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A NEW PROOF OF SAALSCHÜTZ'S THEOREM FOR THE SERIES ${}_3F_2(1)$ AND ITS CONTIGUOUS RESULTS WITH APPLICATIONS

Yong Sup Kim and Arjun Kumar Rathie

Abstract. The aim of this paper is to establish the well-known and very useful classical Saalschütz's theorem for the series ${}_3F_2(1)$ by following a different method. In addition to this, two summation formulas closely related to the Saalschütz's theorem have also been obtained. The results established in this paper are further utilized to show how one can obtain certain known and useful hypergeometric identities for the series ${}_3F_2(1)$ and ${}_4F_3(1)$ already available in the literature.

1. Introduction and results required

We start with the following well-known and useful classical Saalschütz's theorem [4, p. 87, Section 51] for the series ${}_3F_2(1)$. If n is a non-negative integer and if a, b, c are independent of n,

$$(1.1)\qquad {}_3F_2\left[\begin{matrix}-n, \quad a, \quad b \\ c,\ 1+a+b-c-n\end{matrix}\ ;\ 1\right] = \frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n}.$$

As mentioned in almost all the standard books on generalized hypergeometric series that this theorem can be established with the help of the following Euler's transformation formula [4, p. 60, Eq.(5)]. If $|x| < 1$,

$$(1.2)\qquad {}_2F_1\left[\begin{matrix}a,\ b \\ c\end{matrix}\ ;\ x\right] = (1-x)^{c-a-b}\,{}_2F_1\left[\begin{matrix}c-a,\ c-b \\ c\end{matrix}\ ;\ x\right]$$

by equating the coefficient of x^n on both sides. On the other hand, in 1926, Whipple [6] has obtained the following transformation formula between a nearly

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poised ${}_4F_3(1)$ and a Saalschützian ${}_5F_4(1)$ series:

$$(1.3)\quad \begin{aligned} &{}_4F_3\left[\begin{matrix} f,\ 1+f-h,\ h-a,\ d \\ h,\ 1+f+a-h,\ g \end{matrix};\ 1\right] \\ &= \frac{\Gamma(g)\,\Gamma(g-f-d)}{\Gamma(g-f)\,\Gamma(g-d)} \\ &\times {}_5F_4\left[\begin{matrix} a,\quad d,\quad 1+f-g,\quad \frac{1}{2}f,\quad \frac{1}{2}f+\frac{1}{2} \\ h,\ 1+f-a-h,\ \frac{1}{2}+\frac{1}{2}f+\frac{1}{2}d-\frac{1}{2}g,\ 1+\frac{1}{2}f+\frac{1}{2}d-\frac{1}{2}g \end{matrix};1\right], \end{aligned}$$

where either f or d must be a negative integer.

Whipple [6] also showed that, if $f=-N$, the other parameters a, f, g and h in the ${}_5F_4(1)$ series can be chosen in four ways, in order that this series can be reduced to a summable Saalschützian ${}_3F_2(1)$ series and carry out these summations, writing a for f, b for g, we get four nearly-poised summation theorems [5, p. 244, Eq.(III.15,16,17,18)]:

$$(1.4)\quad {}_3F_2\left[\begin{matrix} a,\ 1+\frac{1}{2}a,\ -N \\ \frac{1}{2}a,\quad b \end{matrix};\ 1\right] = \frac{(b-a-1-N)(b-a)_{N-1}}{(b)_N}$$

or equivalently

$$(1.5)\quad {}_3F_2\left[\begin{matrix} a,\ 1+\frac{1}{2}a,\ -N \\ \frac{1}{2}a,\quad b \end{matrix};\ 1\right] = \frac{(2+b-a)_N(b-a-1)_N}{(b)_N(1+a-b)_N},$$

$$(1.6)\quad {}_3F_2\left[\begin{matrix} a,\quad b,\quad -N \\ 1+a-b,\quad 1+2b-N \end{matrix};\ 1\right] = \frac{(a-2b)_N(1+\frac{1}{2}a-b)_N(-b)_N}{(1+a-b)_N(\frac{1}{2}a-b)_N(-2b)_N},$$

$$(1.7)\quad {}_4F_3\left[\begin{matrix} a,\ 1+\frac{1}{2}a,\ b,\ -N \\ \frac{1}{2}a,\ 1+a-b,\ 1+2b-N \end{matrix};\ 1\right] = \frac{(a-2b)_N(-b)_N}{(1+a-b)_N(-2b)_N},$$

