

## GLOBAL ATTRACTOR OF THE WEAKLY DAMPED WAVE EQUATION WITH NONLINEAR BOUNDARY CONDITIONS

CHAOSHENG ZHU

ABSTRACT. In this paper, the main purpose is to study existence of the global attractors for the weakly damped wave equation with nonlinear boundary conditions. To this end, we first show that the existence of a bounded absorbing set by the perturbed energy method. Secondly, we utilize the decomposition of the solution operator to verify the asymptotic compactness.

### 1. Introduction

The main purpose of this work is to study existence of the global attractors for the weakly damped wave equation with nonlinear boundary conditions. To formalize this problem let us take  $\Omega$  an open bounded set of  $\mathbb{R}^n$  with smooth boundary  $\Gamma$  and assume that  $\Gamma$  can be divided into two non-null parts

$$\Gamma = \Gamma_0 \cup \Gamma_1, \quad \overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset.$$

Denote by  $\nu(x)$  the unit normal vector at  $x \in \Gamma$  outside of  $\Omega$  and let us consider the following initial boundary value problems

$$(1.1) \quad u_{tt} - \Delta u + u_t = f(x) \quad \text{in } \Omega \times (0, \infty),$$

$$(1.2) \quad u = 0 \quad \text{on } \Gamma_0 \times (0, \infty),$$

$$(1.3) \quad \frac{\partial u}{\partial \nu} + u_t + H(u) = 0 \quad \text{on } \Gamma_1 \times (0, \infty),$$

$$(1.4) \quad u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x) \quad \text{in } \Omega,$$

here,  $u = u(x, t)$  is unknown function and  $\Delta u = \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2}$ .

The asymptotic behavior of solutions to the wave equations with boundary damping has been studied by many authors (see [3, 5, 4, 6, 7, 8, 10] and further therein), mainly in the framework of the problem of stabilizability arising in control theory. The first stabilizability result for nonlinear equations in an arbitrary domain was obtained by Tataru in [10] using estimates of Carleman

---

Received July 25, 2009; Revised May 21, 2011.

2010 *Mathematics Subject Classification.* 35B40, 35B41, 35B45.

*Key words and phrases.* wave equation, nonlinear boundary conditions, global attractor.

type. He also uses this approach to establish the existence of a locally compact global attractor for some semilinear problems. In [5], author establish the existence of a compactness global attractor for the following semilinear wave equation with boundary damping

$$(1.5) \quad u_{tt} - \Delta u + f(u) = 0 \quad \text{on } \mathbb{R}_+ \times \Omega,$$

$$(1.6) \quad u_t + \nu \cdot \nabla u = 0 \quad \text{on } \mathbb{R}_+ \times \partial\Omega,$$

$$(1.7) \quad u = \phi, \quad u_t = \psi \quad \text{on } \{0\} \times \Omega.$$

In [2], authors study the long time behavior of solutions of weakly coupled reaction diffusion systems with dispersion of the form

$$(1.8) \quad u_t - \text{Div}(a(x)\nabla u) + \sum_{j=1}^N B_j(x) \frac{\partial u}{\partial x_j} + \lambda u + f(u) = 0 \quad \text{in } \Omega,$$

$$(1.9) \quad \frac{\partial u}{\partial n_a} + g(u) = 0 \quad \text{on } \partial\Omega,$$

$$(1.10) \quad u(x, 0) = u_0(x) \quad \text{in } \Omega.$$

They obtain the existence of a compact attractor in the fractional power spaces.

In [1], author studies the asymptotic behavior of solutions of the following reaction diffusion equation with nonlinear boundary conditions

$$(1.11) \quad u_t - d\Delta u + f(u) = 0 \quad \text{in } \Omega,$$

$$(1.12) \quad d \frac{\partial u}{\partial n} + g(u) = 0 \quad \text{on } \partial\Omega,$$

$$(1.13) \quad u(x, 0) = u_0(x) \quad \text{in } \Omega.$$

He gives the proper conditions on the nonlinear terms such that problems (1.11)–(1.13) is globally well posed and moreover has a global compact attractor.

