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WEYL'S TYPE THEOREMS FOR ALGEBRAICALLY (p, k)-QUASIHYPONORMAL OPERATORS

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Abstract. For a bounded linear operator T we prove the following assertions: (a) If T is algebraically (p, k)-quasihyponormal, then T is a-isoloid, polaroid, reguloid and a-polaroid. (b) If T^* is algebraically (p, k)-quasihyponormal, then a-Weyl's theorem holds for $f(T)$ for every $f \in Hol(\sigma(T))$, where $Hol(\sigma(T))$ is the space of all functions that analytic in an open neighborhoods of $\sigma(T)$ of T. (c) If T^* is algebraically (p, k)-quasihyponormal, then generalized a-Weyl's theorem holds for $f(T)$ for every $f \in Hol(\sigma(T))$. (d) If T is a (p, k)-quasihyponormal operator, then the spectral mapping theorem holds for semi-B-essential approximate point spectrum $\sigma_{SBF_+^-}(T)$, and for left Drazin spectrum $\sigma_{lD}(T)$ for every $f \in Hol(\sigma(T))$.

1. Introduction

Throughout this paper let $\mathbf{B}(\mathcal{H})$, denote, the algebra of bounded linear operators acting on an infinite dimensional separable Hilbert space $\mathcal{H}$. If $T \in \mathbf{B}(\mathcal{H})$ we shall write $\ker(T)$ and $\mathcal{R}(T)$ for the null space and range of T, respectively. Also, let $\alpha(T) := \dim \ker(T)$, $\beta(T) := \dim \mathcal{R}(T)$, and let $\sigma(T), \sigma_a(T), \sigma_p(T)$ denote the spectrum, approximate point spectrum and point spectrum of T, respectively. An operator $T \in \mathbf{B}(\mathcal{H})$ is called *Fredholm* if it has closed range, finite dimensional null space, and its range has finite codimension. The index of a Fredholm operator is given by

$$i(T) := \alpha(T) - \beta(T).$$

T is called *Weyl* if it is Fredholm of index 0, and *Browder* if it is Fredholm "of finite ascent and descent".

Recall that the *ascent*, $a(T)$, of an operator T is the smallest non-negative integer p such that $\ker(T^p) = \ker(T^{p+1})$. If such integer does not exist we put $a(T) = \infty$. Analogously, the *descent*, $d(T)$, of an operator T is the smallest non-negative integer q such that $\mathcal{R}(T^q) = \mathcal{R}(T^{q+1})$, and if such integer does

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not exist we put $d(T)=\infty$. The essential spectrum $\sigma_F(T)$, the Weyl spectrum $\sigma_W(T)$ and the Browder spectrum $\sigma_b(T)$ of T are defined by

$$\sigma_F(T)=\{\lambda\in\mathbb{C}:T-\lambda \text{ is not Fredholm}\},$$

$$\sigma_W(T)=\{\lambda\in\mathbb{C}:T-\lambda \text{ is not Weyl}\}$$

and

$$\sigma_b(T)=\{\lambda\in\mathbb{C}:T-\lambda \text{ is not Browder}\}$$

respectively. Evidently

$$\sigma_F(T)\subseteq\sigma_W(T)\subseteq\sigma_b(T)\subseteq\sigma_F(T)\cup acc\sigma(T),$$

where we write $accK$ for the accumulation points of $K\subseteq\mathbb{C}$.

Following [13], we say that *Weyl's theorem* holds for T if $\sigma(T)\setminus\sigma_W(T)=E_0(T)$, where $E_0(T)$ is the set of all eigenvalues λ of finite multiplicity isolated in $\sigma(T)$. And *Browder's theorem* holds for T if $\sigma(T)\setminus\sigma_W(T)=\pi_0(T)$, where π_0 is the set of all poles of T of finite rank.

Let $\Phi_+(\mathcal{H})$ be the class of all upper semi-Fredholm operators, $\Phi_+^-(\mathcal{H})$ be the class of all $T\in\Phi_+(\mathcal{H})$ with $i(T)\leq 0$, and for any $T\in\mathbf{B}(\mathcal{H})$ let

$$\sigma_{SF_+^-}(T)=\left\{\lambda\in\mathbb{C}:T-\lambda I\notin SF_+^-(\mathcal{H})\right\}.$$

Let E_0^a be the set of all eigenvalues of T of finite multiplicity which are isolated in $\sigma_a(T)$. According to [27], we say that T satisfies *a-Weyl's theorem* if $\sigma_{SF_+^-}(T)=\sigma_a(T)\setminus E_0^a(T)$. It follows from [27, Corollary 2.5] a-Weyl's theorem implies Weyl's theorem.

In [12] Berkani define the class of B-Fredholm operators as follows. For each integer n, define T_n to be the restriction of T to $\mathcal{R}(T^n)$ viewed as a map from $\mathcal{R}(T^n)$ into $\mathcal{R}(T^n)$ (in particular $T_0=T$). If for some n the range $\mathcal{R}(T^n)$ is closed and T_n is a Fredholm (resp. semi-Fredholm) operator, then T is called a *B-Fredholm* (resp. *semi-B-Fredholm*) operator. In this case and from [6] T_m is a Fredholm operator and $i(T_m)=i(T_n)$ for each $m\geq n$. The index of a B-Fredholm operator T is defined as the index of the Fredholm operator T_n, where n is any integer such that the range $\mathcal{R}(T^n)$ is closed and T_n is a Fredholm operator (see [12]).

Let $BF(\mathcal{H})$ be the class of all B-Fredholm operators. In [6] Berkani has studied this class of operators and has proved that an operator $T\in\mathbf{B}(\mathcal{H})$ is B-Fredholm if and only if $T=T_0\oplus T_1$, where T_0 is a Fredholm operator and T_1 is a nilpotent operator.

Recall that an operator $T\in\mathbf{B}(\mathcal{H})$ is called a *B-Weyl* operator (see [8]) if it is a B-Fredholm operator of index 0. The B-Weyl spectrum $\sigma_{BW}(T)$ of T is defined by

$$\sigma_{BW}(T)=\{\lambda\in\mathbb{C}:T-\lambda I\quad\text{is not a } B\text{-Weyl operator}\}.$$

In the case of a normal operator T acting on a Hilbert space $\mathcal{H}$, Berkani [12, Theorem 4.5] showed that $\sigma_{BW}(T)=\sigma(T)\setminus E(T)$, where $E(T)$ is the set of all eigenvalues of T which are isolated in the spectrum of T. This result gives a generalization of the classical Weyl's theorem.

Let $SBF_+(\mathcal{H})$ be the class of all upper semi-B-Fredholm operators, and $SBF_+^-(\mathcal{H})$ the class of all $T \in SBF_+(\mathcal{H})$ such that $i(T) \leq 0$, and

$$\sigma_{SBF_+^-}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin SBF_+^-(\mathcal{H})\}.$$

Recall that an operator $T \in \mathbf{B}(\mathcal{H})$ satisfies the *generalized a-Weyl's theorem* if $\sigma_{SBF_+^-}(T) = \sigma_a(T) \setminus E^a(T)$, where $E^a(T)$ is the set of all eigenvalues of T which are isolated in $\sigma_a(T)$. Note that generalized a-Weyl's theorem implies a-Weyl's theorem (see [11]).

Recall that an operator $T \in \mathbf{B}(\mathcal{H})$ is *Drazin invertible* if and only if it has a finite ascent and descent, which is also equivalent to the fact that $T = T_0 \oplus T_1$, where T_0 is a nilpotent operator and T_1 is an invertible operator (see [23, Proposition A]). The Drazin spectrum is given by

$$\sigma_D(T) := \{\lambda \in \mathbb{C} : T - \lambda I \quad \text{is not Drazin invertible}\}.$$

We observe that $\sigma_D(T) = \sigma(T) \setminus \pi(T)$, where $\pi(T)$ is the set of all poles.

