rangle \text{ for } j_\phi(x+y) \in J_\phi(x+y).$$

Remark 1.4. If ϕ is an identity, then we have the following inequality shown by [11];

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, j(x+y)\rangle \text{ for } j(x+y) \in J(x+y).$$

Lemma 1.2 ([10]). *Let $\{a_n\}_{n\ge0}$ and $\{b_n\}_{n\ge0}$ be nonnegative sequences satisfying*

$$a_{n+1} \le (1-\theta_n)a_n + b_n$$

with $\theta_n \in [0,1]$, $\sum_{n=0}^{\infty}\theta_n = \infty$, and $b_n = o(\theta_n)$. Then,

$$\lim_{n\to\infty} a_n = 0.$$

2. Main result

Now, we consider the following main result.

Theorem 2.1. *Let K be a nonempty closed convex subset of a real Banach space E, $T : K \to K$ a uniformly continuous asymptotically ϕ-hemicontractive mapping having a bounded range with a sequence $\{k_n\}_{n\ge0} \subset [1,\infty)$, $\lim_{n\to\infty} k_n = 1$, $S_j : K \to K (j = 1, \ldots, p-1; p \ge 2)$ mappings having bounded range. Let $\{\alpha_n\}_{n\ge0}$, $\{\beta_n^j\}_{n\ge0} \in [0,1)$, $(j = 0, 1, 2, \ldots, p-1; p \ge 2)$ be such that $\sum_{n\ge0}\alpha_n = \infty$, $\lim_{n\to\infty}\alpha_n = 0$ and $\lim_{n\to\infty}\beta_n^1 = 0$. For an arbitrary point $x_0 \in K$, let $\{x_n\}_{n\ge0}$ be an iterative sequence defined by*

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T^n y_n^1, \\ y_n^i &= (1-\beta_n^i)x_n + \beta_n^i S_i^n y_n^{i+1}, \\ y_n^{p-1} &= (1-\beta_n^{p-1})x_n + \beta_n^{p-1} S_{p-1}^n x_n \\ &\quad (n \ge 0, \ \ i = 1, 2, \ldots, p-2; \ \ p \ge 2). \end{aligned} \tag{2.1}$$

Then, $\{x_n\}_{n\ge0}$ converges strongly to a common fixed point of T and S_j.

Proof. Since T and S_j has a bounded range, for $x^* \in F(T)\bigcap(\bigcap_{i=1}^{p-1} F(S_j))$,

$$M_1 := \|x_0 - x^*\| + \sup_{n\ge0}\|T^n y_n^1 - x^*\| + \sup_{n\ge0}\|S_1^n y_n^2 - x^*\|$$

is finite.

Now, we show that $\{x_n - x^*\}_{n\ge0}$ is also bounded. Obviously, $\|x_0 - x^*\| \le M_1$. Assume that $\|x_n - x^*\| \le M_1$. Consider

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|(1-\alpha_n)x_n + \alpha_n T^n y_n^1 - x^*\| \\ &= \|(1-\alpha_n)x_n + \alpha_n T^n y_n^1 - x^* + \alpha_n x^* - \alpha_n x^*\| \\ &= \|(1-\alpha_n)x_n - (1-\alpha_n)x^* + \alpha_n(T^n y_n^1 - x^*)\| \\ &= \|(1-\alpha_n)(x_n - x^*) + \alpha_n(T^n y_n^1 - x^*)\| \end{aligned}$$

$$\begin{aligned} &\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n\|T^n y_n^1 - x^*\| \\ &\leq (1-\alpha_n)M_1 + \alpha_n M_1 = M_1. \end{aligned}$$

Thus, $\{x_n - x^*\}_{n\geq 0}$ is bounded. Let $M_2 = \sup_{n\geq 0}\|x_n - x^*\|$. Denote $M = M_1 + M_2$, then M is finite. Since $\{x_n - x^*\}_{n\geq 0}$ is bounded and ϕ is a continuous strictly increasing function, $M^* := \sup_{n\geq 0}\phi(\|x_{n+1} - x^*\|)$ is also finite. Now, from Lemma 1.1 for all $n \geq 0$, we obtain

(2.2)

