Commun. Korean Math. Soc. **27** (2012), No. 1, pp. 159–166
http://dx.doi.org/10.4134/CKMS.2012.27.1.159

A NOTE ON GENERATING FUNCTIONS OF q-GOTTLIEB POLYNOMIALS

Mumtaz Ahmad Khan and Mohammad Asif

Abstract. The present paper envisages the q-analogue of the Gottlieb polynomials.

1. Introduction

Gottlieb Polynomials are defined as (see [2, Eq.(2.3), p. 454], [4, p. 303], [5, p. 185])

$$\begin{aligned} l_n(x;\lambda) &= e^{-n\lambda}\sum_{k=0}^{n}\binom{n}{k}\binom{x}{k}(1-e^{\lambda})^k \\ &= e^{-n\lambda}\ {}_2F_1\left[\begin{array}{l} -n,\ -x; \\ 1-e^{\lambda} \\ 1; \end{array}\right]. \end{aligned} \tag{1.1}$$

Polynomial (1.1) satisfies the following generating functions (see [4, p. 303], [5, p. 186])

$$\sum_{n=0}^{\infty} l_n(x;\lambda)t^n = (1-t)^x(1-te^{-\lambda})^{-x-1}, \quad |t|<1, \tag{1.2}$$

$$\sum_{n=0}^{\infty} l_n(x;\lambda)\frac{t^n}{n!} = e^t\ {}_1F_1\left[\begin{array}{l} x+1; \\ -(1-e^{-\lambda})t \\ 1; \end{array}\right], \tag{1.3}$$

$$\sum_{n=0}^{\infty}\frac{(c)_n}{n!} l_n(x;\lambda)t^n = (1-te^{-\lambda})^{-c}\ {}_2F_1\left[\begin{array}{l} c,\ -x; \\ \frac{(1-e^{-\lambda})t}{1-te^{-\lambda}} \\ 1; \end{array}\right]. \tag{1.4}$$

Recently, Khan and Akhlaq [3] defined two variable and three variable analogues of the Gottlieb Polynomials. Of which, the two variable analogue of

Received September 6, 2010; Revised December 2, 2010.
2010 *Mathematics Subject Classification.* 33D45.
Key words and phrases. q-calculus, q-hypergeometric functions.

Gottlieb polynomials is given below

$$l_n(x,y;\lambda,\mu) = e^{-n(\lambda+\mu)} F \left[\begin{array}{l} -n : \ -x; \ -y; \\ 1-e^{\lambda}, \ 1-e^{\mu} \\ 1 : \ -; \ -; \end{array} \right] \tag{1.5}$$

or in other words,

$$l_n(x,y;\lambda,\mu) = e^{-n(\lambda+\mu)} \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(-n)_{r+s}(-x)_r(-y)_s(1-e^{\lambda})^r(1-e^{\mu})^s}{r!s!(1)_{r+s}} \tag{1.6}$$

and the three variable analogue as follows [3]

$$l_n(x,y,z;\lambda,\mu,\eta) = e^{-n(\lambda+\mu+\eta)} F \left[\begin{array}{l} -n :: \ -;-;-; \ -x; \ -y; \ -z; \\ 1-e^{\lambda}, \ 1-e^{\mu}, \ 1-e^{\eta} \\ 1 :: \ -;-;-; \ -; \ -; \ -; \end{array} \right] \tag{1.7}$$

or in other words,

$$l_n(x,y,z;\lambda,\mu,\eta) = e^{-n(\lambda+\mu+\eta)} \sum_{r=0}^{n} \sum_{s=0}^{n-r} \sum_{k=0}^{n-r-s} \frac{(-n)_{r+s+k}(-x)_r(-y)_s(-z)_k(1-e^{\lambda})^r(1-e^{\mu})^s(1-e^{\eta})^k}{r!s!k!(1)_{r+s+k}}. \tag{1.8}$$

