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CERTAIN DECOMPOSITION FORMULAS OF GENERALIZED HYPERGEOMETRIC FUNCTIONS $_pF_q$ AND SOME FORMULAS OF AN ANALYTIC CONTINUATION OF THE CLAUSEN FUNCTION $_3F_2$

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Abstract. Here, by using the symbolical method introduced by Burchnall and Chaundy, we aim at constructing certain expansion formulas for the generalized hypergeometric function $_pF_q$. In addition, using our expansion formulas for $_pF_q$, we present formulas of an analytic continuation of the Clausen hypergeometric function $_3F_2$, which are much simpler than an earlier known result. We also give some integral representations for $_3F_2$.

1. Introduction

The well-known generalized hypergeometric function $_pF_q$ $(p,\, q \in \mathbb{N} \cup \{0\};$ $\mathbb{N} := \{1,\, 2,\, 3,\, \ldots\})$ is defined by the following series:

$$(1.1)\qquad {}_pF_q\left[\begin{array}{c}\alpha_1, \alpha_1, \ldots, \alpha_p;\\ \beta_1, \beta_1, \ldots, \beta_q;\end{array} z\right] = \sum_{m=0}^{\infty} \frac{(\alpha_1)_m\,(\alpha_2)_m \cdots (\alpha_p)_m}{(\beta_1)_m\,(\beta_2)_m \cdots (\beta_q)_m} \frac{z^m}{m!}$$
$$= {}_pF_q\left(\alpha_1, \alpha_1, \ldots, \alpha_p;\, \beta_1, \beta_1, \ldots, \beta_q;\, z\right),$$

where $(\lambda)_n$ denotes the Pochhammer symbol defined by

$$(1.2)\qquad (\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} \quad (\lambda \neq 0,\, -1,\, -2, \ldots,).$$

For the convergence conditions, special cases, and further generalizations of (1.1), we refer to a comprehensive book of Srivastava and Karlsson [15] on this subject. It goes without saying that the theory of various kinds of hypergeometric functions has been useful and important for scientists and engineers who have been practically involved in solving certain differential equations. In fact, the solutions of the differential equations concerning thermal conduction and dynamics, electromagnetic oscillations and aerodynamics, quantum mechanics

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and the theory of potentials, have been able to be expressed as hypergeometric functions. In particular, they appear very often in solving partial differential equations which are dealt with separating variables. In view of theory and applications, a large number of hypergeometric functions have been developed, for example, as many as 205 hypergeometric functions are recorded in the monograph [15].

Olsson [9, p. 704, Eq.(3)] presented formulas of analytic continuation of Clausen function ${}_3F_2$ (for its name, see [1, p. 141]) one of which is recalled here:

$$(1.3)\qquad {}_3F_2\left[\begin{array}{c} a_1,a_2,a_3; \\ b_1,b_2; \end{array} x\right] = F_R\left[\begin{array}{c} a_1,a_2,a_3 \\ b_1,b_2 \end{array} x\right] + \xi\left[\begin{array}{c} a_1,a_2,a_3; \\ b_1,b_2; \end{array} x\right],$$

where

(1.4)

$$\begin{aligned} F_R\left[\begin{array}{c} a_1,a_2,a_3; \\ b_1,b_2; \end{array} x\right] =& \frac{\Gamma(b_1)\Gamma(b_2)\Gamma(b_1+b_2-a_1-a_2-a_3)}{\Gamma(a_1)\Gamma(b_1+b_2-a_1-a_2)\Gamma(b_1+b_2-a_1-a_3)} \\ &\cdot \sum_{n=0}^{\infty} \frac{(b_1-a_1)_n(b_2-a_1)_n(b_1+b_2-a_1-a_2-a_3)_n}{(b_1+b_2-a_1-a_2)_n(b_1+b_2-a_1-a_3)_n\, n!} \\ &\cdot {}_2F_1(a_2,a_3;a_1+a_2+a_3-b_1-b_2-n+1;1-x) \end{aligned}$$

and

(1.5)

$$\begin{aligned} &\xi\left[\begin{array}{c} a_1,a_2,a_3; \\ b_1,b_2; \end{array} x\right] = \frac{\Gamma(b_1)\Gamma(b_2)\Gamma(a_1+a_2+a_3-b_1-b_2)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)} \\ &\cdot x^{a_1-b_1-b_2+1}(1-x)^{b_1+b_2-a_1-a_2-a_3}\sum_{n=0}^{\infty}\frac{(b_1-a_1)_n(b_2-a_1)_n}{(b_1+b_2-a_1-a_2-a_3+1)_n\, n!}\left(\frac{x-1}{x}\right)^n \\ &\cdot {}_2F_1(1-a_2,1-a_3;b_1+b_2-a_1-a_2-a_3+n+1;1-x). \end{aligned}$$

