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NONLINEAR CONTRACTIONS IN PARTIALLY ORDERED QUASI b-METRIC SPACES

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Abstract. Using the concept of a g-monotone mapping we prove some common fixed point theorems for g-non-decreasing mappings which satisfy some generalized nonlinear contractions in partially ordered complete quasi b-metric spaces. The new theorems are generalizations of very recent fixed point theorems due to L. Ciric, N. Cakic, M. Rojovic, and J. S. Ume, [*Monotone generalized nonlinear contractions in partailly ordered metric spaces*, Fixed Point Theory Appl. (2008), article, ID-131294] and R. P. Agarwal, M. A. El-Gebeily, and D. O'Regan [*Generalized contractions in partially ordered metric spaces*, Appl. Anal. **87** (2008), 1–8].

1. Introduction

The extension of Banach fixed point theorem for contractive mappings has been done in many directions (cf. [1]-[15]). Recently, Agarwal et al. [1] and Ciric et al. [5], have come up with some new fixed and common fixed point theorems of mappings satisfying certain generalized nonlinear contractions in partially ordered metric spaces. The main idea in [1], [10] and [14] involve combining the ideas of iterative technique in the contraction mapping principle with those in the monotone technique.

The aim of this paper is to extend the results of [1] and [5] to the setting of partially ordered complete quasi b-metric spaces, by using some modified technique of [5]. Based on the concept of a g-monotone mapping we generalize some fixed point and common fixed point theorems for g-non-decreasing mappings satisfying some generalized nonlinear contractions in partially ordered complete quasi b-metric spaces.

Let $(X, \leq)$ be a partially ordered set. A mapping $F : X \to X$ is said to be non-decreasing if $x \leq y$ implies that $F(x) \leq F(y)$ for all $x, y \in X$. For completeness sake, the main results of [1] and [5] are described below.

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Theorem 1.1 ([1, Theorem 2.2]). *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Assume there is a non-decreasing function $\psi : [0, +\infty) \to [0, +\infty)$ with*

$$\lim_{n\to\infty} \psi^n(t) = 0$$

for each $t > 0$ and also suppose F is a non-decreasing mapping with

$$d(F(x), F(y)) \leq \psi\Bigg(\max\Bigg\{d(x,y), d(x,F(x)), d(y,F(y)), \frac{1}{2}[d(x,F(y)) + d(y,F(x))]\Bigg\}\Bigg) \tag{1}$$

for all $x \geq y$. Also suppose either

(a) *F is continuous or*

(b) *if $\{x_n\} \subset X$ is a non-decreasing sequence with $x_n \to x$ in X, then $x_n \leq x$ for all n hold.*

If there exists an $x_0 \in X$ with $x_0 \leq F(x_0)$, then F has a fixed point.

Agarwal, El-Gebeily and O'Regan [1] remove the condition that ψ is non-decreasing in Theorem 1.1 and so they came up with the following fixed point theorem.

Theorem 1.2 ([1, Theorem 2.3]). *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Assume there is a continuous function $\psi : [0, +\infty) \to [0, +\infty)$ with $\psi(t) < t$ for each $t > 0$ and also suppose F is a non-decreasing mapping with*

$$d(F(x), F(y)) \leq \psi(\max\{d(x,y), d(x,F(x)), d(y,F(y))\}) \quad \text{for all} \quad x \geq y. \tag{2}$$

Also suppose either (a) *or* (b) *hold. If there exists an $x_0 \in X$ with $x_0 \leq F(x_0)$, then F has a fixed point.*

The problem to extend Theorem 1.2 to mappings which satisfy (1) was addressed by Ciric et al. in the following theorem.