They [3] obtained the following generating functions for the result (1.5)

$$\sum_{n=0}^{\infty} l_n(x,y;\lambda,\mu)t^n = (1-te^{-\mu})^x(1-te^{-\lambda})^y(1-te^{-\lambda-\mu})^{x-y-1}, \quad |\, t \,|< 1, \tag{1.9}$$

$$\sum_{n=0}^{\infty} (c)_n l_n(x,y;\lambda,\mu)\frac{t^n}{n!} = (1-te^{-\lambda-\mu})^{-c} F \left[\begin{array}{l} c : \ -x; \ -y; \\ \frac{t(e^{\lambda}-1)e^{-(\lambda+\mu)}}{1-te^{-(\lambda+\mu)}}, \ \frac{t(e^{\mu}-1)e^{-(\lambda+\mu)}}{1-te^{-(\lambda+\mu)}} \\ 1 : \ -; \ -; \end{array} \right] \tag{1.10}$$

and for result (1.7) as

$$\sum_{n=0}^{\infty} l_n(x,y,z;\lambda,\mu,\eta)t^n = (1-te^{-(\mu+\eta)})^x(1-te^{-(\lambda+\eta)})^y(1-te^{-(\lambda+\mu)})^z(1-te^{-(\lambda+\mu+\eta)})^{x-y-z-1}, \quad |\, t \,|< 1, \tag{1.11}$$

(1.12)
$$\sum_{n=0}^{\infty}(c)_n l_n(x,y,z;\lambda,\mu,\eta)\frac{t^n}{n!}$$
$$= (1-te^{-(\lambda+\mu+\eta)-c})F\left[\begin{array}{l} c :: \ -;-;-; \ \ -x; \ \ -y; \ \ -z; \\ \frac{t(e^{\lambda}-1)e^{-(\lambda+\mu+\eta)}}{1-te^{-(\lambda+\mu+\eta)}}, \ \frac{t(e^{\mu}-1)e^{-(\lambda+\mu+\eta)}}{1-te^{-(\lambda+\mu+\eta)}}, \ \frac{t(e^{\eta}-1)e^{-(\lambda+\mu+\eta)}}{1-te^{-(\lambda+\mu+\eta)}} \\ 1 :: \ -;-;-; \ \ -; \ \ -; \ \ -; \end{array}\right].$$

Motivated by the above works, we investigate here generating functions of q-Gottlieb polynomials. For this we require the following definitions and notations of the q-theory [1]. The q-analogue of the hypergeometric series or ${}_r\phi_s$ basic hypergeometric series is defined as [1, p. 4, Eq.(1.2.22)]

$$\begin{aligned} & {}_r\phi_s(a_1,a_2,\ldots,a_r;b_1,b_2,\ldots,b_s;q,z) \\ \equiv\ & {}_r\phi_s\left[\begin{array}{l} a_1,a_2,\ldots,a_r \\ ;q,z \\ b_1,b_2,\ldots,b_s \end{array}\right] \\ =\ & \sum_{n=0}^{\infty}\frac{(a_1;q)_n(a_2;q)_n\cdots(a_r;q)_n}{(q;q)_n(b_1;q)_n\cdots(b_s;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} z^n, \end{aligned} \tag{1.13}$$

with $\binom{n}{2} = \frac{n(n-1)}{2}$, where $q \neq 0$ when $r > s+1$. Parameters $b_1, b_2, \ldots, b_s$, are such that the denominator factors in the terms of the series are never zero.

$$(a;q)_n = \begin{cases} 1, \ n=0, \\ (1-a)(1-aq)\cdots(1-aq^{n-1}), \ n=1,2,\ldots, \end{cases}$$

is the q-shifted factorial and it is assumed that the denominator parameters $b \neq q^{-m}$ for $m = 0, 1, \ldots$.

A q-number or basic number is defined as

$$[a]_q = \frac{1-q^a}{1-q}, \ q \neq 1. \tag{1.14}$$

The q-number factorial of $n!$ is defined for a nonnegative integer n by

$$[n]_q! = \prod_{k=1}^{n}[k]_q, \tag{1.15}$$

the corresponding q-shifted factorial is defined by

$$[a]_{q;n} = \prod_{k=0}^{n-1}[a+k]_q. \tag{1.16}$$

Clearly,

$$\lim_{q\to 1}[n]_q! = n!, \ \lim_{q\to 1}[a]_q = a,$$

and

$$[a]_{q;n} = \frac{(q^a;q)_n}{(1-q)^n}, \ \lim_{q\to 1}[a]_{q;n} = (a)_n.$$

Corresponding to

$$(a_1, a_2, \ldots, a_m; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_m; q)_n$$