An analytic continuation of the Appell hypergeometric function (see [1, p. 14, Eq.(12); p. 28, Eq.(2)] and [5])

$$(1.6)\qquad F_2(a;b_1,b_2;c_1,c_2;x,y) = \sum_{i,j=0}^{\infty}\frac{(a)_{i+j}(b_1)_i(b_2)_j}{(c_1)_i(c_2)_j\, i!j!}x^iy^j \qquad (|x|+|y|<1)$$

is achieved by expressing in the following integral form:

(1.7)

$$\begin{aligned} &F_2(a;b_1,b_2;c_1,c_2;x,y) \\ &= \frac{\Gamma(c_1)\Gamma(c_2)}{\Gamma(b_1)\Gamma(c_1-b_1)\Gamma(b_2)\Gamma(c_2-b_1)} \\ &\cdot \int_0^1\int_0^1 \xi^{b_1-1}\eta^{b_2-1}(1-\xi)^{c_1-b_1-1}(1-\eta)^{c_2-b_2-1}(1-x\xi-y\eta)^{-a}\,d\xi d\eta \\ &(\Re(c_i)>\Re(b_i)>0,\ \ i=1,\,2). \end{aligned}$$

Here, by using the symbolical method introduced by Burchnall and Chaundy (see [2, 3] and Chaundy [4]), we aim at constructing certain expansion formulas for the generalized hypergeometric function ${}_pF_q$. In addition, using our expansion formulas for ${}_pF_q$, we present formulas of an analytic continuation of the Clausen hypergeometric function ${}_3F_2$, which are much simpler than (1.3). We also give some integral representations for ${}_3F_2$.

2. Sets of operator and functional identities

Burchnall and Chaundy [2, 3] and (Chaundy [4]) systematically presented a number of expansion and decomposition formulas for double hypergeometric functions in series of simpler hypergeometric functions. Their method is based upon the following inverse pairs of symbolic operators:

$$\nabla_{x,y}(h) := \frac{\Gamma(h)\,\Gamma(\delta+\delta'+h)}{\Gamma(\delta+h)\,\Gamma(\delta'+h)} = \sum_{i=0}^{\infty} \frac{(-\delta)_i\,(-\delta')_i}{(h)_i\,i!} \tag{2.1}$$

and

(2.2)

$$\begin{aligned}\Delta_{x,y}(h) &:= \frac{\Gamma(\delta+h)\,\Gamma(\delta'+h)}{\Gamma(h)\,\Gamma(\delta+\delta'+h)} = \sum_{i=0}^{\infty} \frac{(-\delta)_i\,(-\delta')_i}{(1-h-\delta-\delta')_i\,i!} \\ &= \sum_{i=0}^{\infty} \frac{(-1)^i\,(h)_{2i}\,(-\delta)_i\,(-\delta')_i}{(h+i-1)_i\,(\delta+h)_i\,(\delta'+h)_i\,i!} \quad \left(\delta := x\frac{\partial}{\partial x};\ \delta' := y\frac{\partial}{\partial y}\right).\end{aligned}$$

Indeed, as already observed by Srivastava and Karlsson [15, pp. 332–333], the aforementioned method of Burchnall and Chaundy (cf. [2, 3]) was subsequently applied by Pandey [10] and Srivastava [14] in order to derive the corresponding expansion and decomposition formulas for the triple hypergeometric functions $F_A^{(3)}$, F_E, F_K, F_M, F_P and F_T, H_A, H_C, respectively (see, for definitions, [15, Section 1.5] and [16, p. 66 et seq.]), and by Singhal and Bhati [13], Hasanov and Srivastava [6, 7] for deriving analogous multiple-series expansions associated with several multivariable hypergeometric functions. We now introduce here the following analogues of Burchnall-Chaundy symbolic operators $\nabla_{x,y}$ and $\Delta_{x,y}$ defined by (2.1) and (2.2), respectively:

$$\begin{aligned}H_{x_1,\ldots,x_l}(\alpha,\beta) &:= \frac{\Gamma(\beta)\,\Gamma(\alpha+\delta_1+\cdots+\delta_l)}{\Gamma(\alpha)\,\Gamma(\beta+\delta_1+\cdots+\delta_l)} \\ &= \sum_{k_1,\ldots,k_l=0}^{\infty} \frac{(\beta-\alpha)_{k_1+\cdots+k_l}\,(-\delta_1)_{k_1}\cdots(-\delta_l)_{k_l}}{(\beta)_{k_1+\cdots+k_l}\,k_1!\cdots k_l!}\end{aligned} \tag{2.3}$$