Theorem 1.3 ([5, Theorem 2.1]). *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Assume there is a continuous function $\varphi : [0, +\infty) \to [0, +\infty)$ with $\varphi(t) < t$ for each $t > 0$ and also suppose $F, g : X \to X$ are such that $F(X) \subseteq g(X)$, F is a g-non-decreasing mapping and*

$$d(F(x), F(y)) \leq \max\Bigg\{\varphi(d(g(x), g(y))), \varphi(d(g(x), F(x))), \varphi(d(g(y), F(y))), \varphi\left(\frac{d(g(x), F(y)) + d(g(y), F(x))}{2}\right)\Bigg\} \tag{3}$$

for all $x, y \in X$ for which $g(x) \geq g(y)$. Also suppose if $\{g(x_n)\} \subset X$ is a non-decreasing sequence with $g(x_n) \to g(z)$ in $g(X)$, then $g(x_n) \leq g(z)$ and

$g(z) \leq g(g(z))$ for all n hold. Also suppose $g(X)$ is closed. If there exists an $x_0 \in X$ with $g(x_0) \leq F(x_0)$, then F and g have a coincidence. Further, if F, g commute at their coincidence points, then F and g have a common fixed point.

In this paper we mainly extend Theorem 1.3, to the setting of a partially ordered complete quasi b-metric space, by modifying φ and hence using a somewhat different technique.

2. Main results

The concept of b-metric space was introduced by Czerwik in [6]. Since then several papers deal with fixed point theory for single valued and multivalued operators in b-metric spaces (see [2, 6, 15] and references therein).

Definition 2.1. Let X be a non-empty set. A real-valued function $d : X\times X \to \mathbb{R}^+$ is said to be a quasi b-metric on X with the constant $s \geq 1$ if the following conditions are satisfied:

(M_1) $d(x,y) \geq 0$ for all $x, y \in X$,
(M_2) $d(x,y) = 0$ if and only if $x = y$,
(M_3) $d(x,z) \leq s(d(x,y) + d(y,z))$ for all $x, y, z \in X$.

The pair (X, d) is called a quasi b-metric space. Observe that if $s = 1$, then the ordinary triangle inequality is satisfied, however it does not hold true when $s > 1$. Thus the class of quasi b-metric spaces is effectively larger than that of the ordinary quasi-metric spaces. That is, every quasi-metric space is a quasi b-metric space but the converse need not be true. The following example explains the above mentioned situation.

Example 2.2. Let $X = C([0,1],\mathbb{R})$ with the usual partial order. Define $d : X \times X \to \mathbb{R}^+$ by

$$d(f,g) = \begin{cases} \int_0^1 [g(t) - f(t)]^3\, dt, & \text{if } f \leq g, \\ \int_0^1 [f(t) - g(t)]^3\, dt, & \text{if } f \geq g. \end{cases}$$

Note that $d(f,g) \geq 0$ for all $f, g \in X$, and $d(f,g) = 0$ if and only if $f = g$. Also $d(f,g) = d(g,f)$ if and only if $f = g$ so that d is not symmetric.

Case (a) Let $f(t) = 2t$, $g(t) = 5t$ and $h(t) = 6t$ for $t \in [0,1]$. Then

$$d(f,h) = 16,$$
$$d(f,g) = \tfrac{27}{4},$$
$$d(g,h) = \tfrac{1}{4}.$$

That is,

$$d(f,h) > d(f,g) + d(g,h)$$

so that the usual triangle inequality is not satisfied. Suppose that there exists $s > 1$ such that

$$d(f,h) \leq s[d(f,g) + d(g,h)];$$

then putting the values and simplifying we get, $16 \leq 7s$ or $s \geq \frac{16}{7}$.

Thus for every $f, g, h \in X$, whenever the usual quasi-metric triangle inequality fails to hold, we can find an $s > 1$ such that the triangle inequality of the quasi b-metric is satisfied.

Case (b) Let $f(t) = -2t$, $g(t) = -5t$ and $h(t) = -6t$ for $t \in [0, 1]$. Then following the lines similar to Case (a) we conclude that the usual quasi-metric triangle inequality fails to hold and for every $f, g, h \in X$ we can find an $s > 1$ such that the triangle inequality of the quasi b-metric is satisfied.