Motivated by the paper cited above, in this paper, we investigate the long time behavior of solutions to problems (1.1)–(1.4) and show that the existence of the global attractors. Our problems (1.1)–(1.4) consist of weakly damped in domain and nonlinear conditions on boundary, and the problems (1.1)–(1.4) are differ from problems (1.5)–(1.7). Because of this, our methods are differ from in [5]. That is, we shall firstly show that the existence of a bounded absorbing set by the perturbed energy method. Secondly, we utilize the decomposition of the solution operator to verify the asymptotic compactness.

Now let us state precise assumptions on the function  $H(u)$ .

*The function  $H \in C^1(\mathbb{R})$  satisfies*

$$(1.14) \quad H(0) = 0,$$

$$(1.15) \quad \widehat{H}(u) \geq 0, H(u)u \geq (1 + \delta)\widehat{H}(u) \text{ for some } \delta > 0,$$

$$(1.16) \quad |H(x) - H(y)| \leq \lambda_H(1 + |x|^\rho + |y|^\rho)|x - y|, \quad \forall x, y \in \mathbb{R},$$

where  $\widehat{H}(u) = \int_0^u H(s)ds$  and  $\lambda_H > 0$  are some constant, and

$$0 < \rho \leq \frac{1}{n-2}, \quad \text{if } n \geq 3; \quad \text{or} \quad \rho > 0, \quad \text{if } n = 1, 2.$$

Next let us introduce the functional space. Let

$$V := \{u \in H^1(\Omega); \quad u = 0 \quad \text{on } \Gamma_0\},$$

which is a Hilbert subspace of  $H^1(\Omega)$  equipped with the topology given by the norm  $\|\nabla \cdot\|_{L^2(\Omega)}$ . We denote

$$\begin{aligned} (u, v) &:= \int_{\Omega} uv dx, \quad \|u\|^2 = \int_{\Omega} |u|^2 dx, \\ (u, v)_{\Gamma_1} &:= \int_{\Gamma_1} uv d\Gamma, \quad \|u\|_{p, \Gamma_1}^p = \int_{\Gamma_1} |u|^p d\Gamma, \quad \|u\|_{\Gamma_1}^2 = \|u\|_{2, \Gamma_1}^2, \end{aligned}$$

and let

$$\mathcal{H}_0 = V \times L^2(\Omega), \quad \|(u, u_t)\|_{\mathcal{H}_0} = \|\nabla u\|^2 + \|u_t\|^2.$$

In what follows we will often use the next inequality: for every  $u \in H^1(\Omega)$  and  $\varepsilon > 0$ , there exists a positive constant  $C_\varepsilon$  such that

$$\int_{\Gamma} u^2 d\Gamma \leq \varepsilon \int_{\Omega} |\nabla u|^2 dx + C_\varepsilon \int_{\Omega} u^2 dx.$$

Let  $\lambda_\Omega > 0$  and  $\lambda_{\Gamma_1} > 0$  be two constants such that for  $\forall v \in V$ ,

$$(1.17) \quad \|v\| \leq \lambda_\Omega \|\nabla v\|, \quad \|v\|_{\Gamma_1} \leq \lambda_{\Gamma_1} \|\nabla v\|.$$

Here, the first-order energy of system (1.1)–(1.4) is given by

$$(1.18) \quad E(t) = \frac{1}{2} \|u_t\|^2 + \frac{1}{2} \|\nabla u\|^2 + \int_{\Gamma_1} \widehat{H}(u) d\Gamma.$$

In order to obtain the global attractor for the problems (1.1)–(1.4), we need the following theorem of existence, uniqueness of solution.