An operator $T \in \mathbf{B}(\mathcal{H})$ is called *left Drazin invertible* if $a(T) < \infty$ and $\mathcal{R}(T^{a(T)+1})$ is closed (see [9, Definition 2.4]). The left Drazin spectrum is given by

$$\sigma_{LD}(T) := \{\lambda \in \mathbb{C} : T - \lambda I \quad \text{is not left Drazin invertible}\}.$$

Recall [9, Definition 2.5] that $\lambda \in \sigma_a(T)$ is a left pole of T if $T - \lambda I$ is a left Drazin invertible operator and $\lambda \in \sigma_a(T)$ is a left pole of finite rank if λ is a left pole of T and $\alpha(T - \lambda) < \infty$. We will denote $\pi^a(T)$ the set of all left poles of T, and by $\pi_0^a(T)$ the set of all left poles, of T of finite rank. We have $\sigma_{LD}(T) = \sigma_a(T) \setminus \pi^a(T)$.

Note that if $\lambda \in \pi^a(T)$, then it is easily seen that $T - \lambda$ is an operator of topological uniform descent. Therefore it follows from ([11, Theorem 2.5]) that λ is isolated in $\sigma_a(T)$. Following [9] if $T \in \mathbf{B}(\mathcal{H})$ and $\lambda \in \mathbb{C}$ is isolated in $\sigma_a(T)$, then $\lambda \in \pi^a(T)$ if and only if $\lambda \notin \sigma_{SBF_+^-}(T)$ and $\lambda \in \pi_0^a(T)$ if and only if $\lambda \notin \sigma_{SF_+^-}(T)$.

For the sake of simplicity of notation we introduce the abbreviations gaW, aW, gW and W to signify that an operator $T \in \mathbf{B}(\mathcal{H})$ obeys generalized a-Weyl's theorem, a-Weyl's theorem, generalized Weyl's theorem and Weyl's theorem, respectively. Analogous meaning is attached to the abbreviations gaB, aB, gB and B with respect to Browder's theorem.

In the following diagram, arrows signify implications between various Weyl and Browder type theorems. It is known from [1, 3, 7, 11, 19, 20, 27] that if

$T \in \mathbf{B}(\mathcal{H})$, then we have:

$$\begin{tikzcd}
& gW \arrow[r] & gB \arrow[r, leftrightarrow] & B \\
gaW \arrow[ur] \arrow[r] \arrow[dr] & aW \arrow[r] & aB \arrow[ur] \arrow[dl, leftrightarrow] & \\
& gaB & &
\end{tikzcd}$$

The quasinilpotent part $H_0(T-\lambda)$ and the analytic core $K(T-\lambda)$ of $T-\lambda$ are defined by

$$H_0(T-\lambda) := \{x \in \mathcal{H} : \lim_{n \longrightarrow \infty} \|(T-\lambda)^n x\|^{\frac{1}{n}} = 0\}.$$

and

$$\begin{aligned} K(T-\lambda) = \{&x \in \mathcal{H} : \text{there exists a sequence } \{x_n\} \subset \mathcal{H} \text{ and } \delta > 0 \text{ for which} \\ &x = x_0, (T-\lambda)x_{n+1} = x_n \text{ and } \quad \|x_n\| \leq \delta^n \|x\| \text{ for all } \quad n = 1, 2, \ldots\}. \end{aligned}$$

We note that $H_0(T-\lambda)$ and $K(T-\lambda)$ are generally non-closed hyper-invariant subspaces of $T-\lambda$ such that $(T-\lambda)^{-p}(0) \subseteq H_0(T-\lambda)$ for all $p = 0, 1, \ldots$ and $(T-\lambda)K(T-\lambda) = K(T-\lambda)$. Recall that if $\lambda \in iso(\sigma(T))$, then $H_0(T-\lambda) = \chi_T(\{\lambda\})$, where $\chi_T(\{\lambda\})$ is the glocal spectral subspace consisting of all $x \in \mathcal{H}$ for which there exists an analytic function $f : \mathbb{C} \setminus \{\lambda\} \longrightarrow \mathcal{H}$ that satisfies $(T-\mu)f(\mu) = x$ for all $\mu \in \mathbb{C} \setminus \{\lambda\}$ (see [17]).

Let $Hol(\sigma(T))$ be the space of all functions that analytic in an open neighborhoods of $\sigma(T)$. Following [18] we say that $T \in \mathbf{B}(\mathcal{H})$ has the single-valued extension property (SVEP) at point $\lambda \in \mathbb{C}$ if for every open neighborhood U_λ of λ, the only analytic function $f : U_\lambda \longrightarrow \mathcal{H}$ which satisfies the equation $(T-\mu)f(\mu) = 0$ is the constant function $f \equiv 0$. It is well-known that $T \in \mathbf{B}(\mathcal{H})$ has SVEP at every point of the resolvent $\rho(T) := \mathbb{C} \setminus \sigma(T)$. Moreover, from the identity theorem for analytic function it easily follows that $T \in \mathbf{B}(\mathcal{H})$ has SVEP at every point of the boundary $\partial\sigma(T)$ of the spectrum. In particular, T has SVEP at every isolated point of $\sigma(T)$. In [25, Proposition 1.8], Laursen proved that if T is of finite ascent, then T has SVEP.

Proposition 1.1 ([24])**.** *Let $T \in \mathbf{B}(\mathcal{H})$.*

(i) *If T has the SVEP, then $i(T-\lambda I) \leq 0$ for every $\lambda \in \rho_{SBF}(T)$.*

(ii) *If T^* has the SVEP, then $i(T-\lambda I) \geq 0$ for every $\lambda \in \rho_{SBF}(T)$.*

(iii) *If T^* has the SVEP, then*

$$\text{(a)} \quad \sigma_{SF_+^-}(T) = \omega(T) \quad \textit{and} \quad \text{(b)} \quad \sigma_{SBF_+^-}(T) = \sigma_{B\omega}(T).$$

In [36] H. Weyl examined the spectra of all compact perturbations of a hermitian operator T on a Hilbert space and proved that their intersection coincides with the isolated point of the spectrum $\sigma(T)$ which are the eigenvalues of finite multiplicity. Weyl's theorem has been extended to several classes of

Hilbert space operators including seminormal operators [4, 5]. In [7] M. Berkani introduced the concepts of the generalized Weyl's theorem and generalized Browder's theorem, and they showed that T satisfies the generalized Weyl's theorem whenever T is a normal operator on Hilbert space. More recently, [10] extended this result to hyponormal operators. In [32] extended this result to log-hyponormal operators. Recently, Rashid et al. [31] showed that if T is quasi-class A, then generalized Weyl's theorem holds $f(T)$ for every $f \in Hol(\sigma(T))$. More recently, in [26] Mecheri showed that generalized Weyl's theorem holds for algebraically (p,k)-quasihyponormal operators.

In this paper, we study generalized a-Weyl's theorem for algebraically (p,k)-quasihyponormal operators. Among other things, we prove that the spectral mapping theorem holds for semi-B-essential approximate point spectrum $\sigma_{SBF_+^-}(T)$, and for left Drazin spectrum for every $f \in Hol(\sigma(T))$.

2. Properties of algebraically (p,k)-quasihyponormal operators

Definition 2.1 ([22]). An operator $T \in \mathbf{B}(\mathcal{H})$ is said to be (p,k)-quasihyponormal if

$$T^{k*}((T^*T)^p - (TT^*)^p)T^k \geq 0,$$

where $0 \leq p \leq 1$ and k is a positive integer. Especially, when $p = 1, k = 1, p = k = 1$, T is called k-quasihyponormal, p-quasihyponormal, quasihyponormal, respectively.

Definition 2.2. An operator $T \in \mathbf{B}(\mathcal{H})$ is said to be algebraically (p,k)-quasihyponormal if there exists a non-constant complex polynomial $\mathcal{P}$ such that $\mathcal{P}(T)$ is a (p,k)-quasihyponormal operator.