From the above discussion it follows that (X, d) is a quasi b-metric space which is not an ordinary quasi-metric space.

Following example explains that the class of quasi b-metric spaces contains the class of the usual quasi-metric spaces.

Example 2.3. Let $X = l_p$, where $1 \leq p < \infty$, be defined by

$$l_p = \left\{ (x_n)_{n\geq 1} \subseteq \mathbb{R} : \sum_{n=1}^{\infty} |x_n|^p < \infty \right\}.$$

Define $d : X \times X \to \mathbb{R}^+$ by

$$d(x, y) = \begin{cases} 0 & \text{if } x \leq y, \\ \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}} & \text{if } x \geq y. \end{cases}$$

Then d satisfies all the conditions of a quasi b-metric with the constant $s = p \geq 1$. Indeed, if $p = 1$ the triangle inequality trivially holds; so let $p > 1$ and $x = (x_n)_{n\geq 1}$; $y = (y_n)_{n\geq 1}$; $z = (z_n)_{n\geq 1}$ be sequences in X with $x \neq y \neq z$. Then

$$d(x, z) = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}} = d(x, y); \quad d(y, z) = \left(\sum_{n=1}^{\infty} |y_n|^p\right)^{\frac{1}{p}}.$$

Since

$$|x_n|^p \leq p\,|x_n|^p = p\,|x_n|\,|x_n|^{p-1} \leq p\,(|x_n| + |y_n|)\,|x_n|^{p-1} \text{ for } n \in \mathbb{N},$$

we have

$$\begin{aligned} \sum_{n=1}^{\infty} |x_n|^p &\leq p\left(\sum_{n=1}^{\infty} |x_n|\,|x_n|^{p-1} + \sum_{n=1}^{\infty} |y_n|\,|x_n|^{p-1}\right) \\ &\leq p\left\{\left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{\infty} |y_n|^p\right)^{\frac{1}{p}}\right\}\left(\sum_{n=1}^{\infty} |x_n|^{q(p-1)}\right)^{\frac{1}{q}} \end{aligned}$$

simplifying we get

$$\left(\sum_{n=1}^{\infty} |x_n|^{p)}\right)^{\frac{1}{p}} \leq p\left\{\left(\sum_{n=1}^{\infty} |x_n|^p\right)^{\frac{1}{p}} + \left(\sum_{n=1}^{\infty} |y_n|^p\right)^{\frac{1}{p}}\right\}.$$

Thus

$$d(x,y) \leq p(d(x,y) + d(y,z))$$

and d is a quasi b-metric on X.

Definition 2.4. Suppose $(X, \leq)$ is a partially ordered set and $F, g : X \to X$ are mappings of X into itself. We say F is g-non-decreasing if for $x, y \in X$,

$$g(x) \leq g(y) \ \text{ implies } \ F(x) \leq F(y). \tag{4}$$

The main theoretical result of this paper is the following theorem.

Theorem 2.5. *Let $(X, \leq, d)$ be a partially ordered complete quasi b-metric space with the constant $s \geq 1$. Assume that the function $\varphi : [0, +\infty) \to [0, +\infty)$ is such that $\varphi(t) < \frac{t}{2s}$ for each $t > 0$ and $F, g : X \to X$ are such that $F(X) \subseteq g(X)$, F is a g-non-decreasing mapping and*

$$\begin{aligned} d(F(x), F(y)) \leq \max\Big\{ & \varphi(d(g(x), g(y))), \varphi(d(g(x), F(x))), \\ & \varphi\left(\frac{1}{2}[d(g(y), F(y)) + d(g(x), F(x))]\right), \\ & \varphi\left(\frac{1}{2s}[d(g(x), F(y)) + d(g(y), F(x))]\right) \Big\} \end{aligned} \tag{5}$$

for all $x, y \in X$ for which $g(x) \geq g(y)$. Further, suppose that $g(X)$ is closed and

(6) *if $\{g(x_n)\} \subset X$ is a non-decreasing sequence with $g(x_n) \to g(z)$ in $g(X)$, then $g(x_n) \leq g(z)$ and $g(z) \leq g(g(z))$ for all n hold.*

If there exists an $x_0 \in X$ with $g(x_0) \leq F(x_0)$, then F and g have a coincidence, and if F, g commute at their coincidence points, then F and g have a common fixed point.