and

$$
\begin{aligned}
\bar{H}_{x_1,\ldots,x_l}(\alpha,\beta) &:= \frac{\Gamma(\alpha)\,\Gamma(\beta+\delta_1+\cdots+\delta_l)}{\Gamma(\beta)\,\Gamma(\alpha+\delta_1+\cdots+\delta_l)} \\
&= \sum_{k_1,\ldots,k_l=0}^{\infty} \frac{(\beta-\alpha)_{k_1+\cdots+k_l}\,(-\delta_1)_{k_1}\cdots(-\delta_l)_{k_l}}{(1-\alpha-\delta_1-\cdots-\delta_l)_{k_1+\cdots+k_l}\,k_1!\cdots k_l!}
\end{aligned} \tag{2.4}
$$

$$
\left(\delta_j := x_j\,\frac{\partial}{\partial x_j},\ j=1,\ldots,l;\ l\in\mathbb{N}\right).
$$

By making use of operators (2.3)-(2.4) and the method given by Burchnall and Chaundy [2, 3], we construct functional identities for the generalized hypergeometric function ${}_pF_q$ given in (1.1) as in the following lemma.

Lemma 1. *Each of the following functional identities holds true*:

$$
{}_pF_q\left[\begin{matrix}\alpha_1,\alpha_2,\ldots,\alpha_p;\\ \beta_1,\beta_2,\ldots,\beta_q;\end{matrix}\ x\right] = H_x(\alpha_p,\beta_q)\ {}_{p-1}F_{q-1}\left[\begin{matrix}\alpha_1,\alpha_2,\ldots,\alpha_{p-1};\\ \beta_1,\beta_2,\ldots,\beta_{q-1};\end{matrix}\ x\right] \tag{2.5}
$$

$$
(p\geq 3,\ q\geq 2;\ p,\,q\in\mathbb{N});
$$

$$
{}_{p-1}F_{q-1}\left[\begin{matrix}\alpha_1,\alpha_2,\ldots,\alpha_{p-1};\\ \beta_1,\beta_2,\ldots,\beta_{q-1};\end{matrix}\ x\right] = \bar{H}_x(\alpha_p,\beta_q)\ {}_pF_q\left[\begin{matrix}\alpha_1,\alpha_2,\ldots,\alpha_p;\\ \beta_1,\beta_2,\ldots,\beta_q;\end{matrix}\ x\right] \tag{2.6}
$$

$$
(p\geq 3,\ q\geq 2;\ p,\,q\in\mathbb{N});
$$

$$
{}_pF_q\left[\begin{matrix}\alpha_1,\alpha_2,\ldots,\alpha_p;\\ \beta_1,\beta_2,\ldots,\beta_q;\end{matrix}\ x\right] = H_x(\alpha_p,\beta_q)\,H_x(\alpha_{p-1},\beta_{q-1})\,{}_{p-2}F_{q-2}\left[\begin{matrix}\alpha_1,\alpha_2,\ldots,\alpha_{p-2};\\ \beta_1,\beta_2,\ldots,\beta_{q-2};\end{matrix}\ x\right] \tag{2.7}
$$

$$
(p\geq 4,\ q\geq 3;\ p,\,q\in\mathbb{N});
$$

$$
{}_{p-2}F_{q-2}\left[\begin{matrix}\alpha_1,\alpha_2,\ldots,\alpha_{p-2};\\ \beta_1,\beta_2,\ldots,\beta_{q-2};\end{matrix}\ x\right] = \bar{H}_x(\alpha_p,\beta_q)\,\bar{H}_x(\alpha_{p-1},\beta_{q-1})\ {}_pF_q\left[\begin{matrix}\alpha_1,\alpha_2,\ldots,\alpha_p;\\ \beta_1,\beta_2,\ldots,\beta_q;\end{matrix}\ x\right] \tag{2.8}
$$

$$
(p\geq 4,\ q\geq 3;\ p,\,q\in\mathbb{N}).
$$

Proof. The functional identities (2.5)-(2.8) can be proved by means of Mellin transformation (see, for example, [8]). □

3. Expansion formulas for ${}_pF_q$

Applying operators (2.3)-(2.4) and using the functional identities in Lemma 1, we obtain the following expansions for the generalized hypergeometric function ${}_pF_q$.