the compact notation is used

$$[a_1, a_2, \ldots, a_m]_{q;n} = [a_1]_{q;n} [a_2]_{q;n} \cdots [a_m]_{q;n}.$$

Similarly, the compact notation for Eq.(1.13) is given as

$$\sum_{n=0}^{\infty} \frac{[a_1, a_2, \ldots, a_r]_{q;n}}{[n]_q! [b_1, \ldots, b_s]_{q;n}} \left[(-1)^n q^{\binom{n}{2}} \right]^{1+s-r} z^n \tag{1.17}$$
$$= {}_r\phi_s(q^{a_1}, q^{a_2}, \cdots, q^{a_r}; q^{b_1}, q^{b_2}, \ldots, q^{b_s}; q, z).$$

q-analogue of the binomial theorem is defined as (see [1, p. 8, Eq.(1.3.2)])

$${}_1\phi_0(a; -; q, z) = \sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} z^n = \frac{(az; q)_\infty}{(z; q)_\infty}, \ | z |< 1, \ | q |< 1. \tag{1.18}$$

Gasper and Rahman [1, pp. 10–11, Eqs.(1.3.15) and (1.3.16)] define two q-analogues of the exponential function as follows:

$$e_q(z) = {}_1\phi_0(0; -; q, z) = \sum_{n=0}^{\infty} \frac{z^n}{(q; q)_n} = \frac{1}{(z; q)_\infty}, \ | z |< 1, \tag{1.19}$$

and

$$E_q(z) = {}_0\phi_0(-; -; q, -z) = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}}}{(q; q)_n} z^n = (-z; q)_\infty. \tag{1.20}$$

Then it is seen that

$$e_q(z) E_q(-z) = 1, \quad e_{q^{-1}}(z) = E_q(-qz). \tag{1.21}$$

2. q-Gottlieb polynomials

We define the q-Gottlieb polynomials as follows:

$$l_{n;q}(x; \lambda) = \{E_q(-\lambda)\}^n \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} x \\ k \end{bmatrix}_q q^{k(k-1)-xk} (1 - e_q(\lambda))^k \tag{2.1}$$

$$= \{E_q(-\lambda)\}^n \sum_{k=0}^{n} \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}} \frac{(q; q)_x}{(q; q)_k (q; q)_{x-k}} q^{k(k-1)-xk} (1 - e_q(\lambda))^k. \tag{2.2}$$

Using the identity (see [1, p. 6, Eq.(1.2.37)])

$$(q^{-n}; q)_k = \frac{(q; q)_n}{(q; q)_{n-k}} (-1)^k q^{\binom{k}{2} - nk} \tag{2.3}$$

we find from Eqs.(2.2) and (2.3) that

$$= \{E_q(-\lambda)\}^n \sum_{k=0}^{n} \frac{(-1)^k (q^{-n};q)_k q^{-\binom{k}{2}+nk}}{(q;q)_k} \frac{(-1)^k (q;q)_x q^{-\binom{k}{2}+xk}}{(q;q)_k} q^{k(k-1)-xk}(1-e_q(\lambda))^k$$

$$= \{E_q(-\lambda)\}^n \sum_{k=0}^{n} \frac{(q^{-n};q)_k}{(q;q)_k} \frac{(q^{-x};q)_k}{(q;q)_k} q^{nk}(1-e_q(\lambda))^k$$

or

$$l_{n;q}(x;\lambda) = \{E_q(-\lambda)\}^n \; {}_2\phi_1 \left[\begin{array}{l} q^{-n}, \quad q^{-x} \\ \; ; \; q, \; q^n(1-e_q(\lambda)) \\ q \end{array} \right]. \tag{2.4}$$