Proof. Choose $x_0 \in X$ such that $g(x_0) \leq F(x_0)$. Since $F(X) \subseteq g(X)$, there exists $x_1 \in X$ such that $g(x_1) = F(x_0)$. Again $F(X) \subseteq g(X)$ implies that there exists $x_2 \in X$ such that $g(x_2) = F(x_1)$. Continuing this process we can obtain a sequence $\{x_n\}$ in X such that

$$g(x_{n+1}) = F(x_n) \ \text{ for all } \ n \geq 0. \tag{7}$$

Since $g(x_0) \leq g(x_1)$ from (4),

$$F(x_0) \leq F(x_1).$$

Thus, by (7), $g(x_1) \leq g(x_2)$ and (4),

$$F(x_1) \leq F(x_2),$$

that is, $g(x_2) \leq g(x_3)$. Proceeding in this way, we get

$$F(x_0) \leq F(x_1) \leq F(x_2) \leq F(x_3) \leq \cdots \leq F(x_n) \leq F(x_{n+1}) \leq \cdots. \tag{8}$$

Let $\delta_n = d(F(x_n), F(x_{n+1}))$. We shall prove that

$$\delta_n < \frac{\delta_{n-1}}{2s} \quad \text{for all} \quad n \geq 1. \tag{9}$$

Since $g(x_n) \leq g(x_{n+1})$ for all $n \geq 0$, putting $x = x_n$ and $y = x_{n+1}$ into (5) we get

$$\begin{aligned} & d(F(x_n), F(x_{n+1})) \\ \leq\ & \max\Big\{\varphi(d(g(x_n), g(x_{n+1}))), \varphi(d(g(x_n), F(x_n))), \\ & \varphi\Big(\frac{1}{2}[d(g(x_{n+1}), F(x_{n+1})) + d(g(x_n), F(x_n))]\Big), \\ & \varphi\Big(\frac{1}{2s}[d(g(x_n), F(x_{n+1})) + d(g(x_{n+1}), F(x_n))]\Big)\Big\}. \end{aligned}$$

And by (7),

$$\begin{aligned} & d(F(x_n), F(x_{n+1})) \\ \leq\ & \max\Big\{\varphi(d(F(x_{n-1}), F(x_n))), \varphi(d(F(x_{n-1}), F(x_n))), \\ & \varphi\Big(\frac{1}{2}[d(F(x_n), F(x_{n+1})) + d(F(x_{n-1}), F(x_n))]\Big), \\ & \varphi\Big(\frac{1}{2s}d(F(x_{n-1}), F(x_{n+1}))\Big)\Big\}. \end{aligned}$$

Or

$$\begin{aligned} & d(F(x_n), F(x_{n+1})) \\ \leq\ & \max\Big\{\varphi(d(F(x_{n-1}), F(x_n))), \\ & \varphi\Big(\frac{1}{2}[d(F(x_n), F(x_{n+1})) + d(F(x_{n-1}), F(x_n))]\Big), \\ & \varphi\Big(\frac{1}{2s}d(F(x_{n-1}), F(x_{n+1}))\Big)\Big\}. \end{aligned}$$