$$(1.8)\quad \begin{aligned} &{}_4F_3\left[\begin{matrix} a,\ 1+\frac{1}{2}a,\ b,\ -N \\ \frac{1}{2}a,\ 1+a-b,\ 2+2b-N \end{matrix};\ 1\right] \\ &= \frac{(a-2b-1)_N(\frac{1}{2}+\frac{1}{2}a-b)_N(-b-1)_N}{(1+a-b)_N(\frac{1}{2}a-\frac{1}{2}-b)_N(-2b-1)_N}. \end{aligned}$$

The aim of this paper is to establish the well-known Saalschütz's theorem (1.1) by following a different method. In addition to this, explicit expressions of

$${}_3F_2\left[\begin{matrix} a,\quad b,\quad -n \\ c,\ 1+a+b-c-n+i \end{matrix};\ 1\right]$$

for $i=1,2$ closely related to (1.1) have also been obtained. These results are further utilized to show how one can obtain known identities (1.5), (1.6) and (1.7). For this, the following results will be required in our present investigations.

An integral transformation [4, p. 85, Section 49] :

$$ (1.9) \quad \begin{aligned} &{}_3F_2\left[\begin{matrix}\alpha,\ \beta,\ \rho\\ \gamma,\ \rho+\sigma\end{matrix};\ 1\right] \\ &= \frac{\Gamma(\rho+\sigma)}{\Gamma(\rho)\Gamma(\sigma)}\int_0^1 x^{\rho-1}\,(1-x)^{\sigma-1}\,{}_2F_1\left[\begin{matrix}\alpha,\ \beta\\ \gamma\end{matrix};\ x\right]dx, \end{aligned} $$

provided $\Re(\rho) > 0$, $\Re(\sigma) > 0$ and $\Re(\gamma+\sigma-\alpha-\beta) > 0$.

Kummer's transformation [4, p. 60, Eq.(4)] : If $|x| < 1$ and $|x/(1-x)| < 1$,

$$ (1.10) \quad {}_2F_1\left[\begin{matrix}a,\ b\\ c\end{matrix};\ x\right] = (1-x)^{-b}\,{}_2F_1\left[\begin{matrix}c-a,\ b\\ c\end{matrix};\ -\frac{x}{1-x}\right]. $$

Vandermonde's theorem [5, p. 28, Eq.(1.7.7)]:

$$ (1.11) \quad {}_2F_1\left[\begin{matrix}-n,\ \beta\\ \gamma\end{matrix};\ 1\right] = \frac{(\gamma-\beta)_n}{(\gamma)_n} $$

for $n = 0, 1, 2, \ldots$.

The well-known identity [5, p. 240, Eq.(I.30)] :

$$ (1.12) \quad \Gamma(c-n) = (-1)^n\,\frac{\Gamma(c)}{(1-c)_n} $$

for $n = 0, 1, 2, \ldots$.

2. Proof of Saalschütz's theorem (1.1)

In order to derive (1.1), we proceed as follows. Denoting the left hand side of (1.1) by $\mathbf{S}$ and taking $\alpha = a$, $\beta = -n$, $\gamma = c$, $\rho = b$ and $\sigma = 1 + a - c - n$, in (1.9), we have

$$ (2.1) \quad \begin{aligned} \mathbf{S} &= {}_3F_2\left[\begin{matrix}-n,\ a,\ b\\ c,\ 1+a+b-c-n\end{matrix};\ 1\right] \\ &= \frac{\Gamma(1+a+b-c-n)}{\Gamma(b)\Gamma(1+a-c-n)}\int_0^1 x^{b-1}\,(1-x)^{a-c-n}\,{}_2F_1\left[\begin{matrix}a,\ -n\\ c\end{matrix};\ x\right]dx \\ &= \frac{(c-a)_n\Gamma(1+a+b-c)}{(c-a-b)_n\Gamma(b)\Gamma(1+a-c)} \\ &\quad\times\int_0^1 x^{b-1}\,(1-x)^{a-c-n}\,{}_2F_1\left[\begin{matrix}a,\ -n\\ c\end{matrix};\ x\right]dx \\ &= \frac{(c-a)_n\Gamma(1+a+b-c)}{(c-a-b)_n\Gamma(b)\Gamma(1+a-c)} \\ &\quad\times\int_0^1 x^{b-1}\,(1-x)^{a-c}\,{}_2F_1\left[\begin{matrix}c-a,\ -n\\ c\end{matrix};\ -\frac{x}{1-x}\right]dx, \end{aligned} $$

where we have used (1.12) and (1.10) for the third and fourth equalities, respectively.