Theorem 1. *Each of the following expansion formulas for* ${}_pF_q$ *holds true*:

(3.1)
$$\begin{aligned}&{}_pF_q\left[\begin{array}{c}\alpha_1,\alpha_2,\ldots,\alpha_p;\\ \beta_1,\beta_2,\ldots,\beta_q;\end{array} x\right]\\ &=\sum_{i=0}^{\infty}\frac{(-1)^i(\alpha_1)_i(\alpha_2)_i\cdots(\alpha_{p-1})_i(\beta_q-\alpha_p)_i}{(\beta_1)_i(\beta_2)_i\cdots(\beta_{q-1})_i(\beta_q)_i\,i!}x^i\,{}_{p-1}F_{q-1}\left[\begin{array}{c}\alpha_1+i,\alpha_2+i,\ldots,\alpha_{p-1}+i;\\ \beta_1+i,\beta_2+i,\ldots,\beta_{q-1}+i;\end{array} x\right]\\ &\qquad (p\geq 3,\ q\geq 2;\ p,\,q\in\mathbb{N});\end{aligned}$$

(3.2)
$$\begin{aligned}&{}_{p-1}F_{q-1}\left[\begin{array}{c}\alpha_1,\alpha_2,\ldots,\alpha_{p-1};\\ \beta_1,\beta_2,\ldots,\beta_{q-1};\end{array} x\right]\\ &=\sum_{i=0}^{\infty}\frac{(\alpha_1)_i(\alpha_2)_i\cdots(\alpha_{p-1})_i(\beta_q-\alpha_p)_i}{(\beta_1)_i(\beta_2)_i\cdots(\beta_p)_i\,i!}x^i\,{}_pF_q\left[\begin{array}{c}\alpha_1+i,\alpha_2+i,\ldots,\alpha_{p-1}+i,\alpha_p;\\ \beta_1+i,\beta_2+i,\ldots,\beta_q+i;\end{array} x\right]\\ &\qquad (p\geq 3,\ q\geq 2;\ p,\,q\in\mathbb{N});\end{aligned}$$

(3.3)
$$\begin{aligned}&{}_pF_q\left[\begin{array}{c}\alpha_1,\alpha_2,\ldots,\alpha_p;\\ \beta_1,\beta_2,\ldots,\beta_q;\end{array} x\right]\\ &=\sum_{i,j=0}^{\infty}\frac{(-1)^{i+j}(\alpha_1)_{i+j}(\alpha_2)_{i+j}\cdots(\alpha_{p-2})_{i+j}(\alpha_{p-1})_i(\beta_{q-1}-\alpha_{p-1})_j(\beta_q-\alpha_p)_i}{(\beta_1)_{i+j}(\beta_2)_{i+j}\cdots(\beta_{q-2})_{i+j}(\beta_{q-1})_{i+j}(\beta_q)_i\,i!j!}x^{i+j}\\ &\quad\cdot{}_{p-2}F_{q-2}\left[\begin{array}{c}\alpha_1+i,\alpha_2+i,\ldots,\alpha_{p-2}+i;\\ \beta_1+i,\beta_2+i,\ldots,\beta_{q-2}+i;\end{array} x\right]\quad (p\geq 4,\ q\geq 3;\ p,\,q\in\mathbb{N});\end{aligned}$$

(3.4)
$$\begin{aligned}&{}_{p-2}F_{q-2}\left[\begin{array}{c}\alpha_1,\alpha_2,\ldots,\alpha_{p-2};\\ \beta_1,\beta_2,\ldots,\beta_{q-2};\end{array} x\right]\\ &=\sum_{i,j=0}^{\infty}\frac{(\alpha_1)_{i+j}(\alpha_2)_{i+j}\cdots(\alpha_{p-2})_{i+j}(\alpha_p)_j(\beta_{q-1})_i(\beta_{q-1}-\alpha_{p-1})_{i+j}(\beta_q-\alpha_p)_i}{(\beta_1)_{i+j}(\beta_2)_{i+j}\cdots(\beta_q)_{i+j}(\beta_{q-1}-\alpha_{p-1})_i\,i!j!}x^{i+j}\\ &\quad\cdot{}_pF_q\left[\begin{array}{c}\alpha_1+i+j,\alpha_2+i+j,\ldots,\alpha_{p-2}+i+j,\alpha_{p-1},\alpha_p+j;\\ \beta_1+i+j,\beta_2+i+j,\ldots,\beta_q+i+j;\end{array} x\right]\\ &\qquad (p\geq 4,\ q\geq 3;\ p,\,q\in\mathbb{N});\end{aligned}$$