**Lemma 1.1.** *Assume that conditions (1.14)–(1.16) hold, and  $f(x) \in L^2(\Omega)$ ,  $(u_0, u_1) \in \mathcal{H}_0$ . Then, problems (1.1)–(1.4) possesses a unique solution in the class*

$$u \in C(0, \infty; V), \quad u_t \in C^1(0, \infty; L^2(\Omega)).$$

*Remark 1.2.* Applying the almost same argument as that in [3, 4] we can prove Lemma 1.1.

Now we are in position to state our main result.

**Theorem 1.3.** *Under the hypotheses of Lemma 1.1, the semigroup  $S_t$  associated with problems (1.1)–(1.4) possesses a global attractor  $\mathcal{A}$  in  $\mathcal{H}_0$ .*

It is well known that a compact global attractor exists if the continuous semigroup has a bounded absorbing set and is asymptotically compact [11]. The first difficulty is nonlinear boundary conditions when proving existence of bounded absorbing set. In order to overcome this difficulty, we shall combine the perturbed energy method used in [4, 12] with techniques from [9]. Secondly, for the proof of asymptotically compact, one usually decompose the solution operator into a compact part and an asymptotically small part. We shall utilize the decomposition for solution operator to verify the asymptotic compactness.

Our paper is organized as follows. In Section 2, we shall show that the existence of absorbing set in  $\mathcal{H}_0$ . In Section 3, we shall show that the asymptotic compactness for problems (1.1)–(1.4).

## 2. Absorbing set in $\mathcal{H}_0$

In this section, we shall show that the semigroup  $S_t$  has a bounded absorbing set, i.e., a bounded set  $B \subset \mathcal{H}_0$  satisfying the following condition: for any bounded  $A \subset \mathcal{H}_0$  there exists  $t(A) > 0$  such that  $S_t A \subset B$  for all  $t \geq t(A)$ .

To obtain a bounded absorbing set, we used the perturbed energy method, see Zuazua [4, 12], combined with techniques from Munoz Revera [9]. The derivative of the energy defined in (1.18) is given by

$$(2.1) \quad \begin{aligned} \frac{d}{dt} E(t) &= -\|u_t\|^2 - \|u_t\|_{\Gamma_1}^2 + \int_{\Omega} f u_t dx \\ &\leq -\|u_t\|_{\Gamma_1}^2 + (\varepsilon - 1)\|u_t\|^2 + \frac{1}{4\varepsilon} \|f(x)\|^2 \end{aligned}$$

for all  $\varepsilon > 0$ . We define the perturbed energy by

$$(2.2) \quad E_{\varepsilon}(t) = E(t) + \varepsilon\psi(t),$$

where

$$(2.3) \quad \psi(t) = \int_{\Omega} u u_t dx.$$

From (2.3) we have

$$\begin{aligned} |\psi(t)| &\leq \lambda_{\Omega} \|\nabla u\| \|u_t\| \\ &\leq \lambda_{\Omega} \left( \frac{1}{2} \|u_t\|^2 + \frac{1}{2} \|\nabla u\|^2 \right) \\ &\leq \lambda_{\Omega} E(t). \end{aligned}$$

We can write

$$(2.4) \quad |E_{\varepsilon}(t) - E(t)| \leq \varepsilon \lambda_{\Omega} E(t).$$

**Lemma 2.1.** *There exist  $C_1$ ,  $C_2 = C_2(\varepsilon)$ ,  $C(\lambda_H)$  and  $\varepsilon_1$  positive constants such that*

$$(2.5) \quad \frac{d}{dt} E_{\varepsilon}(t) \leq -\varepsilon C_1 E(t) + C_2 \|f\|^2 + C(\lambda_H), \quad \forall t \geq 0, \forall \varepsilon \in (0, \varepsilon_1].$$