In general, the following implications hold:

$$\begin{aligned} &p\text{-hyponormal} \Rightarrow p\text{-quasihyponormal} \Rightarrow \text{algebraically } p\text{-quasihyponormal} \\ &\qquad \Rightarrow \text{algebraically } (p,k)\text{-quasihyponormal}. \end{aligned}$$

An operator $T \in \mathbf{B}(\mathcal{H})$ is called isoloid if every isolated point of $\sigma(T)$ is an eigenvalue of T. An operator $T \in \mathbf{B}(\mathcal{H})$ is called normaloid if $r(T) = \|T\|$, where $r(T)$ is the spectral radius of T. $X \in \mathbf{B}(\mathcal{H})$ is called a quasiaffinity if it has trivial kernel and dense range. $S \in \mathbf{B}(\mathcal{H})$ is said to be a quasiaffine transform of $T \in \mathbf{B}(\mathcal{H})$ (notation: $S \prec T$) if there is a quasiaffinity $X \in \mathbf{B}(\mathcal{H})$ such that $XS = TX$. If both $S \prec T$ and $T \prec S$, then we say that S and T are quasisimilar.

The following facts follow from the above definition and some well known facts about (p,k)-quasihyponormal operators.

(i) If $T \in \mathbf{B}(\mathcal{H})$ is an algebraically (p,k)-quasihyponormal operator, then so is $T - \lambda I$ for each $\lambda \in \mathbb{C}$.

(ii) If $T \in \mathbf{B}(\mathcal{H})$ is an algebraically (p,k)-quasihyponormal operator and M is a closed T-invariant subspace of $\mathcal{H}$, then $T|_M$ is an algebraically (p,k)-quasihyponormal operator.

Lemma 2.3. *Let $T \in \mathbf{B}(\mathcal{H})$ be a p-quasihyponormal operator for $0 < p \leq 1$. Then the following assertions hold.*

(1) $\|T^n x\|^2 \leq \|T^{n-1}x\|\|T^{n+1}x\|$ *for all unit vector $x \in \mathcal{H}$ and all positive integer n.*
(2) $\|T^n\|^n \leq \|T^{n-1}\|^n r(T^n)$ *for all positive integer n, where $r(T^n)$ denote the spectral radius of T^n. Hence T is normaloid.*
(3) *T is a paranormal operator.*

Proof. (1) It is obvious that if T is p-quasihyponormal, then it is a (p,n)-quasihyponormal operator for each positive integer n, since

$$\begin{aligned}&\langle T^{*n}(TT^*)^p T^n x, x\rangle\\ &= \langle T^{*n}T(T^*T)^{p-1}T^*T^n x, x\rangle\\ &= \langle (T^*T)^{p+1}T^{n-1}x, T^{n-1}x\rangle\\ &\geq \|T^{n-1}x\|^{-2p}\langle T^*TT^{n-1}x, T^{n-1}x\rangle^{p+1} \text{ (by Hölder-McCarthy inequality)}\\ &= \|T^{n-1}x\|^{-2p}\|T^n x\|^{2p+2}\end{aligned}$$

and

$$\begin{aligned}&\langle T^{*n}(T^*T)^p T^n x, x\rangle\\ &= \langle (T^*T)^p T^n x, T^n x\rangle\\ &\leq \|T^n x\|^{2-2p}\langle T^*TT^n x, T^n x\rangle \text{ (Hölder-McCarthy inequality)}\\ &= \|T^n x\|^{2-2p}\|T^{n+1}x\|^{2p}.\end{aligned}$$

But T is a p-quasihyponormal operator. Then

$$\langle T^{*n}((T^*T)^p - (TT^*)^p)T^n x, x\rangle \geq 0.$$

Hence

$$\|T^n x\|^2 \leq \|T^{n-1}x\|\|T^{n+1}x\|.$$

(2) If $T^n = 0$ for some $n > 1$, then $T = 0$, and in this case $r(T) = 0$. Hence (2) is obvious. Hence we may assume $T^n \neq 0$ for all $n \geq 1$. Then

$$\frac{\|T^n\|}{\|T^{n-1}\|} \leq \frac{\|T^{n+1}\|}{\|T^n\|} \leq \cdots \leq \frac{\|T^{mn}\|}{\|T^{mn-1}\|}$$

by (1), and we have

$$\left(\frac{\|T^n\|}{\|T^{n-1}\|}\right)^{mn-n-1} \leq \frac{\|T^{n+1}\|}{\|T^n\|} \times \cdots \times \frac{\|T^{mn}\|}{\|T^{mn-1}\|} = \frac{\|T^{mn}\|}{\|T^{n-1}\|}.$$

Hence

$$\left(\frac{\|T^n\|}{\|T^{n-1}\|}\right)^{n-\frac{n}{m}-\frac{1}{n}} \leq \frac{\|T^{mn}\|^{\frac{1}{m}}}{\|T^{mn-1}\|^{\frac{1}{m}}}.$$

Now letting $m \longrightarrow \infty$. We get

$$\|T^n\|^n \leq \|T^{n-1}\|^n r(T^n).$$

Put $n = 1$, we have $\|T\| \leq r(T)$. So $\|T\| = r(T)$, *i.e.*, T is normaloid.

(3) Put $n=1$ in (1), we have $\|Tx\|^2 \leq \|T^2x\|$, that is, T is paranormal. $\square$

Definition 2.4 ([17])**.** An operator $T \in \mathbf{B}(\mathcal{H})$ is said to be totally hereditarily normaloid, $T \in THN$ if every part of T (*i.e.*, its restriction to an invariant subspace), and T_p^{-1} for every invertible part T_p of T, is normaloid.

Lemma 2.5. *Let $T \in THN$, let $\lambda \in \mathbb{C}$. Assume that $\sigma(T) = \{\lambda\}$. Then $T = \lambda I$.*

Proof. We consider two cases:

case I. $(\lambda = 0)$: Since T is normaloid. Therefore $T = 0$.

case II. $(\lambda \neq 0)$: Here T is invertible, and since $T \in THN$, we see that T, T^{-1} are normaloid. On the other hand $\sigma(T^{-1}) = \{\frac{1}{\lambda}\}$, so $\|T\|\|T^{-1}\| = |\lambda||\frac{1}{\lambda}| = 1$. It follows that T is convexoid, so $W(T) = \{\lambda\}$. Therefore $T = \lambda I$. $\square$

In [14], Curto and Han proved that quasinilpotent algebraically paranormal operators are nilpotent. We now establish a similar result for algebraically (p,k)-quasihyponormal operators.

Proposition 2.6. *Let T be a quasinilpotent (p,k)-quasihyponormal operator. Then T is nilpotent.*

Proof. Assume that $p(T)$ is a totally hereditarily normaloid operator for some nonconstant polynomial p. Since $\sigma(p(T)) = p(\sigma(T))$, the operator $p(T) - p(0)$ is quasinilpotent. Thus Lemma 2.5 would imply that

$$cT^m(T - \lambda_1 I) \cdots (T - \lambda_n I) \equiv p(T) - p(0) = 0,$$

where $m \geq 1$. Since $T - \lambda_j I$ is invertible for every $\lambda_j \neq 0$, we must have $T^m = 0$. $\square$

Lemma 2.7. *Let T be an invertible p-quasihyponormal operator. Then $\mathcal{H} = \mathcal{R}(T) \oplus \ker(T)$. Moreover T_1, the restriction of T to $\mathcal{R}(T)$ is one-one and onto.*

Proof. Suppose that $y \in \mathcal{R}(T) \cap \ker(T)$ then $y = Tx$ for some $x \in \mathcal{H}$ and $Ty = 0$. It follows that $T^2x = 0$. However, $d(T) = 1$ and so $x \in \ker(T^2) = \ker(T)$. Hence $y = Tx = 0$ and so $\mathcal{R}(T) \cap \ker(T) = \{0\}$. Also, $T\mathcal{R}(T) = \mathcal{R}(T)$.