(3.5)
$$\begin{aligned}&{}_pF_q\left[\begin{array}{c}\alpha_1,\alpha_2,\ldots,\alpha_p;\\ \beta_1,\beta_2,\ldots,\beta_q;\end{array} x+y-xy\right]\\ &=\sum_{i=0}^{\infty}\frac{(-1)^i(\alpha_1)_i(\alpha_2)_i\cdots(\alpha_p)_i}{(\beta_1)_i(\beta_2)_i\cdots(\beta_q)_i\,i!}x^iy^i\,{}_pF_q\left[\begin{array}{c}\alpha_1+i,\alpha_2+i,\ldots,\alpha_p+i;\\ \beta_1+i,\beta_2+i,\ldots,\beta_q+i;\end{array} x+y\right];\end{aligned}$$

(3.6)
$$\begin{aligned}&{}_pF_q\left[\begin{array}{c}\alpha_1,\alpha_2,\ldots,\alpha_p;\\ \beta_1,\beta_2,\ldots,\beta_q;\end{array} x+y\right]\\ &=\sum_{i=0}^{\infty}\frac{(\alpha_1)_i(\alpha_2)_i\cdots(\alpha_p)_i}{(\beta_1)_i(\beta_2)_i\cdots(\beta_q)_i\,i!}x^iy^i\,{}_pF_q\left[\begin{array}{c}\alpha_1+i,\alpha_2+i,\ldots,\alpha_p+i;\\ \beta_1+i,\beta_2+i,\ldots,\beta_q+i;\end{array} x+y-xy\right].\end{aligned}$$

Proof. We begin by proving the identity (3.1). Applying the operator (2.3) to the functional identity (2.5), we obtain

(3.7)
$$ {}_pF_q\left[\begin{array}{c}\alpha_1,\alpha_2,\ldots,\alpha_p;\\ \beta_1,\beta_2,\ldots,\beta_q;\end{array}\ x\right]=\sum_{i=0}^{\infty}\frac{(\beta_q-\alpha_p)_i\,(-\delta)_i}{(\beta_q)_i\, i!}\ {}_{p-1}F_{q-1}\left[\begin{array}{c}\alpha_1,\alpha_2,\ldots,\alpha_{p-1};\\ \beta_1,\beta_2,\ldots,\beta_{q-1};\end{array}\ x\right]. $$

It is shown (see [11, p. 93]) that, for any analytic function $f(z)$, the following identity is true:

$$ (-\delta)_i\, f(z)=(-\delta)(1-\delta)\cdots(i-1-\delta)\,f(z)=(-1)^i\,z^i\frac{d^i}{dz^i}f(z), \tag{3.8} $$

where the operator δ is given in (2.2). Then on the basis of identity (3.8) and by virtue of the differentiation formula for the generalized hypergeometric function (see [1, p. 153, Eq.(31)])

(3.9)
$$ \frac{d^i}{dx^i}{}_pF_q\left[\begin{array}{c}\alpha_1,\alpha_2,\ldots,\alpha_p;\\ \beta_1,\beta_2,\ldots,\beta_q;\end{array}\ x\right]=\frac{(\alpha_1)_i\,(\alpha_2)_i\cdots(\alpha_p)_i}{(\beta_1)_i\,(\beta_2)_i\cdots(\beta_q)_i}{}_pF_q\left[\begin{array}{c}\alpha_1+i,\alpha_2+i,\ldots,\alpha_p+i;\\ \beta_1+i,\beta_2+i,\ldots,\beta_q+i;\end{array}\ x\right], $$

we find

(3.10)
$$ \begin{aligned}&(-\delta)_i\ {}_{p-1}F_{q-1}\left[\begin{array}{c}\alpha_1,\alpha_2,\ldots,\alpha_{p-1};\\ \beta_1,\beta_2,\ldots,\beta_{q-1};\end{array}\ x\right]\\ &=x^i\frac{(-1)^i\,(\alpha_1)_i\,(\alpha_2)_i\cdots(\alpha_{p-1})_i}{(\beta_1)_i\,(\beta_2)_i\cdots(\beta_{q-1})_i}{}_{p-1}F_{q-1}\left[\begin{array}{c}\alpha_1+i,\alpha_2+i,\ldots,\alpha_{p-1}+i;\\ \beta_1+i,\beta_2+i,\ldots,\beta_{q-1}+i;\end{array}\ x\right].\end{aligned} $$

Substituting identities (3.10) for the formula (3.7) gives the desired identity (3.1).