If $d(F(x_n), F(x_{n+1})) \leq \varphi(d(F(x_{n-1}), F(x_n)))$, then (9) holds, as $\varphi(t) < \frac{t}{2s}$ for $t > 0$. If

$$d(F(x_n), F(x_{n+1})) \leq \varphi\left(\frac{1}{2}[d(F(x_n), F(x_{n+1})) + d(F(x_{n-1}), F(x_n))]\right),$$

then we have

$$\begin{aligned} d(F(x_n), F(x_{n+1}))) &\leq \varphi\left(\frac{1}{2}[d(F(x_n), F(x_{n+1})) + d(F(x_{n-1}), F(x_n))]\right) \\ &< \frac{\frac{1}{2}[d(F(x_n), F(x_{n+1})) + d(F(x_{n-1}), F(x_n))]}{2s} \\ &= \frac{1}{4s}[d(F(x_n), F(x_{n+1})) + d(F(x_{n-1}), F(x_n))] \end{aligned}$$

or

$$d(F(x_n),F(x_{n+1}))) < \frac{1}{4s-1}d(F(x_{n-1}),F(x_n) \leq \frac{1}{2s}d(F(x_{n-1}),F(x_n)).$$

Thus

$$\delta_n = d(F(x_n),F(x_{n+1})) < \frac{1}{2s}d(F(x_{n-1}),F(x_n) = \frac{\delta_{n-1}}{2s}.$$

Lastly, if $d(F(x_n),F(x_{n+1})) \leq \varphi(d(F(x_{n-1}),F(x_{n+1}))/2s)$, then we have

$$\begin{aligned} d(F(x_n),F(x_{n+1})) &\leq \varphi\left(\frac{1}{2s}d(F(x_{n-1}),F(x_{n+1}))\right) \\ &< \frac{1}{4s^2}d(F(x_{n-1}),F(x_{n+1})) \\ &\leq \frac{1}{4s^2}(s[d(F(x_{n-1}),F(x_n))+d(F(x_n),F(x_{n+1}))]) \\ &= \frac{1}{4s}[d(F(x_{n-1}),F(x_n))+d(F(x_n),F(x_{n+1}))] \end{aligned}$$

simplifying we get

$$d(F(x_n),F(x_{n+1})) < \frac{1}{4s-1}d(F(x_{n-1}),F(x_n)) \leq \frac{\delta_{n-1}}{2s}.$$

Therefore, we have proved that (9) holds. It follows from (9) that for every $s \geq 1$

$$0 \leq \delta_n < \frac{\delta_{n-1}}{(2s)} < \frac{\delta_{n-2}}{(2s)^2} < \frac{\delta_{n-3}}{(2s)^3} < \cdots < \frac{\delta_0}{(2s)^n}$$

and so

$$\lim_{n\to\infty}\delta_n = 0. \tag{10}$$

Now we prove that $\{F(x_n)\}$ is a Cauchy sequence. Let $m > n$. Then we have

$$\begin{aligned} d(F(x_n),F(x_m)) &\leq sd(F(x_n),F(x_{n+1})) + s^2d(F(x_{n+1}),F(x_{n+2})) \\ &\quad + s^3d(F(x_{n+2}),F(x_{n+3})) + \cdots + s^md(F(x_{m-1}),F(x_m)) \\ &= s\delta_n + s^2\delta_{n+1} + s^3\delta_{n+2} + \cdots + s^m\delta_{m-1} \\ &\leq s\delta_n + \left(\frac{s\delta_n}{2} + \frac{s\delta_n}{2^2} + \cdots + \frac{s\delta_n}{2^{m-n-1}}\right) \\ &\leq s\delta_n\left(1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots\right) = 2s\delta_n \to 0 \text{ as } n\to\infty. \end{aligned}$$

Therefore, $\{F(x_n)\}$ is a Cauchy sequence. Since $\{F(x_n)\} = \{g(x_{n+1})\} \subseteq g(X)$ and $g(X)$ is closed, there exists $z \in X$ such that

$$\lim_{n\to\infty} g(x_n) = g(z)\left(= \lim_{n\to\infty} F(x_{n-1})\right). \tag{11}$$