*Proof.* Taking the derivative of  $\psi(t)$  and using (1.1), it follows that

$$\frac{d}{dt}\psi(t) = (\Delta u, u) - (u_t, u) + (f(x), u) + \|u_t\|^2.$$

From the generalized Green's formula and taking (1.3) into account we obtain

$$(2.6) \quad \begin{aligned} \frac{d}{dt}\psi(t) = & -\|\nabla u\|^2 - (u_t, u) + \|u_t\|^2 \\ & + (f(x), u) - (u_t, u)_{\Gamma_1} - (H(u), u)_{\Gamma_1}. \end{aligned}$$

Subtracting and adding the term  $\|\nabla u\|^2$  in the equality (2.6) and taking (1.18) into account we get

$$(2.7) \quad \begin{aligned} \frac{d}{dt}\psi(t) \leq & -2E(t) - (u_t, u) + 2\|u_t\|^2 + \int_{\Gamma_1} \widehat{H}(u) d\Gamma \\ & + (f(x), u) - (u_t, u)_{\Gamma_1} - (H(u), u)_{\Gamma_1}. \end{aligned}$$

Now, since  $V \hookrightarrow L^{2\rho+2}(\Gamma_1)$  then, by (1.17) and (1.18) we have

$$-(u_t, u) \leq \lambda_\Omega \|\nabla u\| \|u_t\| \leq \frac{1}{8} \|\nabla u\|^2 + 2\lambda_\Omega^2 \|u_t\|^2,$$

$$(f, u) \leq \lambda_\Omega \|\nabla u\| \|f\| \leq \frac{1}{8} \|\nabla u\|^2 + 2\lambda_\Omega^2 \|f\|^2,$$

$$-(u_t, u)_{\Gamma_1} \leq \lambda_{\Gamma_1} \|\nabla u\| \|u_t\|_{\Gamma_1} \leq \frac{1}{8} \|\nabla u\|^2 + 2\lambda_{\Gamma_1}^2 \|u_t\|_{\Gamma_1}^2,$$

$$\begin{aligned} \int_{\Gamma_1} \widehat{H}(u) d\Gamma - (H(u), u)_{\Gamma_1} & \leq \int_{\Gamma_1} 2\lambda_H (1 + |u|^\rho) |u|^2 d\Gamma \\ & \leq \frac{7}{8} E(t) + C(\lambda_H). \end{aligned}$$

Now, by above inequalities and (2.7) we obtain

$$(2.8) \quad \begin{aligned} \frac{d}{dt}\psi(t) \leq & -\frac{3}{8} E(t) + 2(1 + \lambda_\Omega^2) \|u_t\|^2 + 2\lambda_\Omega^2 \|f\|^2 \\ & + 2\lambda_{\Gamma_1}^2 \|u_t\|_{\Gamma_1}^2 + C(\lambda_H). \end{aligned}$$

Thus from (2.1), (2.2) and (2.8) we can write

$$(2.9) \quad \begin{aligned} \frac{d}{dt} E_\varepsilon(t) & = \frac{d}{dt} E(t) + \varepsilon \frac{d}{dt} \psi(t) \\ & \leq -\frac{3\varepsilon}{8} E(t) + ((3 + 2\lambda_\Omega^2)\varepsilon - 1) \|u_t\|^2 + C(\lambda_H)\varepsilon \\ & \quad + \left( \frac{1}{4\varepsilon} + 2\lambda_\Omega^2 \varepsilon \right) \|f(x)\|^2 + (2\lambda_{\Gamma_1}^2 \varepsilon - 1) \|u_t\|_{\Gamma_1}^2. \end{aligned}$$

Defining  $\varepsilon_1 = \min\{\frac{1}{3+2\lambda_\Omega^2}, \frac{1}{2\lambda_{\Gamma_1}^2}\}$  and considering  $\varepsilon \in (0, \varepsilon_1]$  from (2.9) we conclude the inequality (2.5). The proof of Lemma 2.1 is completed.  $\square$

*Proof of the existence for bounded absorbing set.* Let

$$\varepsilon_0 = \min \left\{ \frac{1}{2\lambda_\Omega}, \varepsilon_1 \right\},$$

and let us consider  $\varepsilon \in (0, \varepsilon_0]$ . From (2.4) we have

$$(2.10) \quad (1 - \lambda_\Omega \varepsilon)E(t) \leq E_\varepsilon(t) \leq (1 + \lambda_\Omega \varepsilon)E(t).$$