Using the identity (3.8), we find the following Taylor series for ${}_pF_q(x-h)$ (h is a variable):

(3.11)
$$ \begin{aligned}&{}_pF_q\left[\begin{array}{c}\alpha_1,\alpha_2,\ldots,\alpha_p;\\ \beta_1,\beta_2,\ldots,\beta_q;\end{array}\ x-h\right]\\ &=\sum_{i=0}^{\infty}\frac{h^i}{i!}x^{-i}\,(-\delta)_i\,{}_pF_q\left[\begin{array}{c}\alpha_1,\alpha_2,\ldots,\alpha_p;\\ \beta_1,\beta_2,\ldots,\beta_q;\end{array}\ x\right]\\ &=\sum_{i=0}^{\infty}\frac{(-1)^i\,(\alpha_1)_i\,(\alpha_2)_i\cdots(\alpha_p)_i}{(\beta_1)_i\,(\beta_2)_i\cdots(\beta_q)_i\, i!}h^i\,{}_pF_q\left[\begin{array}{c}\alpha_1+i,\alpha_2+i,\ldots,\alpha_p+i;\\ \beta_1+i,\beta_2+i,\ldots,\beta_q+i;\end{array}\ x\right].\end{aligned} $$

Replacing the arguments x and h in (3.11) by $x+y$ and xy, and by $x+y-xy$ and $-xy$ respectively, we prove (3.5) and (3.6). A similar argument will establish the other formulas (3.2), (3.3), and (3.4). □

The special cases of the formulas (3.1)-(3.4) are easily shown as in the following corollary.

Corollary 1. *Each of the following formulas for holds true*:

(3.12)
$$
{}_pF_q\left[\begin{array}{c}\alpha_1,\alpha_2,\ldots,\alpha_p;\\ \beta_1,\beta_2,\ldots,\beta_q;\end{array}\ x\right]
= \sum_{j=0}^{\infty}\frac{(\alpha_1)_j(\alpha_2)_j\cdots(\alpha_{p-1})_j}{(\beta_1)_j(\beta_2)_j\cdots(\beta_{q-1})_j\, j!}x^j\,{}_pF_q\left[\begin{array}{c}\alpha_1+j,\alpha_2+j,\ldots,\alpha_{p-1}+j,\beta_q-\alpha_p;\\ \beta_1+j,\beta_2+j,\ldots,\beta_q+j,\beta_q;\end{array}\ -x\right];
$$

(3.13)
$$
{}_pF_q\left[\begin{array}{c}\alpha_1,\alpha_2,\ldots,\alpha_p;\\ \beta_1,\beta_2,\ldots,\beta_q;\end{array}\ x\right]=F^{p-1,1,0}_{q-1,1,0}\left[\begin{array}{ccc}\alpha_1,\alpha_2,\ldots,\alpha_{p-1}; & \beta_q-\alpha_p; & -;\\ \beta_1,\beta_2,\ldots,\beta_{q-1}; & \beta_q; & -;\end{array}\ -x,x\right];
$$

(3.14)
$$
{}_3F_2\left[\begin{array}{c}\alpha_1,\alpha_2,\alpha_3\\ \beta_1,\beta_2\end{array}\ x\right]=(1-x)^{-\alpha_1}F^{1,2,1}_{1,1,0}\left[\begin{array}{ccc}\alpha_1; & \alpha_2,\beta_2-\alpha_3; & \beta_1-\alpha_2;\\ \beta_1; & \beta_2; & -;\end{array}\ \tfrac{x}{x-1},\tfrac{x}{x-1}\right];
$$

where $F^{p;q;k}_{l;i;j}$ *denotes the Kampé de Fériet function defined by* (*see* [1, p. 150, Eq.(29)]; [15, p. 27]):

(3.15)
$$
F^{p;q;k}_{l;i;j}\left[\begin{array}{ccc}(a_p); & (b_q); & (c_k);\\ (\alpha_l); & (\beta_m); & (\gamma_n);\end{array}\ x,y\right]=\sum_{r,s=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{r+s}\prod_{j=1}^{q}(b_j)_r\prod_{j=1}^{k}(c_j)_s}{\prod_{j=1}^{l}(\alpha_j)_{r+s}\prod_{j=1}^{m}(\beta_j)_r\prod_{j=1}^{n}(\gamma_j)_s\, r!s!}x^ry^s
$$

for the convergence conditions of which are given in [15, p. 27];

(3.16)
$$
\sum_{j=0}^{\infty}\frac{(a)_j(b)_j}{(c)_j\, j!}\,{}_3F_2\left[\begin{array}{c}a+j,b+j,1+a+b-c;\\ c+j,1+a+b-c-n;\end{array}\ -1\right]=\frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n}\quad (n\in\mathbb{N}).
$$

It is noted that the closed form evaluation of the left hand series in (3.16) *follows from Saalschütz's formula* (*see* [5]; *see also* [12, p. 87]):

$$
{}_3F_2\left[\begin{array}{c}a,b,-n;\\ c,1+a+b-c-n;\end{array}\ 1\right]=\frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n}\quad (n\in\mathbb{N}). \tag{3.17}
$$

It is not difficult to find integral representations of Clausen hypergeometric function ${}_3F_2$ as in the following theorem.