If $x \in \mathcal{H}$, there is $u \in \mathcal{R}(T)$ such that $Tu = Tx$. Now if $z = x - u$, then $Tz = 0$. Hence

$$\mathcal{H} = \mathcal{R}(T) \oplus \ker(T).$$

Since $a(T) = 1$, T maps $\mathcal{R}(T)$ onto itself. If $y \in \mathcal{R}(T)$ and $Ty = 0$, then $y \in \mathcal{R}(T) \cap \ker(T) = \{0\}$. Hence T_1 is one-one and onto. $\square$

Observe that $\{\lambda_0\}$ is a clopen subset of $\sigma(T)$. Let $T \in \mathbf{B}(\mathcal{H})$. The $R_\lambda(T) = (T - \lambda)^{-1}$ is analytic on $\rho(T)$, and an isolated point λ_0 of $\sigma(T)$ is an isolated singular point of the resolvent of T. Here there is a Laurent expansion of this function in powers of $\lambda - \lambda_0$. We write this in the form

$$(T - \lambda)^{-1} = \sum_{n=0}^{\infty} (\lambda - \lambda_0)^n A_n + \sum_{n=1}^{\infty} (\lambda - \lambda_0)^{-n} B_n.$$

The coefficients A_n and B_n are members of $\mathbf{B}(\mathcal{H})$ and given by the standard formulas

$$A_n = \frac{1}{2\pi i}\int_\Gamma (\lambda-\lambda_0)^{-n-1}(\lambda-T)^{-1}\,d\lambda, \tag{2.1}$$

$$B_n = \frac{1}{2\pi i}\int_\Gamma (\lambda-\lambda_0)^{n-1}(\lambda-T)^{-1}\,d\lambda, \tag{2.2}$$

where Γ is any circle $|\lambda-\lambda_0| = \rho$ with $0<\rho<\delta$ described once counterclockwise.
The function f_n defined by

$$f_n(\lambda) = \begin{cases} (\lambda-\lambda_0)^{n-1}, & \text{if} \quad |\lambda-\lambda_0| \le \rho < \delta, \\ 0, & \text{otherwise.} \end{cases}$$

is in $Hol(\sigma(T))$ and moreover

$$B_n = f_n(T),\ n = 1, 2, \ldots.$$

For each positive integer n, we have

$$(\lambda-\lambda_0)f_n(\lambda) = f_{n+1}.$$

So

$$(T-\lambda_0)B_n = B_{n+1} \tag{2.3}$$

and by induction

$$(T-\lambda_0)^n B_1 = B_{n+1}. \tag{2.4}$$

We note in passing that

$$B_1 = E(\lambda_0) \tag{2.5}$$

the spectral projection corresponding to the clopen set λ_0 of $\sigma(T)$.

Consider for each non-negative integer n the function g_n defined by

$$g_n(\lambda) = \begin{cases} 0, & \text{if} \quad |\lambda-\lambda_0| \le \rho < \delta, \\ (\lambda-\lambda_0)^{-n-1}, & \text{otherwise.} \end{cases}$$

is in $Hol(\sigma(T))$. Moreover,

$$A_n = -g_n(T)$$

for each non-negative integer n. We have

$$(\lambda-\lambda_0)g_{n+1}(\lambda) = g_n(\lambda) \tag{2.6}$$

and so

$$(\lambda-\lambda_0)A_{n+1} = A_n. \tag{2.7}$$

Similarly $(\lambda-\lambda_0)g_0(\lambda) + f_1(\lambda) = 1$ and so

$$(T-\lambda)A_0 = B_0 - 1. \tag{2.8}$$

Recall that if $T \in \mathbf{B}(\mathcal{H})$ and λ_0 is an isolated point of $\sigma(T)$, then λ_0 is called a pole of order m if and only if $E(\lambda_0)(\lambda_0 - T)^m = 0$ and $E(\lambda_0)(\lambda_0 - T)^{m-1} \neq 0$.

Lemma 2.8. *Let T be a (p,k)-quasihyponormal operator and $\lambda_0 \in iso\sigma(T)$. Let $\tau = \sigma(T) \setminus \{\lambda_0\}$. Then λ_0 is an eigenvalue of T. The ascent and descent of $T - \lambda_0$ are both equal to k. Also*

$$\mathcal{R}(E(\lambda_0)) = \ker((T - \lambda_0)^k),$$

$$\mathcal{R}(E(\tau)) = \mathcal{R}((T - \lambda_0)^k).$$

Proof. For convenience we denote the null-space and range of $(\lambda_0 - T)^k$ by $\ker_k$ and $\mathcal{R}_k$, respectively. If $x \in \ker_k$, where $k \geq 1$, we see by (2.7), induction and (2.8) that

$$0 = A_{k-1}(T - \lambda_0)^k x = (T - \lambda_0)^k A_{k-1} x = (T - \lambda_0) A_0 x = B_1 x - x.$$

So that by (2.5), we have $x = B_1 x \in \mathcal{R}(E(\lambda_0))$. Thus $\ker_k \subseteq \mathcal{R}(E(\lambda_0))$ if $k \geq 1$. On the other hand, it follows from (2.4) that if $x \in \mathcal{R}(E(\lambda_0))$, then $x = B_1 x$ and $(T - \lambda_0)^k x = B_{k+1} x$. Since $B_{n+1} x = 0$ if $n \geq k$. It follows that $\mathcal{R}(E(\lambda_0)) \subseteq \ker_k$ and $\ker_n = \mathcal{R}(E(\lambda_0))$ if $n \geq k$. However, $\ker_{k-1}$ is a proper subset of $\ker_k$ because $B_k \neq 0$. The equations $\ker_{k-1} = \ker_k = \mathcal{R}(E(\lambda_0))$ imply that $B_k = 0$ in view of the relation $B_k = (T - \lambda_0)^{k-1} B_1$. We have now proved that the ascent of $\lambda_0 - T$ is k and $\ker_k = \mathcal{R}(E(\lambda_0))$. In particular, since $k > 0$, λ_0 is an eigenvalue of T.

Now let T_1 and T_2 be the restrictions of T to $\mathcal{R}(E(\tau))$ and $\mathcal{R}(E(\lambda_0))$, respectively. $\lambda_0 \in \sigma(T_2)$ but $\lambda_0 \notin \sigma(T_1)$. Hence, the descent of $\lambda_0 - T_1$ is 0 and $\mathcal{R}((\lambda_0 - T_1)^k) = \mathcal{R}(E(\tau))$ when $k \geq 1$. Thus $\mathcal{R}(E(\tau)) \subseteq \mathcal{R}_k$. Now if $n \geq k$, the only point common to $\mathcal{R}_n$ and $\ker_n$ is 0. For, if $x \in \mathcal{R}_n \cap \ker_n$, then $(\lambda_0 - T)^n x = 0$ and there is $y \in \mathcal{H}$ such that $x = (\lambda_0 - T)^n y$. Hence $y \in \ker_{2n} = \ker$ and so $x = 0$. Now suppose that $n \geq k$ and $x \in \mathcal{R}_n$. Let $x_1 = E(\tau)x$ and $x_2 = E(\lambda_0)$, then $x_2 = x - x_1 \in \mathcal{R}_n$ because $\mathcal{R}(E(\tau)) \subseteq \mathcal{R}_n$. However, $x_2 \in \mathcal{R}(E(\lambda_0)) = \ker_n$, and so $x_2 = 0$ whence $x = x_1 \in \mathcal{R}(E(\tau))$. Thus $\mathcal{R}_n \subseteq \mathcal{R}(E(\tau))$ if $n \geq k$ and therefore that the descent of $\lambda_0 - T$ is less that or equal to k. Then by [15, Proposition 1.49] shows that the descent is exactly k, which know to be the ascent. □

Corollary 2.9. *Let $T \in \mathbf{B}(\mathcal{H})$ be a (p,k)-quasihyponormal operator. Then T is of finite ascent.*

An operator $T \in \mathbf{B}(\mathcal{H})$ is said to be polaroid if $iso\sigma(T) \subseteq \pi(T)$, where $\pi(T)$ is the set of all poles of T. In general, if T is polaroid, then it is isoloid. However, the converse is not true. Consider the following example. Let $T \in \ell^2(\mathbb{N})$ be defined by

$$T(x_1, x_2, \ldots) = \left(\frac{x_2}{2}, \frac{x_3}{3}, \ldots\right).$$

Then T is a compact quasinilpotent operator with $\alpha(T) = 1$, and so T is isoloid. However, since T does not have finite ascent, T is not polaroid.