Since  $\varepsilon \leq \frac{1}{2\lambda_\Omega}$ , then

$$(2.11) \quad \frac{1}{2}E(t) \leq E_\varepsilon(t) \leq \frac{3}{2}E(t) \leq 2E(t), \quad \forall t \geq 0,$$

and therefore

$$(2.12) \quad -\varepsilon C_1 E(t) \leq -\frac{\varepsilon}{2} C_1 E_\varepsilon(t).$$

Hence, from (2.12) and considering Lemma 2.1 we obtain

$$\frac{d}{dt} E_\varepsilon(t) \leq -\frac{\varepsilon}{2} C_1 E_\varepsilon(t) + C_2 \|f(x)\|^2 + C(\lambda_H).$$

Thus by Gronwall's inequality and (2.11) we obtain

$$(2.13) \quad \begin{aligned} \frac{1}{2}E(t) &\leq E_\varepsilon(t) \\ &\leq E_\varepsilon(0) \exp\left(-\frac{\varepsilon}{2} C_1 t\right) + \frac{2C_2 \|f(x)\|^2 + 2C(\lambda_H)}{\varepsilon C_2} \left(1 - \exp\left(-\frac{\varepsilon}{2} C_1 t\right)\right). \end{aligned}$$

For any bounded subset  $B$  of  $\mathcal{H}_0$ ,  $(u_0, u_1) \in B$ ,  $\widehat{M}(\|\nabla u_0\|^2)$ ,  $\int_{\Gamma_1} \widehat{H}(u_0) d\Gamma$  and  $\int_\Omega u_0 u_1 dx$  are bounded, too. Hence

$$\begin{aligned} R &= R(B) = \sup_{(u_0, u_1) \in B} E_\varepsilon(0) \\ &= \sup_{(u_0, u_1) \in B} \left\{ \|u_1\|^2 + \|\nabla u_0\|^2 + \int_{\Gamma_1} \widehat{H}(u_0) d\Gamma + \varepsilon \int_\Omega u_0 u_1 dx \right\} < \infty \end{aligned}$$

and

$$(2.14) \quad \lim_{t \rightarrow \infty} \sup_{(u_0, u_1) \in B} E(t) \leq \frac{4C_2 \|f(x)\|^2 + 4C(\lambda_H)}{\varepsilon C_1} \equiv \mu_0^2.$$

Let  $\mu_1 > \mu_0$  be fixed, and

$$t_0 = t_0(R, \mu_1) = \frac{1}{\alpha} \ln \frac{R}{\mu_1^2 - \mu_0^2}$$

for any  $t \geq t_0$ , then we have  $E(t) \leq \mu_1^2$  and

$$(2.15) \quad \|\nabla u(t)\|^2 + \|u_t(t)\|^2 \leq \mu_1^2 \quad \text{for } t \geq t_0.$$

Thus we obtain a bounded absorbing set in  $\mathcal{H}_0$ . □

### 3. Asymptotic compactness

In this section, we show the asymptotic compactness for the semigroup  $S_t$ . By definition, the semigroup  $S_t$  is asymptotically compact if for any bounded  $A \subset \mathcal{H}_0$  and any  $\varepsilon > 0$  there exist a precompact set  $K \subset \mathcal{H}_0$  and a time  $t$  such that  $dist(S_t A, K) < \varepsilon$ . To establish the asymptotic compactness of the semigroup  $S_t$  generated by problems (1.1)–(1.4), we shall utilize a decomposition for solution operator. The idea is to decompose the solution operator into two parts

$$S_t(u_0, u_1) = V_t(u_0, u_1) + W_t(u_0, u_1),$$

where  $V_t$  is a contraction in the sense that  $V_t(u_0, u_1) \rightarrow 0$  as  $t \rightarrow +\infty$  uniformly in  $(u_0, u_1) \in A$ , and  $W_t$  is a compact mapping for all  $t$ . Then choosing  $t$  sufficiently large so that  $\|V_t(u_0, u_1)\|_{\mathcal{H}_0} < \varepsilon$  for all  $(u_0, u_1) \in A$ , we have  $dist(S_t A, W_t A) < \varepsilon$ , which proves the asymptotic compactness.