Now, expressing ${}_2F_1$ as a series, changing the order of integration and summation (which is seen to be justified due to the uniform convergence of the series), evaluate the integral and then using the result (1.12), we get, after a little simplification,

$$\mathbf{S} = \frac{(c-a)_n}{(c-a-b)_n} \sum_{r=0}^{n} \frac{(b)_r(-n)_r}{(c)_r r!}, \tag{2.2}$$

which can be written in the form

$$\mathbf{S} = \frac{(c-a)_n}{(c-a-b)_n} \, {}_2F_1 \left[\begin{matrix} -n, \ b \\ c \end{matrix} ; \ 1 \right].$$

Finally, using Vandermonde's theorem (1.11), we get

$$\mathbf{S} = \frac{(c-a)_n (c-b)_n}{(c)_n (c-a-b)_n}.$$

This completes the proof of (1.1).

3. Contiguous results

The results contiguous to the Saalschütz's theorem (1.1) to be established are

(3.1)

$$\begin{aligned} &{}_3F_2 \left[\begin{matrix} a, \ b, \ -n \\ c, \ 2+a+b-c-n \end{matrix} ; \ 1 \right] \\ &= \frac{(c-a-1)_n (c-b)_n}{(c)_n (c-a-b-1)_n} \left\{ 1 - \frac{nb}{(c-a-1)(c-b-1+n)} \right\} \\ &= \frac{1}{(a-b)(c)_n (c-a-b-1)_n} \left\{ a(c-a-1)_n (c-b)_n - b(c-a)_n (c-b-1)_n \right\} \end{aligned}$$

and

$$\begin{aligned} &{}_3F_2 \left[\begin{matrix} a, \ b, \ -n \\ c, \ 3+a+b-c-n \end{matrix} ; \ 1 \right] \\ &= \sum_{a \leftrightarrow b} \left\{ \frac{a(a+1)(c-a-2)_n (c-b)_n}{(a-b)(a-b+1)(c)_n (c-a-b-2)_n} \right. \\ &\quad \left. - \frac{2ab(c-a-1)_n (c-b-1)_n}{(a-b-1)(a-b+1)(c)_n (c-a-b-2)_n} \right\}, \end{aligned} \tag{3.2}$$

where $\sum_{a \leftrightarrow b} f(a,b) \equiv f(a,b) + f(b,a)$.

4. Proofs of (3.1) and (3.2)

In order to derive the result (3.1), we proceed as follows. Denoting the left hand side of (3.1) by S_1 and in (1.9), taking $\alpha = a, \beta = -n, \gamma = c, \rho = b$ and $\rho + \sigma = 2 + a + b - c - n$ so that $\sigma = 2 + a - c - n$, we have

(4.1)
$$S_1 = \frac{\Gamma(2+a+b-c-n)}{\Gamma(b)\Gamma(2+a-c-n)} \int_0^1 x^{b-1} (1-x)^{1+a-c-n} {}_2F_1 \left[\begin{matrix} a, \ -n \\ c \end{matrix}; \ x\right] dx.$$

Using (1.12) and then (1.10), we have

(4.2)
$$\begin{aligned} S_1 = & \frac{(c-a-1)_n \Gamma(2+a+b-c)}{(c-a-b-1)_n \Gamma(b)\Gamma(2+a-c)} \qquad \times \int_0^1 x^{b-1} (1-x)^{1+a-c} \\ & \times {}_2F_1 \left[\begin{matrix} c-a, \ -n \\ c \end{matrix}; \ -\frac{x}{1-x}\right] dx. \end{aligned}$$

Now, proceeding similarly as in case of the derivation of Saalschütz's theorem (1.1), we easily arrive at

(4.3)
$$S_1 = \frac{(c-a-1)_n}{(c-a-b-1)_n} \sum_{r=0}^{n} \frac{(b)_r (-n)_r (c-a)_r}{(c)_r (c-a-1)_r r!}.$$