Proposition 2.10. *Let T be an algebraically (p, k)-quasihyponormal operator. Then T is polaroid.*

Proof. Suppose T is an algebraically (p, k)-quasihyponormal operator. Then $p(T)$ is (p, k)-quasihyponormal for some nonconstant polynomial p. Let $\lambda \in iso(\sigma(T))$. Using the spectral projection $P := \frac{1}{2i\pi}\int_{\partial D}(\mu - T)^{-1}\, d\mu$, where D is a closed disk of center λ which contains no other points of $\sigma(T)$, we can represent T as the direct sum

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}, \quad \text{and} \quad \sigma(T_1) = \{\lambda\} \quad \text{and} \quad \sigma(T_2) = \sigma(T) \setminus \{\lambda\}.$$

Since T_1 is algebraically (p, k)-quasihyponormal and $\sigma(T_1) = \{\lambda\}$. But $\sigma(T_1 - \lambda I) = \{0\}$ it follows from Proposition 2.6 that $T_1 - \lambda I$ is nilpotent. Therefore $T_1 - \lambda$ has finite ascent and descent. On the other hand, since $T_2 - \lambda I$ is invertible, clearly it has finite ascent and descent. Therefore $T - \lambda I$ has finite ascent and descent. Therefore λ is a pole of the resolvent of T. Thus if $\lambda \in iso(\sigma(T))$ implies $\lambda \in \pi(T)$, and so $iso(\sigma(T)) \subset \pi(T)$. Hence T is polaroid. $\square$

Corollary 2.11. *Let T be an algebraically (p, k)-quasihyponormal operator. Then T is isoloid.*

For $T \in \mathbf{B}(\mathcal{H})$, $\lambda \in \sigma(T)$ is said to be a regular point if there exists $S \in \mathbf{B}(\mathcal{H})$ such that $T - \lambda I = (T - \lambda I)S(T - \lambda I)$. T is is called reguloid if every isolated point of $\sigma(T)$ is a regular point. It is well known [19, Theorems 4.6.4 and 8.4.4] that $T - \lambda I = (T - \lambda I)S(T - \lambda I)$ for some $S \in \mathbf{B}(\mathcal{H}) \iff T - \lambda I$ has a closed range.

Theorem 2.12. *Let T be an algebraically (p, k)-quasihyponormal operator. Then T is reguloid.*

Proof. Suppose T is an algebraically (p, k)-quasihyponormal operator. Then $p(T)$ is a (p, k)-quasihyponormal operator for some nonconstant polynomial p. Let $\lambda \in iso(\sigma(T))$. Using the spectral projection $P := \frac{1}{2i\pi}\int_{\partial D}(\mu - T)^{-1}\, d\mu$, where D is a closed disk of center λ which contains no other points of $\sigma(T)$, we can represent T as the direct sum

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}, \quad \text{and} \quad \sigma(T_1) = \{\lambda\} \quad \text{and} \quad \sigma(T_2) = \sigma(T) \setminus \{\lambda\}.$$

Since T_1 is algebraically (p, k)-quasihyponormal and $\sigma(T_1) = \{\lambda\}$, it follows from Lemma 2.5 that $T_1 = \lambda I$. Therefore by [34, Theorem 6],

$$\mathcal{H} = E(\mathcal{H}) \oplus E(\mathcal{H})^{\perp} = \ker(T - \lambda I) \oplus \ker(T - \lambda I)^{\perp}. \tag{2.9}$$

Relative to decomposition 2.9, $T = \lambda I \oplus T_2$. Therefore $T - \lambda I = 0 \oplus T - \lambda I$ and hence $\operatorname{ran}(T - \lambda I) = (T - \lambda I)(\mathcal{H}) = 0 \oplus (T_2 - \lambda I)(\ker(T - \lambda I)^{\perp})$. Since $T_2 - \lambda I$ is invertible, $T - \lambda I$ has closed range. $\square$

Theorem 2.13. *Let $T^* \in \mathbf{B}(\mathcal{H})$ be an algebraically (p, k)-quasihyponormal operator. Then T is a-isoloid.*

Proof. Suppose T^* is algebraically (p, k)-quasihyponormal. Since T^* has SVEP, then $\sigma(T) = \sigma_a(T)$. Let $\lambda \in iso(\sigma_a(T)) = iso(\sigma(T))$. But T^* is polaroid, hence T is also polaroid. Therefore it is isoloid, and hence $\lambda \in \sigma_p(T)$. Thus T is a-isoloid. □

3. Weyl's type theorem

Lemma 3.1. *If T is a (p, k)-quasihyponormal operator and $S \prec T$, then S has SVEP.*

Proof. Since T is a (p, k)-quasihyponormal operator, then it has a SVEP. So the result follows from [14, Lemma 3.1]. □

Theorem 3.2. *Let $S, T \in \mathbf{B}(\mathcal{H})$. If T has SVEP and $S \prec T$, then $f(S) \in gaB$ for every $f \in Hol(\sigma(T))$. In particular, if T has SVEP, then $T \in gaB$.*

Proof. Suppose that T has SVEP. Since $S \prec T$, it follows from the proof of [14] that S has SVEP. We now show that $S \in gaB$. Let $\lambda \in \sigma_a(S) \setminus \sigma_{SBF_+^-}(S)$; then $S - \lambda I \in SBF_+^-(S)$ but not bounded below. Since $S - \lambda I \in SBF_+^-(S)$, it follows from from [11, Corollary 2.10] that $S - \lambda I = S_1 \oplus S_2$, where S_1 is an upper semi-Fredholm operator with $i(S_1) \le 0$, and S_2 is nilpotent. Since S has SVEP, S_1 and S_2 also have SVEP. Therefore a-Browder's theorem holds for S_1, and hence $\sigma_{ab}(S_1) = \sigma_{SF_+^-}(S_1)$. Since S_1 is semi-Fredholm with $i(S_1) \le 0$, S_1 is a-Browder's. Hence λ is an isolated point of $\sigma_a(S)$. It follows that $S \in gaB$.

Now let $f \in Hol(\sigma(T))$. Since the SVEP is stable under the functional calculus, then $f(S)$ has the SVEP. Therefore $f(S) \in gaB$, by the first part of the proof. □

We now recall that the generalized a-Weyl's theorem may not hold for quasinilpotent operators, and that it does not necessarily transfer to or from adjoints.

Example 3.3. Let $T \in \mathbf{B}(\mathcal{H})$ defined on ℓ^2 by

$$T(x_1, x_2, \ldots) = \left(\frac{x_2}{2}, \frac{x_3}{3}, \ldots\right).$$

Then T is a quasinilpotent operator and $\sigma(T) = \sigma_{SBF_+^-}(T) = E^a(T) = \{0\}$. Thus T does not obey generalized a-Weyl's theorem.

Now $\sigma(T^*) = \sigma_{SBF_+^-}(T^*) = \{0\}$ and $E^a(T^*) = \emptyset$. Therefore $T^* \in gaW$.

As a consequence of [17, Theorem 2.4] and [16, Lemma 2.5] we have:

Theorem 3.4. *Let $T \in \mathbf{B}(\mathcal{H})$ be a (p, k)-quasihyponormal operator. Then T is of stable index.*

Let $T \in \mathbf{B}(\mathcal{H})$. It is well known that the inclusion $\sigma_{SF_+^-}(f(T)) \subseteq f(\sigma_{SF_+^-}(T))$ holds for every $f \in Hol(\sigma(T))$ with no restriction on T [29]. The next theorem shows that the spectral mapping theorem holds for the essential approximate point spectrum for algebraically (p, k)-quasihyponormal operator.