The proof of the asymptotic compactness consists of two parts, i.e., Lemmas 3.1 and 3.2 below. Firstly, let us define  $V_t$  as the solution operator of the following problems

$$(3.1) \quad v_{tt} - \Delta v + v_t = 0 \quad \text{in } \Omega \times (0, \infty),$$

$$(3.2) \quad v = 0 \quad \text{on } \Gamma_0 \times (0, \infty),$$

$$(3.3) \quad \frac{\partial v}{\partial \nu} + v_t = 0 \quad \text{on } \Gamma_1 \times (0, \infty),$$

$$(3.4) \quad v(x, 0) = u_0(x), v_t(x, 0) = u_1(x) \quad \text{in } x \in \Omega.$$

**Lemma 3.1.** *Assume that  $(u_0, u_1) \in \mathcal{H}_0$ , then the problems (3.1)–(3.4) admits a unique global solution  $v$  satisfying*

$$v \in L^\infty(0, +\infty; V), \quad v_t \in L^\infty(0, +\infty; L^2(\Omega)).$$

Moreover, for each bounded  $A \subset \mathcal{H}_0$ ,

$$(3.5) \quad \sup_{(u_0, u_1) \in A} \|V_t(u_0, u_1)\|_{\mathcal{H}_0} \rightarrow 0 \quad \text{as } t \rightarrow +\infty.$$

*Proof.* For

$$(3.6) \quad \theta \in (0, \theta_0], \quad \theta_0 = \min\{\eta > 0 \mid \eta \leq \frac{1}{2}, \eta(\eta + 1)^2 \lambda_\Omega^2 \leq 2\},$$

it is easy to obtain

$$(v_t + \theta v)_t - \Delta v + (1 - \theta)(v_t + \theta v) + (\theta^2 - \theta)v = 0.$$

Then

$$(3.7) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} \{ \|v_t + \theta v\|^2 + \|\nabla v\|^2 \} + \theta \|\nabla v\|^2 + \|v_t\|_{\Gamma_1}^2 \\ & + (1 - \theta) \|v_t + \theta v\|^2 + (\theta^2 - \theta)(v, v_t + \theta v) = 0. \end{aligned}$$

We note that

$$\begin{aligned}
& \theta \|\nabla v\|^2 + (1-\theta)\|v_t + \theta v\|^2 + (\theta^2 - \theta)(v, v_t + \theta v) \\
(3.8) \quad & \geq \theta \|\nabla v\|^2 + (1-\theta)\|v_t + \theta v\|^2 - \lambda_\Omega(\theta^2 + \theta)\|\nabla v\|\|v_t + \theta v\| \\
& \geq \left(\frac{1}{2} - \theta\right)\|v_t + \theta v\|^2 + \left(\theta - \frac{1}{2}\lambda_\Omega^2(\theta^2 + \theta)^2\right)\|\nabla v\|^2.
\end{aligned}$$

By (3.6)–(3.8), there is a  $\alpha > 0$  such that

$$(3.9) \quad \frac{d}{dt}\{\|v_t + \theta v\|^2 + \|\nabla v\|^2\} + \alpha\{\|v_t + \theta v\|^2 + \|\nabla v\|^2\} \leq 0.$$

By Gronwall's inequality we can get

$$(3.10) \quad \|v_t(t) + \theta v(t)\|^2 + \|\nabla v(t)\|^2 \leq (\|u_1 + \theta u_0\|^2 + \|\nabla u_0\|^2)e^{-\alpha t}.$$