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&= \|(1-\alpha_n)x_n + \alpha_n T^n y_n^1 - x^*\|^2 \\
&= \|(1-\alpha_n)(x_n - x^*) + \alpha_n(T^n y_n^1 - x^*)\|^2 \\
&\leq (1-\alpha_n)^2\|x_n - x^*\|^2 + 2\alpha_n \left\langle T^n y_n^1 - x^*, \frac{\|x_{n+1} - x^*\|}{\phi(\|x_{n+1} - x^*\|)} j_\phi(x_{n+1} - x^*) \right\rangle \\
&= (1-\alpha_n)^2\|x_n - x^*\|^2 \\
&\quad + 2\alpha_n \frac{\|x_{n+1} - x^*\|}{\phi(\|x_{n+1} - x^*\|)} \langle T^n y_n^1 - T^n x_{n+1} + T^n x_{n+1} - x^*, j_\phi(x_{n+1} - x^*)\rangle \\
&= (1-\alpha_n)^2\|x_n - x^*\|^2 + 2\alpha_n \frac{\|x_{n+1} - x^*\|}{\phi(\|x_{n+1} - x^*\|)} \langle T^n x_{n+1} - x^*, j_\phi(x_{n+1} - x^*)\rangle \\
&\quad + 2\alpha_n \frac{\|x_{n+1} - x^*\|}{\phi(\|x_{n+1} - x^*\|)} \langle T^n y_n^1 - T^n x_{n+1}, j_\phi(x_{n+1} - x^*)\rangle \\
&\leq (1-\alpha_n)^2\|x_n - x^*\|^2 + 2\alpha_n k_n \|x_{n+1} - x^*\|\phi(\|x_{n+1} - x^*\|) \\
&\quad + 2\alpha_n \frac{\|x_{n+1} - x^*\|}{\phi(\|x_{n+1} - x^*\|)} \|T^n y_n^1 - T^n x_{n+1}\|\phi(\|x_{n+1} - x^*\|) \\
&\leq (1-\alpha_n)^2\|x_n - x^*\|^2 + 2\alpha_n k_n M^*\|x_{n+1} - x^*\| + 2\alpha_n M_1\|T^n y_n^1 - T^n x_{n+1}\| \\
&= (1-\alpha_n)^2\|x_n - x^*\|^2 + 2\alpha_n k_n M^*\|x_{n+1} - x^*\| + 2\alpha_n\delta_n,
\end{aligned}$$

where $\delta_n = M_1\|T^n y_n^1 - T^n x_{n+1}\|$. From (2.1), we have

(2.3)

$$\begin{aligned}
&\|y_n^1 - x_{n+1}\| \\
&= \|y_n^1 - x_n + x_n - x_{n+1}\| \\
&\leq \|y_n^1 - x_n\| + \|x_n - x_{n+1}\| \\
&= \|(1-\beta_n^1)x_n + \beta_n^1 S_1^n y_n^2 - x_n\| + \|x_n - \{(1-\alpha_n)x_n + \alpha_n T^n y_n^1\}\| \\
&= \|-\beta_n^1(x_n - S_1^n y_n^2)\| + \|\alpha_n(x_n - T^n y_n^1)\| \\
&= \beta_n^1\|x_n - x^* + x^* - S_1^n y_n^2\| + \alpha_n\|x_n - x^* + x^* - T^n y_n^1\| \\
&\leq \beta_n^1(\|x_n - x^*\| + \|x^* - S_1^n y_n^2\|) + \alpha_n(\|x_n - x^*\| + \|x^* - T^n y_n^1\|) \\
&\leq 2M\beta_n^1 + 2M\alpha_n = 2M(\alpha_n + \beta_n^1).
\end{aligned}$$

By the condition that $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}\beta_n^1 = 0$, from (2.3), we obtain

$$\lim_{n\to\infty}\|y_n^1 - x_{n+1}\| = 0$$

and by the uniform continuity of T, we also obtain

$$\lim_{n\to\infty}\|T^n y_n^1 - T^n x_{n+1}\| = 0.$$

Thus, we have

$$\lim_{n\to\infty}\delta_n = 0. \tag{2.4}$$

On the other hand,

$$\begin{aligned}\|x_{n+1} - x^*\| &= \|(1-\alpha_n)x_n + \alpha_n T^n y_n^1 - x^*\| \\ &= \|(1-\alpha_n)(x_n - x^*) + \alpha_n(T^n y_n^1 - x^*)\| \\ &\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n\|T^n y_n^1 - x^*\| \\ &\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n M.\end{aligned} \tag{2.5}$$

Since $\lim_{n\to\infty}\alpha_n = 0$ for all $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that $\alpha_n \le \epsilon$ for all $n \ge k$.