Theorem 3.5. *Suppose T^* or T is an algebraically (p,k)-quasihyponormal operator. Then*

$$\sigma_{SF_+^-}(f(T)) = f(\sigma_{SF_+^-}(T)).$$

Proof. Assume first that T is an algebraically (p,k)-quasihyponormal operator and let $f \in Hol(\sigma(T))$. It suffices to show that $\sigma_{SF_+^-}(f(T)) \supseteq f(\sigma_{SF_+^-}(T))$. Suppose that $\lambda \notin \sigma_{SF_+^-}(f(T))$. Then $f(T) - \lambda I \in SF_+^-(\mathcal{H})$ and

$$f(T) - \lambda I = c(T - \mu_1 I)(T - \mu_2 I) \cdots (T - \mu_n I)g(T),$$

where $c, \mu_1, \mu_2, \dots, \mu_n \in \mathbb{C}$, and $g(T)$ is invertible. Since T is an algebraically (p,k)-quasihyponormal operator, it has SVEP. It follows from [2, Theorem 2.6] that $i(T - \mu_j) \leq 0$ for each $j = 1, 2, \dots, n$. Therefore $\lambda \notin f(\sigma_{SF_+^-}(T))$, and hence $\sigma_{SF_+^-}(f(T)) = f(\sigma_{SF_+^-}(T))$. Suppose now that T^* is an algebraically (p,k)-quasihyponormal operator. Then T^* has SVEP, and so by [2, Theorem 2.6] $i(T - \mu_j I) \geq 0$ for each $j = 1, 2, \dots, n$. Since

$$0 \leq \sum_{j=1}^{n} i(T - \mu_j I) = i(f(T) - \lambda I) \leq 0,$$

$T - \mu_j I$ is Weyl for each $j = 1, 2, \dots, n$. Hence $\lambda \notin f(\sigma_{SF_+^-}(T))$, and so $\sigma_{SF_+^-}(f(T)) = f(\sigma_{SF_+^-}(T))$. This completes the proof. □

Theorem 3.6. *Suppose T^* is an algebraically (p,k)-quasihyponormal operator. Then a-Weyls theorem holds for $f(T)$ for every $f \in Hol(\sigma(T))$.*

Proof. Suppose T^* is an algebraically (p,k)-quasihyponormal operator. We first show that a-Weyls theorem holds for T. Suppose that $\lambda \in \sigma_a(T) \backslash \sigma_{SF_+^-}(T)$. Then $T - \lambda I$ is upper semi-Fredholm and $i(T - \lambda I) \leq 0$. Since T^* is an algebraically (p,k)-quasihyponormal operator, T^* has SVEP. Therefore by [2, Theorem 2.6] that $i(T - \lambda I) \geq 0$, and hence $T - \lambda I$ is Weyl. Since T^* has SVEP, it follows from [18, Corollary 7] that $\sigma_a(T) = \sigma(T)$. Also, since Weyls theorem holds for T by [26], $\lambda \in \pi_0^a(T)$.

Conversely, suppose that $\lambda \in \pi_0^a(T)$. Since T^* has SVEP, it follows from [18, Corollary 7] that $\sigma_a(T) = \sigma(T)$. Therefore λ is an isolated point of $\sigma(T)$, and hence $\bar{\lambda}$ is an isolated point of $\sigma(T^*)$. But T^* is an algebraically (p,k)-quasihyponormal operator, hence by Proposition 2.10 that $\bar{\lambda} \in \pi(T^*)$. Therefore there exists a natural number n_0 such that $n_0 = a(T^* - \bar{\lambda} I) = d(T^* - \bar{\lambda} I)$. Hence we have $\mathcal{H} = \ker((T^* - \bar{\lambda} I)^{n_0}) \oplus \operatorname{ran}((T^* - \bar{\lambda} I)^{n_0})$ and $\operatorname{ran}((T^* - \bar{\lambda} I)^{n_0})$ is closed. Therefore $\operatorname{ran}((T - \lambda I)^{n_0})$ is closed and $\mathcal{H} = \ker((T^* - \bar{\lambda} I)^{n_0})^\perp \oplus \operatorname{ran}((T^* - \bar{\lambda} I)^{n_0})^\perp = \ker((T - \lambda I)^{n_0}) \oplus \operatorname{ran}((T - \lambda I)^{n_0})$. So $\lambda \in \sigma_p(T)$, and hence $T - \lambda I$ is Weyl. Consequently, $\lambda \in \sigma_a(T) \setminus \sigma_{SF_+^-}(T)$. Thus a-Weyls theorem holds for T.

Now we show that T is a-isoloid. Let λ be an isolated point of $\sigma_a(T)$. Since T^* has SVEP, λ is an isolated point of $\sigma(T)$. But T^* is polaroid, hence T is also polaroid. Therefore it is isoloid, and hence $\lambda \in \sigma_p(T)$. Thus T is a-isoloid.

Finally, we shall show that a-Weyls theorem holds for $f(T)$ for every $f \in Hol(\sigma(T))$. Let $f \in Hol(\sigma(T))$. Since a-Weyls theorem holds for T, it satisfies a-Browders theorem. Therefore $\sigma_{ab}(T) = \sigma_{SF_+^-}(T)$. It follows from Theorem 3.5 that

$$\sigma_{ab}(f(T)) = f(\sigma_{ab}(T)) = f(\sigma_{SF_+^-}(T)) = \sigma_{SF_+^-}(f(T)),$$

and hence a-Browders theorem holds for $f(T)$. So $\sigma_a()f(T) \setminus \sigma_{SF_+^-}(f(T)) \subset \pi_0^a(T)$. Conversely, suppose that $\lambda \in \pi_0^a(f(T))$. Then λ is an isolated point of $\sigma_a(f(T))$ and $0 < \alpha(f(T) - \lambda I) < 1$. Since λ is an isolated point of $f(\sigma_a(T))$, if $\mu_j \in \sigma_a(T)$, then μ_j is an isolated point of $\sigma_a(T)$. Since T is a-isoloid, $0 < \alpha(T - \mu_j) < 1$ for each $j = 1, 2, \ldots, n$. Since a-Weyls theorem holds for T, $T - \mu_j$ is upper semi-Fredholm and $i(T - \mu_j) \leq 0$ for each $j = 1, 2, \ldots, n$. Therefore $f(T) - \lambda I$ is upper semi-Fredholm and $f(T) - \lambda I = \sum_{j=1}^n i(T - \mu_j I) \leq 0$. Hence $\lambda \in \sigma_a()f(T) \setminus \sigma_{SF_+^-}(f(T))$, and so a-Weyls theorem holds for $f(T)$ for each $f \in Hol(\sigma(T))$. This completes the proof. □

Theorem 3.7. *Let T be an algebraically (p,k)-quasihyponormal operator. Then $\sigma_{lD}(T) = \sigma_{SBF_+^-}(T) \cup acc(\sigma_a(T))$.*

Proof. Suppose that $\lambda \in \sigma_a(T) \setminus \sigma_{lD}(T)$. Then $T - \lambda I$ is left Drazin invertible but not bounded below. In particular, $T - \lambda I$ is semi-B-Fredholm. Therefore $d = a(T - \lambda) < \infty$ and $\mathrm{ran}((T - \lambda I)^{d+1})$ is closed. On the other hand, since $d = a(T - \lambda I) < \infty$ and $(\mathrm{ran}(T - \lambda)^{d+1})$ is closed, λ is an isolated point of $\sigma_a(T)$. Hence $\lambda \in \sigma_a(T) \setminus (\sigma_{SBF_+^-}(T) \cup acc(\sigma_a(T)))$.

Conversely, suppose that $\lambda \in \sigma_a(T) \setminus (\sigma_{SBF_+^-}(T) \cup acc(\sigma_a(T)))$. Then $T - \lambda I$ is semi-B-Fredholm and λ is an isolated point of $\sigma_a(T)$. Since $T - \lambda I$ is upper semi-Fredholm, it follows from [11, Corollary 2.10] that $T - \lambda I$ can be decompose as $T - \lambda I = T_1 \oplus T_2$, where T_1 is an upper semi-Fredholm operator with $i(T_1) \leq 0$ and T_2 is nilpotent. We consider two cases.

Case I. Suppose that T_1 is bounded below. Then $T - \lambda I$ is left Drazin invertible, and so $\lambda \notin \sigma_{lD}(T)$.

Case II. Suppose that T_1 is not bounded below. Then 0 is an isolated point of $\sigma_a(T_1)$. But T_1 is an upper semi-Fredholm operator, hence it follows from the punctured neighborhood theorem that T_1 is a-Browder. Therefore there exists a finite rank operator S_1 such that $T_1 + S_1$ is bounded below and $T_1 S_1 = S_1 T_1$. Put $F := S_1 \oplus 0$. Then F is a finite rank operator, $TF = FT$ and $T - \lambda I + F = T_1 \oplus T_2 + S_1 \oplus 0 = (T_1 + S_1) \oplus T_2$ is left Drazin invertible. Hence $\lambda \notin \sigma_{lD}(T)$. □

As shown in [12] that the spectral mapping theorem holds for the Drazin spectrum. We prove here the spectral mapping theorem holds for left Drazin spectrum.