Now we show that z is a coincidence of F and g. Since from (6) and (11) we have $g(x_n) \leq g(z)$ for all n, then by the triangle inequality and (5) we get

$$\begin{aligned}
&d(g(z), F(z)) \\
&\leq s[d(g(z), F(x_n)) + d(F(x_n), F(z))] \\
&\leq sd(g(z), F(x_n)) + s\Bigg[\max\Bigg\{\varphi(d(g(x_n), g(z))), \varphi(d(g(x_n), F(x_n))), \\
&\qquad \varphi\left(\frac{1}{2}d(g(z), F(z)) + d(g(x_n), F(x_n))\right), \\
&\qquad \varphi\left(\frac{d(g(x_n), F(z)) + d(g(z), F(x_n))}{2s}\right)\Bigg\}\Bigg] \\
&< sd(g(z), F(x_n)) + s\Bigg[\max\Bigg\{\frac{1}{2s}d(g(x_n), g(z)), \frac{1}{2s}(d(g(x_n), F(x_n)), \\
&\qquad \frac{1}{4s}(d(g(z), F(z)) + d(g(x_n), F(x_n))), \\
&\qquad \frac{d(g(x_n), F(z)) + d(g(z), F(x_n))}{4s^2}\Bigg\}\Bigg].
\end{aligned}$$

Applying limit on both sides as $n \to \infty$ and simplifying, we get

$$d(g(z), F(z)) \leq \max\left\{\frac{1}{2}d(g(z), F(z)), \frac{1}{4s}d(g(z), F(z))\right\} = \frac{1}{2}d(g(z), F(z)).$$

Hence $d(g(z), F(z)) = 0$, and so $F(z) = g(z)$. Thus F and g have a coincidence z.

Suppose that $Fg(x) = gF(x)$ for all $x \in X$. Set $w = g(z) = F(z)$. Then

$$F(w) = F(g(z)) = g(F(z)) = g(w).$$

Since $\{g(x_n)\}$ is a non-decreasing with $\lim_{n\to\infty} g(x_n) = g(z)$, from (6) we have $g(z) \leq g(g(z)) = g(w)$ and as $g(z) = F(z)$ and $g(w) = F(w)$, from (5) we get

$$\begin{aligned}
d(w, F(w)) &= d(F(z), F(w)) \\
&\leq \max\Bigg\{\varphi(d(g(z), g(w))), \varphi(d(g(z), F(z))), \\
&\qquad \varphi\left(\frac{1}{2}d(g(w), F(w)) + d(g(z), F(z))\right), \\
&\qquad \varphi\left(\frac{d(g(z), F(w)) + d(g(w), F(z))}{2s}\right)\Bigg\} \\
&< \max\left\{\frac{1}{2s}d(g(z), g(w)), \frac{d(g(z), F(w)) + d(g(w), F(z))}{4s^2}\right\}.
\end{aligned} \tag{12}$$

If

$$d(F(z), F(w)) < \frac{1}{2s}d(g(z), g(w)),$$

then clearly $d(F(z), F(w)) = 0$. If

$$d(F(z), F(w)) \leq \frac{d(g(z), F(w)) + d(g(w), F(z))}{4s^2},$$

then this gives us

$$d(F(z), F(w)) < \frac{d(F(w), F(z))}{4s^2 - 1}.$$

Interchanging w and z in the equation (12) and simplifying we get

$$d(F(w), F(z)) < \frac{d(F(z), F(w))}{4s^2 - 1}. \tag{13}$$

Thus from equations (12) and (13), we get

$$d(F(z), F(w)) < \frac{d(F(w), F(z))}{4s^2 - 1} < \frac{d(F(z), F(w))}{(4s^2 - 1)^2}.$$

This implies that $d(F(z), F(w)) = 0$. Thus we get $d(w, F(w)) = 0$, implying that

$$F(w) = g(w) = w$$

and hence F, g have a common fixed point. □

Remark 2.6. Theorem 2.5 also holds true if F is g-non-decreasing be replaced with F is g-non-increasing and $g(x_0) \leq F(x_0)$ be replaced with $F(x_0) \geq g(x_0)$.