The following generating functions hold for q-Gottlieb polynomials (2.4):

$$(I) \qquad \sum_{n=0}^{\infty} l_{n;q}(x;\lambda) t^n = (1-tE_q(-\lambda))^{-1} \; {}_1\phi_1 \left[\begin{array}{l} q^{-x} \\ \; ; \; q, \; -(1-E_q(-\lambda))t \\ qtE_q(-\lambda) \end{array} \right],$$

$$(II) \qquad \sum_{n=0}^{\infty} l_{n;q}(x;\lambda) \frac{t^n}{(q;q)_n} = e_q(tE_q(-\lambda)) \; {}_1\phi_1 \left[\begin{array}{l} q^{-x} \\ \; ; \; q, \; -(1-E_q(-\lambda))t \\ q \end{array} \right],$$

$$(III) \qquad \begin{aligned} &\sum_{n=0}^{\infty} \frac{(q^c;q)_n}{(q;q)_n} l_{n;q}(x;\lambda) t^n \\ &= \frac{(1-tq^cE_q(-\lambda))_\infty}{(1-tE_q(-\lambda))_\infty} \; {}_2\phi_2 \left[\begin{array}{l} q^c, \; q^{-x} \\ \; ; \; q, \; -(1-E_q(-\lambda))t \\ q, \; tq^cE_q(-\lambda) \end{array} \right]. \end{aligned}$$

Let us prove one by one the generating functions from (I) to (III). The proof of (I) is given below

(2.5)

$$\begin{aligned} \sum_{n=0}^{\infty} l_{n;q}(x;\lambda) t^n &= \sum_{n=0}^{\infty} \{E_q(-\lambda)\}^n t^n \; {}_2\phi_1 \left[\begin{array}{l} q^{-n}, \quad q^{-x} \\ \; ; \; q, \; q^n(1-e_q(\lambda)) \\ q \end{array} \right] \\ &= \sum_{n=0}^{\infty} \{E_q(-\lambda)\}^n t^n \sum_{k=0}^{n} \frac{(q^{-n};q)_k (q^{-x};q)_k}{(q;q)_k (q;q)_k} q^{nk}(1-e_q(\lambda))^k \end{aligned}$$

from Eqs.(2.3) and (2.5), we have

(2.6)

$$= \sum_{n=0}^{\infty} \{E_q(-\lambda)\}^n t^n \sum_{k=0}^{n} \frac{(q;q)_n (-1)^k (q^{-x};q)_k}{(q;q)_{n-k} (q;q)_k (q;q)_k} q^{nk} q^{\binom{k}{2}-nk} (1-e_q(\lambda))^k$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(q;q)_{n+k} \{E_q(-\lambda)\}^{n+k} t^{n+k} (-1)^k q^{\binom{k}{2}} (q^{-x};q)_k}{(q;q)_n (q;q)_k (q;q)_k} (1-e_q(\lambda))^k.$$

Making use of the following identity (see [1, p. 6, Eq.(1.2.33)])

$$(a;q)_{n+k} = (a;q)_n(aq^n;q)_k \tag{2.7}$$

in (2.6), we find

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(q^{k+1};q)_n(q^{-x};q)_k(-1)^k q^{\binom{k}{2}}\{tE_q(-\lambda)\}^k(1-e_q(\lambda))^k\{tE_q(-\lambda)\}^n}{(q;q)_n(q;q)_k}$$

$$= \sum_{k=0}^{\infty} \frac{(q^{-x};q)_k(-1)^k q^{\binom{k}{2}}(E_q(-\lambda)-1)^k t^k}{(q;q)_k}\sum_{n=0}^{\infty}\frac{(q^{k+1};q)_n\{tE_q(-\lambda)\}^n}{(q;q)_n}$$

$$= \sum_{k=0}^{\infty} \frac{(q^{-x};q)_k(-1)^k q^{\binom{k}{2}}(E_q(-\lambda)-1)^k t^k}{(q;q)_k}\frac{[1-tq^{k+1}E_q(-\lambda)]_\infty}{[1-tE_q(-\lambda)]_\infty}$$

$$= \sum_{k=0}^{\infty} \frac{(q^{-x};q)_k(-1)^k q^{\binom{k}{2}}(E_q(-\lambda)-1)^k t^k}{(q;q)_k}\frac{1}{[1-tE_q(-\lambda)]_{k+1}}$$

$$= [1-tE_q(-\lambda)]^{-1}\sum_{k=0}^{\infty} \frac{(q^{-x};q)_k(-1)^k q^{\binom{k}{2}}(E_q(-\lambda)-1)^k t^k}{(q;q)_k}\frac{1}{[1-tE_q(-\lambda)]_k}$$

or

$$(I) \qquad \sum_{n=0}^{\infty} l_{n;q}(x;\lambda)t^n = (1-tE_q(-\lambda))^{-1}\ {}_1\phi_1\left[\begin{matrix} q^{-x} \\ ;\ q,\ -(1-E_q(-\lambda))t \\ qtE_q(-\lambda)\end{matrix}\right].$$

This completes the proof.