Theorem 3.8. *Let T be an algebraically (p,k)-quasihyponormal operator and let $f \in Hol(\sigma(T))$. Then $\sigma_{lD}(f(T)) = f(\sigma_{lD}(T))$.*

Proof. Suppose that $\mu \notin f(\sigma_{lD}(T))$ and set $h(\lambda) = f(\lambda) - \mu I$. Then h has no zeros in $\sigma_{lD}(T)$. Since $\sigma_{lD}(T) = \sigma_{SBF_+^-}(T) \cup acc(\sigma_a(T))$ by Theorem 3.7, we conclude that h has finitely many zeros in $\sigma_a(T)$. Now we consider two cases.

Case I. Suppose that h has no zeros in $\sigma_a(T)$. Then $h(T) = f(T) - \mu I$ is bounded below, and so $\mu \notin \sigma_{lD}(f(T))$.

Case II. Suppose that h has at least one zero in $\sigma_a(T)$. Then

$$h(\lambda) = c(\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n)g(\lambda),$$

where $c, \lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$ and $g(\lambda)$ is a nonvanishing analytic function on an open neighborhood. Therefore

$$h(T) = c(T - \lambda_1 I)(T - \lambda_2 I)\cdots(T - \lambda_n I)g(T),$$

where $g(T)$ is bounded below. Since $\mu \notin f(\sigma_{lD}(T))$, $\lambda_1, \lambda_2, \ldots, \lambda_n \notin \sigma_{lD}(T)$. Therefore $T - \lambda_j I$ is left Drazin invertible, and hence each $T - \lambda_j I \in SBF_+^-(r), j = 1, 2, \ldots, n$. But each λ_j is an isolated point of $\sigma_a(T)$, it follows from [11, Theorem 2.8] that each λ_j is a left pole of the resolvent of T. Therefore $a(T - \lambda_j I) = d < \infty$ and $\text{ran}(T - \lambda_j I)^{d+1}$ is closed $(j = 1, 2, \ldots, n)$, so $a((T - \lambda_1)(T - \lambda_2)\cdots(T - \lambda_n)) = s < \infty$ and $\text{ran}((T - \lambda_1)(T - \lambda_2)\cdots(T - \lambda_n))^{s+1}$ is closed. Since $g(T)$ is bounded below, $a(h(T)) = t < \infty$ and $\text{ran}((h(T)^{t+1}))$ is closed. Therefore $h(T)$ is left Drazin invertible, and so $0 \notin \sigma_{lD}(h(T))$. Hence $\mu \notin \sigma_{lD}(f(T))$. It follows from Cases I and II that $\sigma_{lD}(f(T)) \subseteq f(\sigma_{lD}(T))$.

Conversely, suppose that $\lambda \notin \sigma_{lD}(f(T))$. Then $f(T) - \lambda I$ is left Drazin invertible. We again consider two cases.

Case I. Suppose that $f(T) - \lambda I$ is bounded below. Then $\lambda \notin \sigma_a(f(T)) = f(\sigma_a(T))$, and hence $\lambda \notin f(\sigma_{lD}(T))$.

Case II. Suppose that $\lambda \in \sigma_a(f(T)) \setminus \sigma_{lD}(f(T))$. Write

$$f(T) = c(T - \lambda_1 I)(T - \lambda_2 I)\cdots(T - \lambda_n I)g(T),$$

where $c, \lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$ and $g(T)$ is bounded below. Since $f(T) - \lambda I$ is left Drazin invertible, $f(T) = c(T - \lambda_1 I)(T - \lambda_2 I)\cdots(T - \lambda_n I)g(T)$ has finite ascent say r and $\text{ran}(f(T))^{r+1}$ is closed. Hence $T - \lambda_j I$ has finite ascent say r_j and $\text{ran}(T - \lambda_j)^{r_j+1}$ is closed for every $j = 1, 2, \ldots, n$. Therefore each $T - \lambda_j I$ is left Drazin invertible, and so $\lambda_1, \ldots, \lambda_n \notin \sigma_{lD}(T)$.

We now wish to prove that $\lambda \notin f(\sigma_{lD}(T))$. Assume not; then there exists $\mu \in \sigma_{aD}(T)$ such that $f(\mu) = \lambda$. Since $g(\mu) \neq 0$, we must have $\mu = \mu_j$ for some $j = 1, 2, \ldots, n$, which implies $\mu_j \in \sigma_{lD}(T)$, a contradiction. Hence $\lambda \notin f(\sigma_{lD}(T))$, and so $f(\sigma_{lD}(T)) \subseteq \sigma_{lD}(f(T))$. This completes the proof. $\square$

Theorem 3.9. *Suppose T or T^* is an algebraically (p,k)-quasihyponormal operator. Then $f(\sigma_{SBF_+^-}(T)) = \sigma_{SBF_+^-}(f(T))$ for all $f \in Hol(\sigma(T))$.*

Proof. Let $\lambda \notin \sigma_{SBF_+^-}(f(T))$. Then $f(T)-\lambda I \in SBF_+^-(T)$ and

$$f(T)-\lambda I = \prod_{j=1}^{m}(T-\lambda_j I)g(T),$$

where $\lambda_1,\dots,\lambda_m \in \mathbb{C}$ and $g(T)$ is invertible. Since $f(T)-\lambda I$ is an upper semi-B-Fredholm operator, it follows from [7, Theorem 3.2] that $T-\lambda_j I$ is upper semi-B-Fredholm for each $1 \le j \le m$. Hence

$$i(f(T)-\lambda I) = \sum_{j=1}^{m} i(T-\lambda_j I) \le 0.$$

Now from [7, Remark A] there exists some integer k such that for each, $1 \le j \le m$, $T-(\lambda_j+\frac{1}{k})I$ is an upper semi-B-Fredholm operator and $i(T-(\lambda_j+\frac{1}{k})I) = i(T-\lambda_j I)$. If T is an algebraically (p,k)-quasihyponormal operator, then it follows from Proposition 1.1 that $i(T-\lambda_j I) \le 0$. Hence $\lambda \notin f(\sigma_{SBF_+^-}(T))$.

Now if T^* is an algebraically (p,k)-quasihyponormal operator, then we have from Proposition 1.1 that $i(T-\lambda_j I) = 0$ and so $T-\lambda_j I$ is a B-Fredholm operator of index 0. Thus $\lambda \notin f(\sigma_{SBF_+^-}(T))$.

For the converse inclusion. Let $\lambda \in \sigma_{SBF_+^-}(f(T)) \setminus f(\sigma_{SBF_+^-}(T))$. Suppose that

$$f(T)-\lambda I = \prod_{j=1}^{m}(T-\lambda_j I)g(T),$$

where $\lambda_1,\dots,\lambda_m \in \mathbb{C}\setminus\sigma_{SBF_+^-}(T)$ and $g(T)$ is invertible. Hence $f(T)-\lambda I$ is upper semi-B-Fredholm and $i(f(T)-\lambda I) = \sum_{j=1}^{m} i(T-\lambda_j I) \le 0$. Therefore $\lambda \notin \sigma_{SBF_+^-}(f(T))$, so a contradiction. □

Lemma 3.10. *Suppose that $T \in \mathbf{B}(\mathcal{H})$ is algebraically (p,k)-quasihyponormal. Then for any $f \in Hol(\sigma(T))$ we have*

$$\sigma_a(f(T)) \setminus E^a(f(T)) = f(\sigma_a(T)) \setminus E^a(T)).$$

Proof. Let $\lambda \in \sigma_a(f(T)) \setminus E^a(f(T))$. Then $\lambda \in \sigma_a(f(T)) = f(\sigma_a(T))$. We distinguish two cases:

Case I. $\lambda \notin iso(f(\sigma_a(T)))$, then there is an infinite sequence $\{\eta_n\}_{n\in\mathbb{N}} \in \sigma_a(T)$ such that $\lambda = f(\eta_0)$ and $\eta_n \longrightarrow \eta_0$. But $f \in Hol(\sigma(T))$, therefore $f(\eta_n) \longrightarrow f(\eta_0) = \lambda$ and $\lambda \in f(\sigma_a(T) \setminus E^a(T))$.