Substituting (2.5) into (2.2), we get

(2.6)

$$\begin{aligned}&\|x_{n+1} - x^*\|^2 \\ &\le (1-\alpha_n)^2\|x_n - x^*\|^2 + 2\alpha_n k_n M^*\|x_{n+1} - x^*\| + 2\alpha_n\delta_n \\ &\le (1-\alpha_n)^2\|x_n - x^*\|^2 + 2\alpha_n k_n M^*\{(1-\alpha_n)\|x_n - x^*\| + \alpha_n M\} + 2\alpha_n\delta_n \\ &= (1-\alpha_n)^2\|x_n - x^*\|^2 + 2\alpha_n k_n(1-\alpha_n)M^*\|x_n - x^*\| + 2\alpha_n^2 k_n M M^* + 2\alpha_n\delta_n \\ &\le (1-\alpha_n)^2\|x_n - x^*\|^2 + 2\alpha_n k_n(1-\alpha_n)M^*\{(1-\alpha_{n-1})\|x_{n-1} - x^*\| + \alpha_{n-1}M\} \\ &\quad + 2\alpha_n(\alpha_n k_n M M^* + \delta_n) \\ &\le (1-\alpha_n)^2\|x_n - x^*\|^2 + 2\alpha_n k_n(1-\alpha_n)(1-\alpha_{n-1})M^*\|x_{n-1} - x^*\| \\ &\quad + 2\alpha_n[k_n M M^*\{\alpha_n + \alpha_{n-1}(1-\alpha_n)\} + \delta_n] \\ &\le \cdots \\ &\le (1-\alpha_n)^2\|x_n - x^*\|^2 + 2\alpha_n k_n\prod_{j=k}^{n}(1-\alpha_j)M^*\|x_k - x^*\| \\ &\quad + 2\alpha_n\{2\alpha_n k_n M M^* + k_n M M^*\sum_{j=k}^{n-1}(\alpha_{n-1-j}\prod_{j=k}^{n-1}(1-\alpha_{n-j})) + \delta_n\} \\ &\le (1-\alpha_n)^2\|x_n - x^*\|^2 + 2\alpha_n\{k_n\prod_{j=k}^{n}(1-\alpha_j)M^* M\end{aligned}$$

$$+ 2\alpha_n k_n MM^* + k_n MM^* \sum_{j=k}^{n-1} (\alpha_{n-1-j} \prod_{j=k}^{n-1} (1-\alpha_{n-j})) + \delta_n\}$$

$$\leq (1-\alpha_n)\|x_n - x^*\|^2 + 2\alpha_n \pi_n,$$

where $\pi_n = [\prod_{j=k}^{n}(1-\alpha_j) + 2\alpha_n + \sum_{j=k}^{n-1}\{\alpha_{n-1-j}\prod_{j=k}^{n-1}(1-\alpha_{n-j})\}]k_n MM^* + \delta_n$. Here, we check $\{\pi_n\}_{n\geq 0}$ converges to 0 as $n \to \infty$. In fact,

$$\prod_{j=k}^{n}(1-\alpha_j) \leq e^{-\sum_{j=k}^{n}\alpha_j} \to 0 \text{ as } n \to \infty$$

and

$$\sum_{j=k}^{n-1}\{\alpha_{n-1-j}\prod_{j=k}^{n-1}(1-\alpha_{n-j})\} \leq \sum_{j=k}^{n-1}\epsilon \to 0 \text{ as } \epsilon \to 0.$$

Let $a_n = \|x_n - x^*\|^2$, $\theta_n = \alpha_n$ and $b_n = 2\alpha_n\pi_n$. Since $\lim_{n\to\infty}\pi_n = 0$, by (2.4) and Lemma 1.2, we obtain from (2.6) that

$$\lim_{n\to\infty}\|x_n - x^*\| = 0.$$

□

Remark 2.1. (1) For $p = 2$, $\beta_n^1 = 0$ and $S_j = T$, we can obtain the results with Mann iteration [6, 9].

(2) For $p = 3$, $\beta_n^2 = 0$ and $S_j = T$, we can obtain the results with Ishikawa iteration [4].

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