Theorem 2. *Each of the following integral representations for* ${}_3F_2$ *holds true*:

(3.18)
$$
{}_3F_2\left[\begin{array}{c}\alpha_1,\alpha_2,\alpha_3;\\ \beta_1,\beta_2;\end{array}\ x\right]=\frac{1}{\Gamma(\alpha_1)}\int_0^{\infty}e^{-\xi}\xi^{\alpha_1-1}\,{}_2F_2\left[\begin{array}{c}\alpha_2,\alpha_3;\\ \beta_1,\beta_2;\end{array}\ x\xi\right]d\xi\quad (\Re(\alpha_1)>0);
$$

(3.19)
$$
{}_3F_2\left[\begin{array}{c}\alpha_1,\alpha_2,\alpha_3;\\ \beta_1,\beta_2;\end{array}\ x\right]=\frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^{\infty}\int_0^{\infty}e^{-\xi}e^{-\eta}\xi^{\alpha_1-1}\eta^{\alpha_2-1}\,{}_1F_2\left[\begin{array}{c}\alpha_3;\\ \beta_1,\beta_2;\end{array}\ x\xi\eta\right]d\xi d\eta
$$
$$
(\Re(\alpha_1)>0,\ \Re(\alpha_2)>0);
$$

(3.20)
$$
{}_3F_2\left[\begin{matrix}\alpha_1,\alpha_2,\alpha_3;\\ \beta_1,\beta_2;\end{matrix}\ x\right]
= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\xi-\eta-\zeta}\xi^{\alpha_1-1}\eta^{\alpha_2-1}\zeta^{\alpha_3-1}{}_0F_2\left[\begin{matrix}-;\\ \beta_1,\beta_2;\end{matrix}\ x\xi\eta\zeta\right]d\xi d\eta d\zeta
$$
$$
(\Re(\alpha_i)>0;\ i=1,\,2,\,3).
$$

4. Formulas of an analytic continuation for Clausen function

By decomposing a function into other functions, we can find a formula of an analytic continuation of the function. Here we find formulas of analytic continuation of the Clausen hypergeometric function ${}_3F_2$ as in the following theorem. Let $\mathbb{Z}$ and $\mathbb{C}$ denote the sets of integers and complex numbers, respectively.

Theorem 3. *Each of the following formulas for* ${}_3F_2$ *holds true for* $\beta_1,\ \beta_2,\ \alpha_2-\alpha_1,\ \beta_1-\alpha_1-\alpha_2\in\mathbb{C}\setminus\mathbb{Z}$:

$$
\begin{aligned}
&{}_3F_2\left[\begin{matrix}\alpha_1,\alpha_2,\alpha_3;\\ \beta_1,\beta_2;\end{matrix}\ x\right]\\
&= B_1\,(-x)^{-\alpha_1}F_2\left(\alpha_1;\beta_2-\alpha_3,1-\beta_1+\alpha_1;\beta_2,1-\alpha_2+\alpha_1;1,\frac{1}{x}\right)\\
&\quad+B_2\,(-x)^{-\alpha_2}F_2\left(\alpha_2;\beta_2-\alpha_3,1-\beta_1+\alpha_2;\beta_2,1-\alpha_1+\alpha_2;1,\frac{1}{x}\right);
\end{aligned}\tag{4.1}
$$

(4.2)
$$
\begin{aligned}
&{}_3F_2\left[\begin{matrix}\alpha_1,\alpha_2,\alpha_3;\\ \beta_1,\beta_2;\end{matrix}\ x\right]\\
&= B_1\,(1-x)^{-\alpha_1}F_2\left(\alpha_1;\beta_2-\alpha_3,\beta_1-\alpha_2;\beta_2,1-\alpha_1-\alpha_2;\frac{x}{x-1},\frac{1}{1-x}\right)\\
&\quad+B_2\,(1-x)^{-\alpha_2}F_2\left(\alpha_2;\beta_2-\alpha_3,\beta_1-\alpha_1;\beta_2,1-\alpha_2-\alpha_1;\frac{x}{x-1},\frac{1}{1-x}\right);
\end{aligned}
$$

$$
\begin{aligned}
{}_3F_2\left[\begin{matrix}\alpha_1,\alpha_2,\alpha_3;\\ \beta_1,\beta_2;\end{matrix}\ x\right] &= B_1\,F_2\left(\alpha_1;\alpha_3,\beta_1-\alpha_2;\beta_2,1-\alpha_2+\alpha_1;x,1\right)\\
&\quad+B_2\,F_2\left(\alpha_2;\alpha_3,\beta_1-\alpha_1;\beta_2,1-\alpha_1+\alpha_2;x,1\right);
\end{aligned}\tag{4.3}
$$