Case II. $\lambda \in iso(f(\sigma_a(T)))$, since $\lambda \notin E^a(f(T))$ then λ is not an eigenvalue of $f(T)$. Then

$$f(T)-\lambda I = (T-\eta_1 I)^{t_1}(T-\eta_2 I)^{t_2}\cdots(T-\eta_m I)^{t_m} g(T),$$

where $\eta_1, \dots, \eta_m$ are scalars and g is invertible. Since λ is not an eigenvalue of $f(T)$, then for each $j \in \{1, \dots, m\}$, η_j is not an eigenvalue of T. Hence $\eta_j \in \sigma_a(T) \setminus E^a(T)$ and $\lambda = f(\eta_j) \in f(\sigma_a(T) \setminus E^a(T))$.

Conversely, Let $\lambda \in f(\sigma_a(T) \setminus E^a(T))$ then $\lambda \in \sigma_a(f(T)) = f(\sigma_a(T))$. Assume that $\lambda \in E^a(f(T))$. Then

$$f(T) - \lambda I = (T - \eta_1 I)^{t_1}(T - \eta_2 I)^{t_2} \cdots (T - \eta_m I)^{t_m} g(T),$$

where $\eta_1, \dots, \eta_m$ are scalars and g is invertible. If $\eta_j \in \sigma_a(T)$, then $\eta_j \in iso(\sigma_a(T))$. Since T is a-isoloid, η_j is an eigenvalue of T. Hence $\eta_j \in E^a(T)$. So $\lambda = f(\eta_j)$ this leads a contraction to the fact that $\lambda \in f(\sigma_a(T) \setminus E^a(T))$. $\square$

Theorem 3.11. *Let $T^* \in \mathbf{B}(\mathcal{H})$ be algebraically (p,k)-quasihyponormal. Then generalized a-Weyl's theorem holds for $f(T)$, for every $f \in Hol(\sigma(T))$.*

Proof. If T^* is an algebraically (p,k)-quasihyponormal operator, then T^* has SVEP $\sigma(T) = \sigma_a(T)$ and consequently $E(T) = E^a(T)$.

Let $\lambda \notin \sigma_{SBF_+^-}(T)$ be given. Then $T - \lambda$ is semi-B-Fredholm and $i(T - \lambda) \leq 0$. Then Proposition 1.1 implies that $i(T - \lambda) = 0$ and consequently $T - \lambda$ is B-Weyl's. Hence $\lambda \notin \sigma_{B\omega}(T)$. Hence it follows from [37, Theorem3.1] that $\lambda \in E(T) = E^a(T)$.

For the converse, let $\lambda \in E^a(T)$. Then $\lambda \in iso\sigma_a(T)$. Since T^* has SVEP, we have $\sigma(T) = \sigma_a(T)$. Hence $\overline{\lambda} \in \sigma(T^*)$. Now we represent T^* as the direct sum $T^* = T_1 \oplus T_2$, where $\sigma(T_1) = \{\overline{\lambda}\}$ and $\sigma(T_2) = \sigma(T) \setminus \{\overline{\lambda}\}$. Since $T \in \mathbf{\Upsilon}(\mathcal{H})$ then so does T_1, and so we have two cases:

Case I. ($\overline{\lambda} = 0$): then T_1 is quasinilpotent. Hence it follows that T_1 is nilpotent. Since T_2 is invertible, Then T^* is B-Weyl's.

Case II. ($\overline{\lambda} \neq 0$): Since $\sigma(T_1) = \{\overline{\lambda}\}$, then $T_1 - \overline{\lambda}$ is nilpotent and $T_2 - \overline{\lambda}$ is invertible, it follows from [37, Theorem 3.1] that $T^* - \overline{\lambda}$ is B-Weyl's. Thus in any case $\lambda \in \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$.

Let $f \in Hol(\sigma(T))$. Since T is a-isoloid, then it follows from Theorem 3.9 that $\sigma_{SBF_+^-}(f(T)) = f(\sigma_{SBF_+^-}(T)) = f(\sigma_a(T) \setminus E^a(T)) = \sigma_a(f(T)) \setminus E^a(f(T))$. Thus generalized a-Weyl's theorem holds for $f(T)$. $\square$

Corollary 3.12. *Let $T^* \in \mathbf{B}(\mathcal{H})$ be an algebraically (p,k)-quasihyponormal. Then $E^a(T) = \pi^a(T)$.*

Proof. If T^* is an algebraically (p,k)-quasihyponormal operator, then $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E^a(T)$. Let $\lambda \in E^a(T)$. Then λ is isolated in $\sigma_a(T)$, and $\lambda \notin \sigma_{SBF_+^-}(T)$. So $T - \lambda I$ is in $SBF_+^-(\mathcal{H})$. It follows from [11, Theorem 2.8] that λ is a left pole of T, and so $\lambda \in \pi^a(T)$. As we have always $\pi^a(T) \subset E^a(T)$, then $E^a(T) = \pi^a(T)$. $\square$

Definition 3.13. Let $T \in \mathbf{B}(\mathcal{H})$ and let $k \in \mathbb{N}$. Then T has a uniform descent for $n \geq k$ if $\mathcal{R}(T) + \ker(T^n) = \mathcal{R}(T) + \ker(T^k)$ for all $n \geq k$. If, in addition,

$\mathcal{R}(T) + \ker(T^k)$ is closed, then T is said to have a topological uniform descent for $n \geq k$.

An operator $T \in \mathbf{B}(\mathcal{H})$ is called a-polaroid if $iso\sigma_a(T) \subset \pi^a(T)$. In general, if T is a-polaroid, then it is polaroid. However, the converse is not true. Consider the following example.

Example 3.14. Let R be the unilateral right shift on $\ell^2(\mathbb{N})$ and define

$$U(x_1, x_2, \ldots) := (0, x_2, x_3, \ldots) \quad \text{for all } x_n \in \ell^2(\mathbb{N}).$$

Clearly, U is a quasi-nilpotent operator. Let $T := R \oplus U$. We have $\sigma(T) = \mathbf{D}$, $\mathbf{D}$ is the unit disc of $\mathbb{C}$, so $iso(\sigma(T)) = E_0(T) = \emptyset$ and hence T is polaroid. Moreover, $\sigma_a(T) = \partial\mathbf{D} \cup \{0\}$. Since $\sigma_a(T)$ does not cluster at 0, then T has the SVEP at 0, as well as at the points $\lambda \notin \sigma_a(T)$. Since T has SVEP at all points $\lambda \in \partial\sigma(T)$ it then follows that T has SVEP. Finally, $\sigma_{SBF_+^-}(T) = \partial\sigma(T)$ so $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = \{0\}$. Hence T is not a-polaroid.

Theorem 3.15. *Let $T^* \in \mathbf{B}(\mathcal{H})$ be an algebraically (p, k)-quasihyponormal operator. Then T is a-polaroid.*

Proof. Suppose T^* is algebraically (p, k)-quasihyponormal. Since T^* has the SVEP, then $\sigma_a(T) = \sigma(T)$. Let $\lambda \in iso(\sigma_a(T)) = iso(\sigma(T))$. Since a-Weyl's theorem holds for T by Theorem 3.6, then λ is a left pole of finite rank of T. Therefore $T - \lambda I$ has a finite ascent $k = a(T - \lambda I)$ and $\mathcal{R}(T - \lambda I)^{k+1}$ is closed. Since $T - \lambda I$ is also an operator of topological uniform descent for $n \geq 0$, then it follows from [9, Lemma 2.8] that $T - \lambda I$ is injective. So $a(T - \lambda I) = 0$ and $\mathcal{R}(T - \lambda I)$ is closed. Since $\pi^a(T) = E^a(T)$, we see that λ is a left pole of T. That is, all isolated points of the approximate point spectrum of T are left poles of the resolvent of T. $\square$

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