Now, writing
$$\frac{(c-a)_r}{(c-a-1)_r} = 1 + \frac{r}{c-a-1}$$
and separating into two parts and adjusting the second series and then summing up the two series, we get

(4.4)
$$\begin{aligned} S_1 = & \frac{(c-a-1)_n}{(c-a-b-1)_n} \\ & \times \left\{ {}_2F_1 \left[\begin{matrix} -n, \ b \\ c \end{matrix}; \ 1\right] - \frac{nb}{c(c-a-1)} {}_2F_1 \left[\begin{matrix} -(n-1), \ b+1 \\ c+1 \end{matrix}; \ 1\right] \right\}. \end{aligned}$$

Now, using Vandermonde's theorem (1.11) in both ${}_2F_1$, we can easily arrive at the right hand side in the first form of (4.1). In a similar way, the result (3.2) can be established, so we prefer to omit the details.

Remark. The results (3.1) and (3.2) have already been obtained by Arora and Rathie [1] by following a different method.

5. Applications

In this section, as an applications, we shall show, how one can obtain the known and useful hypergeometric identities (1.6), (1.7) and (1.8) with the help of Saalschütz's theorem (1.1) and its contiguous result (3.1) and (3.2).

Proof of (1.6). In (3.1), if we take $a = a, b = b, n = N$ and $c = 1 + a + b$, and after a little simplification, we easily obtained the identity (1.6). □

Proof of (1.7). In order to prove the identity (1.7), we shall use the following result (which can be derived without any difficulty)

$$
\begin{aligned}
&{}_4F_3\left[\begin{matrix} a,\ 1+\frac{1}{2}a,\ b,\ -N \\ \frac{1}{2}a,\ 1+a-b,\ 1+2b-N \end{matrix}\ ;\ 1\right] \\
&= {}_3F_2\left[\begin{matrix} a,\ \ b,\ \ -N \\ 1+a-b,\ \ 1+2b-N \end{matrix}\ ;\ 1\right] - \frac{2bN}{(1+a-b)(1+2b-N)} \\
&\quad \times {}_3F_2\left[\begin{matrix} a+1,\ \ b+1,\ \ -(N-1) \\ 2+a-b,\ \ 2+2b-N \end{matrix}\ ;\ 1\right].
\end{aligned} \tag{5.1}
$$

Now it is easy to see that the first ${}_3F_2$ on the right hand side of (5.1) can be evaluated with the help of (1.6) and the second ${}_3F_2$ on the right hand side of (5.1) can be evaluated with the help of Saalschütz's theorem (1.1) and after a little simplification, we arrive at the right hand side of (1.7). This completes the proof of (1.7). □

Proof of (1.8). In order to prove the identity (1.8), we shall use the following result (which can be derived without any difficulty)

$$
\begin{aligned}
&{}_4F_3\left[\begin{matrix} a,\ 1+\frac{1}{2}a,\ b,\ -N \\ \frac{1}{2}a,\ 1+a-b,\ 2+2b-N \end{matrix}\ ;\ 1\right] \\
&= {}_3F_2\left[\begin{matrix} a,\ \ b,\ \ -N \\ 1+a-b,\ \ 2+2b-N \end{matrix}\ ;\ 1\right] - \frac{2bN}{(1+a-b)(2+2b-N)} \\
&\quad \times {}_3F_2\left[\begin{matrix} a+1,\ \ b+1,\ \ -(N-1) \\ 2+a-b,\ \ 3+2b-N \end{matrix}\ ;\ 1\right].
\end{aligned} \tag{5.2}
$$

Now it is easy to see that the first ${}_3F_2$ on the right hand side of (5.2) can be evaluated with the help of (3.2) and the second ${}_3F_2$ on the right hand side of (5.2) can be evaluated with the help of the result (3.1) and after some simplification, we arrive at the right hand side of (1.8). This completes the proof of (1.8). □

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Yong Sup Kim
Department of Mathematics Education
Wonkwang University
Iksan 570-749, Korea
E-mail address: yspkim@wonkwang.ac.kr

Arjun Kumar Rathie
Department of Mathematics
Vedant College of Engineering and Technology, Tulsi
323021, Distt. Bundi, Rajasthan State, India
E-mail address: akrathie@rediffmail.com