Proof of the generating function (II)

$$\sum_{n=0}^{\infty} l_{n;q}(x;\lambda)\frac{t^n}{(q;q)_n}$$

$$= \sum_{n=0}^{\infty}\{E_q(-\lambda)\}^n\frac{t^n}{(q;q)_n}\ {}_2\phi_1\left[\begin{matrix} q^{-n},\quad q^{-x} \\ ;\ q,\ q^n(1-e_q(\lambda)) \\ q\end{matrix}\right] \tag{2.8}$$

$$= \sum_{n=0}^{\infty}\{E_q(-\lambda)\}^n\frac{t^n}{(q;q)_n}\sum_{k=0}^{n}\frac{(q^{-n};q)_k(q^{-x};q)_k}{(q;q)_k(q;q)_k}q^{nk}(1-e_q(\lambda))^k$$

using (2.3) and (2.8), we have

$$= \sum_{n=0}^{\infty}\{E_q(-\lambda)\}^n\frac{t^n}{(q;q)_n}\sum_{k=0}^{n}\frac{(q;q)_n(-1)^k(q^{-x};q)_k}{(q;q)_{n-k}(q;q)_k(q;q)_k}q^{nk}q^{\binom{k}{2}-nk}(1-e_q(\lambda))^k$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{\{tE_q(-\lambda)\}^{n+k}(-1)^k(q^{-x};q)_k}{(q;q)_n(q;q)_k(q;q)_k}q^{\binom{k}{2}}(1-e_q(\lambda))^k$$

$$= \sum_{n=0}^{\infty} \frac{\{tE_q(-\lambda)\}^n}{(q;q)_n} \sum_{k=0}^{\infty} \frac{(q^{-x};q)_k}{(q;q)_k (q;q)_k} (-1)^k q^{\binom{k}{2}} \{t(E_q(-\lambda)-1)\}^k$$

or

$$(II) \qquad \sum_{n=0}^{\infty} l_{n;q}(x;\lambda) \frac{t^n}{(q;q)_n} = e_q(tE_q(-\lambda))\ {}_1\phi_1 \left[\begin{matrix} q^{-x} \\ ;\ q,\ -(1-E_q(-\lambda))t \\ q \end{matrix} \right].$$

This completes the proof.

Proof of the generating function (III)

$$\sum_{n=0}^{\infty} \frac{(q^c;q)_n}{(q;q)_n} l_{n;q}(x;\lambda) t^n$$

$$(2.9) \qquad = \sum_{n=0}^{\infty} \frac{(q^c;q)_n t^n}{(q;q)_n} \{E_q(-\lambda)\}^n\ {}_2\phi_1 \left[\begin{matrix} q^{-n}, \quad q^{-x} \\ ;\ q,\ q^n(1-e_q(\lambda)) \\ q \end{matrix} \right]$$

$$= \sum_{n=0}^{\infty} \frac{(q^c;q)_n t^n}{(q;q)_n} \{E_q(-\lambda)\}^n \sum_{k=0}^{n} \frac{(q^{-n};q)_k (q^{-x};q)_k}{(q;q)_k (q;q)_k} q^{nk} (1-e_q(\lambda))^k$$

using (2.3) and (2.9), we obtain

(2.10)

$$= \sum_{n=0}^{\infty} \frac{(q^c;q)_n t^n}{(q;q)_n} \{E_q(-\lambda)\}^n \sum_{k=0}^{n} \frac{(q;q)_n (-1)^k (q^{-x};q)_k}{(q;q)_{n-k} (q;q)_k (q;q)_k} q^{nk} q^{\binom{k}{2} - nk} (1-e_q(\lambda))^k$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(q^c;q)_{n+k} \{E_q(-\lambda)\}^{n+k} t^{n+k} (-1)^k q^{\binom{k}{2}} (q^{-x};q)_k}{(q;q)_n (q;q)_k (q;q)_k} (1-e_q(\lambda))^k.$$