(4.4)
$$
\begin{aligned}
&{}_3F_2\left[\begin{matrix}\alpha_1,\alpha_2,\alpha_3;\\ \beta_1,\beta_2;\end{matrix}\ x\right]\\
&= A_1\,F^{2,1,0;}_{1,1,0;}\left[\begin{matrix}\alpha_1,\alpha_2; & \beta_2-\alpha_3; & -;\\ 1+\alpha_1+\alpha_2-\beta_1; & \beta_2; & -;\end{matrix}\ x,1-x\right]\\
&\quad+A_2\,(1-x)^{\beta_1-\alpha_1-\alpha_2}H_2\left(\alpha_1+\alpha_2-\beta_1;\beta_2-\alpha_3,\beta_1-\alpha_1,\beta_1-\alpha_2;\beta_2;\frac{x}{x-1},x-1\right),
\end{aligned}
$$

where F_2 *denotes the Appell hypergeometric function given in* (1.6),

$$H_2(\alpha;\beta,\gamma,\delta;\varepsilon;x,y)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m-n}(\beta)_m(\gamma)_n(\delta)_n}{(\varepsilon)_m\, m!n!}x^m y^n,$$

$$A_1=\frac{\Gamma(\beta_1)\Gamma(\beta_1-\alpha_1-\alpha_2)}{\Gamma(\beta_1-\alpha_1)\Gamma(\beta_1-\alpha_2)},\quad A_2=\frac{\Gamma(\beta_1)\Gamma(\alpha_1+\alpha_2-\beta_1)}{\Gamma(\alpha_1)\Gamma(\alpha_2)},$$

$$B_1=\frac{\Gamma(\beta_1)\Gamma(\alpha_2-\alpha_1)}{\Gamma(\alpha_2)\Gamma(\beta_1-\alpha_1)},\quad B_2=\frac{\Gamma(\beta_1)\Gamma(\alpha_1-\alpha_2)}{\Gamma(\alpha_1)\Gamma(\beta_1-\alpha_2)}.$$

Proof. By applying the formula of an analytic continuation of Gauss function ${}_2F_1:=F$ (see [5, p. 108, Eq.(2)]): For $b-a\in\mathbb{C}\setminus\mathbb{Z}$,

$$\begin{aligned}F(a,b;c;x)=&\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-x)^{-a}F\left(a,1-c+a;1-b+a;\frac{1}{x}\right)\\&+\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-x)^{-b}F\left(b,1-c+b;1-a+b;\frac{1}{x}\right),\end{aligned}\tag{4.5}$$

to the expansion formula of ${}_pF_q$ when $p=3$ and $q=2$, we find

$$\begin{aligned}&{}_3F_2\left[\begin{matrix}\alpha_1,\alpha_2,\alpha_3;\\ \beta_1,\beta_2;\end{matrix}\ x\right]\\&=(-x)^{-\alpha_1}\frac{\Gamma(\beta_1)\Gamma(\alpha_2-\alpha_1)}{\Gamma(\alpha_2)\Gamma(\beta_1-\alpha_1)}\sum_{i=0}^{\infty}\frac{(\alpha_1)_i(\beta_2-\alpha_3)_i}{(\beta_2)_i\, i!}F\left(\alpha_1+i,1-\beta_1+\alpha_1;1-\alpha_2+\alpha_1;\frac{1}{x}\right)\\&\quad+(-x)^{-\alpha_2}\frac{\Gamma(\beta_1)\Gamma(\alpha_1-\alpha_2)}{\Gamma(\alpha_1)\Gamma(\beta_1-\alpha_2)}\sum_{i=0}^{\infty}\frac{(\alpha_2)_i(\beta_2-\alpha_3)_i}{(\beta_2)_i\, i!}F\left(\alpha_2+i,1-\beta_1+\alpha_2;1-\alpha_1+\alpha_2;\frac{1}{x}\right).\end{aligned}\tag{4.6}$$

Using the definition of the Gauss function F in (4.6) and considering the definition of Appell function F_2 given in (1.6), we prove the formula (4.1) of analytic continuation for Clausen function ${}_3F_2$. A similar argument will establish the other formulas (4.2), (4.3), and (4.4). □

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