On the other hand, we have

$$\begin{aligned}
(3.11) \quad & \|v_t(t)\|^2 = \|v_t(t) + \theta v(t) - \theta v(t)\|^2 \\
& \leq \|v_t(t) + \theta v(t)\|^2 + \theta^2\|v(t)\|^2 \\
& \leq \|v_t(t) + \theta v(t)\|^2 + \theta^2\lambda_\Omega^2\|\nabla v(t)\|^2.
\end{aligned}$$

From (3.10)–(3.11) we get, for some  $C > 0$

$$(3.12) \quad \|v_t(t)\|^2 + \|\nabla v(t)\|^2 \leq C(\|u_1 + \theta u_0\|^2 + \|\nabla u_0\|^2)e^{-\alpha t}.$$

The proof of Lemma 3.1 is completed.  $\square$

Secondly, we pass to the proof of the compactness of mapping  $W_t = S_t - V_t$ . Clearly, if  $w$  is the first component of  $W_t(u_0, u_1)$ , then its second component is  $w_t$  and the function  $w$  satisfies the following problems

$$(3.13) \quad w_{tt} - \Delta w + w_t = f(x) \quad \text{in } \Omega \times (0, \infty),$$

$$(3.14) \quad w = 0 \quad \text{on } \Gamma_0 \times (0, \infty),$$

$$(3.15) \quad \frac{\partial w}{\partial \nu} + w_t + H(u) = 0 \quad \text{on } \Gamma_1 \times (0, \infty),$$

$$(3.16) \quad w(x, 0) = 0, \quad w_t(x, 0) = 0 \quad \text{in } \Omega.$$

**Lemma 3.2.** *For each  $t \in \mathbb{R}_+$  the mapping  $W_t: \mathcal{H}_0 \rightarrow \mathcal{H}_0$  is compact.*

*Proof.* We consider the difference  $\tilde{w} = w - \bar{w}$  of two solutions  $w, \bar{w}$  of the problems (3.13)–(3.16). Then  $\tilde{w}$  satisfies

$$(3.17) \quad \tilde{w}_{tt} - \Delta \tilde{w} + \tilde{w}_t = 0 \quad \text{in } \Omega \times (0, \infty),$$

$$(3.18) \quad \tilde{w} = 0 \quad \text{on } \Gamma_0 \times (0, \infty),$$

$$(3.19) \quad \frac{\partial \tilde{w}}{\partial \nu} + \tilde{w}_t + H(u) - H(\bar{u}) = 0 \quad \text{on } \Gamma_1 \times (0, \infty),$$

$$(3.20) \quad \tilde{w}(x, 0) = 0, \quad \tilde{w}_t(x, 0) = 0 \quad \text{in } \Omega.$$



Multiplying equation (3.17) by  $\tilde{w}_t$ , we get

$$\frac{1}{2} \frac{d}{dt} (\|\tilde{w}_t\|^2 + \|\tilde{w}\|^2) + \|\nabla \tilde{w}_t\|^2 = -\|\tilde{w}_t\|_{\Gamma_1}^2 + \int_{\Gamma_1} (H(\bar{u}) - H(u)) \tilde{w}_t d\Gamma.$$

Using Young's inequality, we obtain

$$(3.21) \quad \frac{d}{dt} (\|\tilde{w}_t\|^2 + \|\nabla \tilde{w}\|^2) \leq \int_{\Gamma_1} |H(\bar{u}) - H(u)|^2 d\Gamma.$$

Integrating (3.21) over  $(0, t)$ ,  $t \in (0, T)$  and taking (3.20) into account, we obtain

$$(3.22) \quad \|\tilde{w}_t(t)\|^2 + \|\nabla \tilde{w}(t)\|^2 \leq \int_0^t \int_{\Gamma_1} |H(\bar{u}(s)) - H(u(s))|^2 d\Gamma ds.$$