Making use of the following identity (see [1, p. 6, Eq.(1.2.33)])

$$(2.11) \qquad (a;q)_{n+k} = (a;q)_n (aq^n;q)_k$$

in (2.10), we find

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(q^c;q)_k (q^{c+k};q)_n (q^{-x};q)_k (-1)^k q^{\binom{k}{2}} \{tE_q(-\lambda)\}^k \{tE_q(-\lambda)\}^n}{(q;q)_n (q;q)_k (q;q)_k} (1-e_q(\lambda))^k$$

$$= \sum_{k=0}^{\infty} \frac{(q^c;q)_k (q^{-x};q)_k (E_q(-\lambda)-1)^k}{(q;q)_k (q;q)_k} (-1)^k q^{\binom{k}{2}} \sum_{n=0}^{\infty} \frac{(q^{c+k};q)_n \{tE_q(-\lambda)\}^n}{(q;q)_n}$$

$$= \sum_{k=0}^{\infty} \frac{(q^c;q)_k (q^{-x};q)_k (E_q(-\lambda)-1)^k}{(q;q)_k (q;q)_k} (-1)^k q^{\binom{k}{2}} \frac{[1-tq^{c+k}E_q(-\lambda)]_\infty}{[1-tE_q(-\lambda)]_\infty}$$

$$= \sum_{k=0}^{\infty} \frac{(q^c;q)_k (q^{-x};q)_k (E_q(-\lambda)-1)^k}{(q;q)_k (q;q)_k} (-1)^k q^{\binom{k}{2}} \frac{[1-tq^cE_q(-\lambda)]_k [1-tq^{c+k}E_q(-\lambda)]_\infty}{[1-tq^cE_q(-\lambda)]_k [1-tE_q(-\lambda)]_\infty}$$

$$= \frac{[1-tq^cE_q(-\lambda)]_\infty}{[1-tE_q(-\lambda)]_\infty}\sum_{k=0}^{\infty}\frac{(q^c;q)_k(q^{-x};q)_k(E_q(-\lambda)-1)^k}{(q;q)_k[1-tq^cE_q(-\lambda)]_k(q;q)_k}(-1)^k q^{\binom{k}{2}}$$

or

(III)

$$\sum_{n=0}^{\infty}\frac{(q^c;q)_n}{(q;q)_n}l_{n;q}(x;\lambda)t^n = \frac{(1-tq^cE_q(-\lambda))_\infty}{(1-tE_q(-\lambda))_\infty}\,{}_2\phi_2\left[\begin{array}{l} q^c,\ q^{-x} \\ ;\ q,\ -(1-E_q(-\lambda))t \\ q,\ tq^cE_q(-\lambda)\end{array}\right].$$

This completes the proof.

Acknowledgement. The second author wishes to express his heartfelt thanks to the Human Resource Development Group Council of Scientific & Industrial Research of India for awarding Senior Research Fellowship (NET)(F.No. 10-2(5)/ 2005(i)- E.U.II).

References

[1] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, Cambridge University Press, Cambridge, 1990.

[2] M. J. Gottlieb, *Concerning some polynomials orthogonal on a finite or enumerable set of points*, Amer. J. Math. **60** (1938), no. 2, 453–458.

[3] M. A. Khan and M. Akhlaq, *Some new generating functions for Gottlieb polynomials of several variables*, Inter. Trans. Appl. Sci. **1** (2009), no. 4, 567–570.

[4] E. D. Rainville, *Special Functions*, Macmillan, New York 1960.

[5] H. M. Srivastava and H. L. Manocha, *A Treatise on Generating Functions*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane, and Toronto, 1984.

Mumtaz Ahmad Khan
Department of Applied Mathematics
Faculty of Engineering
Aligarh Muslim University
Aligarh-202002, India
E-mail address: mumtaz_ahmad_khan_2008@yahoo.com

Mohammad Asif
Department of Applied Mathematics
Faculty of Engineering
Aligarh Muslim University
Aligarh-202002, India
E-mail address: mohdasiff@gmail.com