Example 2.7. Let $X = [-1, 1]$ with the usual partial order. Define $d : X \times X \to \mathbb{R}^+$ by

$$d(x, y) = \begin{cases} 0 & \text{if and only if } x = y \\ \left|x - \frac{y}{2}\right|^2 & \text{otherwise.} \end{cases}$$

Note that $d(x, y) \geq 0$ for all $x, y \in X$, and $d(x, y) = 0$ if and only if $x = y$. Also $d(x, y) = d(y, x)$ if and only if $x = y$ so that d is not symmetric. Let $x = 1$, $y = 0$ and $z = -1$. Then

$$d(x, z) = 2.25,$$
$$d(x, y) = 1,$$
$$d(y, z) = \tfrac{1}{4},$$

so that the usual triangle inequality is not satisfied. However if $p \in (0, 1]$, then we have

$$d(x, z) \leq 2^{\frac{1}{p}} \left(d(x, y) + d(y, z)\right).$$

Since $s = 2^{\frac{1}{p}} \geq 2$ for $p \in (0, 1]$, so d is a quasi b-metric on X which is not a usual quasi-metric on X. Thus $(X, \leq, d)$ is a partially ordered complete quasi b-metric space with the constant $s \geq 2$. Define $F, g : X \to X$ by $F(x) = \frac{x}{6}$ and $g(x) = x$. Let $\varphi : [0, +\infty) \to [0, +\infty)$ be defined by $\varphi(t) = \frac{t}{3}$. Observe that

$$\text{for } x, y \in X, \quad g(x) \leq g(y) \implies F(x) \leq F(y)$$

and hence F is g-nondecreasing. Also for $g(x) \geq g(y)$

$$d(F(x), F(y)) = \left|F(x) - \frac{F(y)}{2}\right|^2 = \frac{1}{36}\left|x - \frac{y}{2}\right|^2 \leq \frac{1}{16}$$

when $x = 1$ and $y = -1$. On the other hand we have

$$\begin{aligned} \varphi(d(g(x), g(y))) &\leq \frac{3}{4}, \\ \varphi(d(g(x), F(x))) &\leq \frac{1}{16}\left(\frac{121}{27}\right), \\ \varphi\left(\frac{1}{2}[d(g(y), F(y)) + d(g(x), F(x))]\right) &\leq \frac{1}{16}\left(\frac{121}{27}\right), \\ \varphi\left(\frac{1}{2s}[d(g(x), F(y)) + d(g(y), F(x))]\right) &\leq \frac{1}{16s}\left(\frac{169}{27}\right). \end{aligned}$$

Thus

$$d(F(x), F(y)) \leq \max\left\{\begin{array}{c} \varphi(d(g(x), g(y))), \varphi(d(g(x), F(x))), \\ \varphi\left(\frac{1}{2}[d(g(y), F(y)) + d(g(x), F(x))]\right), \\ \varphi\left(\frac{1}{2s}[d(g(x), F(y)) + d(g(y), F(x))]\right) \end{array}\right\}$$

and so F satisfy the contraction condition. Also g is continuous and non-decreasing and $x_n \to x$ implies that $g(x_n) \to g(x)$ so that by Theorem 2.5 $g(x_n) \leq g(x)$ for all $n \geq 1$ and $g(x) \leq gg(x)$. Note that $g(X) = [-1, 1]$ and $F(X) = [-\frac{1}{6}, \frac{1}{6}] \subseteq g(X)$. Let $x_0 = -\frac{1}{2}$; then since

$$g\left(-\frac{1}{2}\right) = -\frac{1}{2} < -\frac{1}{12} = F\left(-\frac{1}{2}\right),$$

by Theorem 2.5, F and g have a coincidence. Since

$$Fg\left(-\frac{1}{2}\right) = -\frac{1}{12} = gF\left(-\frac{1}{2}\right),$$

by Theorem 2.5, F and g have a common fixed point.