Fix an arbitrary bounded sequence  $(u_0^k, u_1^k) \in \mathcal{H}_0$ . Let  $u^k(x, t)$ ,  $v^k(x, t)$ ,  $w^k(x, t)$  denote the first components of  $S_t(u_0^k, u_1^k)$ ,  $V_t(u_0^k, u_1^k)$ , and  $W_t(u_0^k, u_1^k)$  respectively. Then applying the inequality (3.22) we have

$$(3.23) \quad \begin{aligned} & \|W_t(u_0^{k_i}, u_1^{k_i}) - W_t(u_0^{k_j}, u_1^{k_j})\|_{\mathcal{H}_0}^2 \\ & \leq \int_0^t \int_{\Gamma_1} |H(\bar{u}(s)) - H(u(s))|^2 d\Gamma ds \\ & \rightarrow 0 \quad \text{as } i, j \rightarrow \infty. \end{aligned}$$

Thus the sequence  $W_t(u_0^k, u_1^k)$  contains a convergent subsequence, which completes the proof of Lemma 3.2. □

**Acknowledgments.** The author would like to thank the referee for the careful reading of this paper and for the valuable suggestions to improve the presentation and style of the paper. This work was supported by Postdoctor Research Fund of Southwest University and Fundamental Research Funds for the Central Universities(No. XDJK2009C070).

### References

- [1] R. B. Anibal, *Attractors for parabolic equations with nonlinear boundary conditions, critical exponents, and singular initial data*, J. Differential Equations **181** (2002), no. 1, 165–196.
- [2] A. N. Carvalho, S. M. Oliva, A. L. Pereira, and R. B. Anibal, *Attractors for parabolic problems with nonlinear boundary conditions*, J. Math. Anal. Appl. **207** (1997), no. 2, 409–461.
- [3] G. Chen, *Energy decay estimates and exact boundary value controllability for the wave equation in a bounded domain*, J. Math. Pures Appl. (9) **58** (1979), no. 3, 249–273.
- [4] V. Komornik and E. Zuazua, *A direct method for the boundary stabilization of the wave equation*, J. Math. Pures Appl. (9) **69** (1990), no. 1, 33–54.
- [5] I. N. Kostin, *Long-time behavior of solutions to a semilinear wave equation with boundary damping*, Dynam. Control **11** (2001), no. 4, 371–388.
- [6] J. Lagnese, *Decay of solutions of wave equations in a bounded region with boundary dissipation*, J. Differential Equations **50** (1983), no. 2, 163–182.
- [7] I. Lasiecka, *Stabilization of wave and plate-like equations with nonlinear dissipation on the boundary*, J. Differential Equations **79** (1989), no. 2, 340–381.

- [8] I. Lasiecka and R. Triggiani, *Uniform Stabilization of the wave equation with Dirichlet or Neumann feedback control without geometrical conditions*, Appl. Math. Optim. **25** (1992), no. 2, 189–224.
- [9] J. E. Muñoz Rivera, *Smoothness effect and decay on a class of nonlinear evolution equation*, Ann. Fac. Sci. Toulouse Math. (6) **1** (1992), no. 2, 237–260.
- [10] D. Tataru, *Uniform decay rates and attractors for evolution PDE's with boundary dissipation*, J. Differential Equations **121** (1995), no. 1, 1–27.
- [11] R. Temam, *Infinite Dimensional Dynamical Systems in Mechanics and Physics*, New York, Springer, 1988.
- [12] E. Zuazua, *Stability and decay for a class of nonlinear hyperbolic problems*, Asymptotic Anal. **1** (1988), no. 2, 161–185.

SCHOOL OF MATHEMATICS AND STATISTICS  
SOUTHWEST UNIVERSITY  
CHONGQING, 400715, P. R. CHINA  
E-mail address: [zhu.cauchy@yahoo.com.cn](mailto:zhu.cauchy@yahoo.com.cn); [zcs@swu.edu.cn](mailto:zcs@swu.edu.cn)