Corollary 2.8. *Let $(X, \leq, d)$ be a partially ordered complete quasi b-metric space with the constant $s \geq 1$. Assume there is a function $\varphi : [0, +\infty) \to [0, +\infty)$ with $\varphi(t) < \frac{t}{2s}$ for each $t > 0$ and also suppose $F : X \to X$ is a non-decreasing mapping and*

$$d(F(x), F(y)) \leq \max\left\{\varphi(d(x, y)), \varphi(d(x, F(x))), \varphi\left(\frac{d(y, F(y))) + d(x, F(x))}{2}\right), \varphi\left(\frac{d(x, F(y)) + d(y, F(x))}{2s}\right)\right\}$$

for all $x, y \in X$ for which $x \leq y$. Also suppose either (i) *if $\{x_n\} \subset X$ is a non-decreasing sequence with $x_n \to z$ in X, then $x_n \leq z$ for all n hold or* (ii) *F is continuous.*

If there exists an $x_0 \in X$ with $x_0 \leq F(x_0)$, then F has a fixed point.

Proof. Suppose (i) holds, then the corollary follows by taking $g = I$ (the identity mapping on X) in Theorem 2.5. If (ii) holds, then from (11) with $g = I$ we get

$$z = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} F(x_n) = F(\lim_{n\to\infty} x_n) = F(z). \qquad \square$$

Corollary 2.9. *Let $(X, \leq, d)$ be a partially ordered complete quasi b-metric space with the constant $s \geq 1$. Assume there is a function $\varphi : [0,+\infty) \to [0,+\infty)$ with $\varphi(t) < \frac{t}{2s}$ for each $t > 0$ and $F : X \to X$ is a non-decreasing mapping such that*

$$\begin{aligned} &d(F(x), F(y)) \\ (14) \qquad &\leq \max\left\{\varphi(d(x,y)), \varphi(d(x,F(x))), \varphi\left(\frac{d(y,F(y)) + d(x,F(x))}{2s}\right)\right\} \end{aligned}$$

for all $x, y \in X$ for which $x \leq y$. Further, suppose either (i) *if $\{x_n\} \subset X$ is a non-decreasing sequence with $x_n \to z$ in X, then $x_n \leq z$ for all n hold or* (ii) *F is continuous.*

If there exists an $x_0 \in X$ with $x_0 \leq F(x_0)$, then F has a fixed point.

Proof. Follows from Theorem 2.5 with $g = I$, where I is the identity mapping on X. $\square$

Corollary 2.10. *Let $(X, \leq, d)$ be a partially ordered complete quasi b-metric space with the constant $s \geq 1$. Suppose that $F : X \to X$ is a non-decreasing mapping such that*

$$\begin{aligned} d(F(x), F(y)) \leq \frac{1}{3s} \max\Bigg\{ &d(x,y), d(x,F(x)), \frac{d(y,F(y)) + d(x,F(x))}{2}, \\ &\frac{d(x,F(y)) + d(y,F(x))}{2s} \Bigg\} \end{aligned}$$

for all $x, y \in X$ for which $x \leq y$. Also suppose either (i) *if $\{x_n\} \subset X$ is a non-decreasing sequence with $x_n \to z$ in X, then $x_n \leq z$ for all n hold or* (ii) *F is continuous.*

If there exists an $x_0 \in X$ with $x_0 \leq F(x_0)$, then F has a fixed point.

Proof. It follows from Theorem 2.5 with $\varphi(t) = \frac{t}{3s}$ and $g = I$, where I is the identity mapping on X. $\square$

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