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ON LEFT α-MULTIPLIERS AND COMMUTATIVITY OF SEMIPRIME RINGS

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Abstract. Let R be a ring, and α be an endomorphism of R. An additive mapping $H : R \longrightarrow R$ is called a left α-multiplier (centralizer) if $H(xy) = H(x)\alpha(y)$ holds for all $x, y \in R$. In this paper, we shall investigate the commutativity of prime and semiprime rings admitting left α-multipliers satisfying any one of the properties: (i) $H([x, y])-[x, y] = 0$, (ii) $H([x, y])+[x, y] = 0$, (iii) $H(x \circ y) - x \circ y = 0$, (iv) $H(x \circ y) + x \circ y = 0$, (v) $H(xy) = xy$, (vi) $H(xy) = yx$, (vii) $H(x^2) = x^2$, (viii) $H(x^2) = -x^2$ for all x, y in some appropriate subset of R.

1. Introduction

This research has been motivated by the works of S. Ali, C. Haetinger [2] and M. Ashraf, S. Ali [4]. Throughout the present paper R will represent an associative ring with center $Z(R)$. For any $x, y \in R$, the symbol $[x, y]$ and $x \circ y$ stand for the commutator $xy - yx$ and anti-commutator $xy + yx$, respectively. A ring R is called n-torsion free, where $n > 1$ is an integer, if whenever $nx = 0$, with $x \in R$, then $x = 0$. Recall that a ring R is prime if for any $a, b \in R$, $aRb = \{0\}$ implies $a = 0$ or $b = 0$ and is semiprime if for any $a \in R$, $aRa = \{0\}$ implies $a = 0$. If S is a subset of R, then we can define the left (resp. right) annihilator of S as $l(S) = \{x \in R \mid xs = 0 \text{ for all } s \in S\}$ (resp. $r(S) = \{x \in R \mid sx = 0 \text{ for all } s \in S\}$). It is well-known that if I is an ideal of a semiprime ring R, then $l(I) = r(I)$. An additive mapping $d : R \longrightarrow R$ is called a derivation if $d(xy) = d(x)y+xd(y)$ holds for all $x, y \in R$. Following [13], an additive mapping $H : R \longrightarrow R$ is called a left (resp. right) multiplier (centralizer) of R if $H(xy) = H(x)y$ (resp. $H(xy) = xH(y)$) holds for all $x, y \in R$. If H is both left as well as a right multiplier, then it is called a multiplier. The concept of generalized derivation has been introduced by M. Bresar [6], an additive mapping $F : R \longrightarrow R$ is called a generalized derivation if there exists a derivation $d : R \longrightarrow R$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$, and d is called the associated derivation of F. Obviously,

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generalized derivation with $d = 0$ covers the concept of left multipliers. It is easy to see that $F : R \longrightarrow R$ is a generalized derivation if and only if F is of the form $F = d + H$, where d is a derivation and H is a left multiplier.

Recently, E. Albas [1] introduced the notion of α-multipliers (centralizers) of R, i.e., an additive mapping $H : R \longrightarrow R$ is called a left (resp. right) α-multiplier(centralizer) of R if $H(xy) = H(x)\alpha(y)$ (resp. $H(xy) = \alpha(x)H(y)$) holds for all $x, y \in R$, where α is an endomorphism of R. If H is both left as well as right α-centralizer, then it is natural to call H an α-multiplier. It is clear that for an additive mapping $H : R \longrightarrow R$ associated with a homomorphism $\alpha : R \longrightarrow R$, if $L_a(x) = a\alpha(x)$ and $R_a(x) = \alpha(x)a$ for a fixed element $a \in R$ and for all $x \in R$, then L_a is a left α-centralizer and R_a is a right α-multiplier. Clearly every multiplier is a special case of an α-multiplier with $\alpha = id_R$, the identity map on R. Following [2], suppose $H : R \longrightarrow R$ is an additive mapping and α is an endomorphism of R, if $H(x^2) = H(x)\alpha(x)$ (resp. $H(x^2) = \alpha(x)H(x)$) holds for all $x \in R$, then H is called a Jordan left (resp. right) α-multiplier. Obviously every left (resp. right) α-multiplier is a Jordan left (resp. right) α-multiplier. The converse in general is not true (see [2], Example 2.1). In [2], S. Ali and C. Haetinger proved that every Jordan left (resp. right) α-multiplier on a 2-torsion-free semiprime ring is a left (resp. right) α-multiplier, where α is an automorphism of R. Considerable work has been done on this topic during the last couple of decades (cf., [1, 4, 7, 8, 11, 12, 13] where further references can be found).

In [10], M. A. Quadri et al. established that a prime ring R must be commutative if it admits a generalized derivation F associated with a nonzero derivation d such that $F([x, y]) = [x, y]$ (resp. $F([x, y]) + [x, y] = 0$) hold for all $x, y \in I$. In this direction, it seems natural to ask what we can say about the commutativity of a prime R if the generalized derivation F in the above conditions is replaced by a left multiplier. In the year 2008, the first author together with M. Ashraf [4] have investigated this problem for certain situations involving left multipliers. The purpose of this paper is to extend above mentioned results for semiprime rings admitting left α-multipliers. Some related results have also been discussed in the setting of left α-multipliers. We shall make extensive use of the following basic commutator identities throughout the discussion without any specific mention:

$$[xy, z] = x[y, z] + [x, z]y \text{ and } [x, yz] = y[x, z] + [x, y]z,$$

$$x \circ (yz) = (x \circ y)z - y[x, z] = y(x \circ z) + [x, y]z,$$

$$(xy) \circ z = x(y \circ z) - [x, z]y = (x \circ z)y + x[y, z].$$

We shall restrict our attention to left α-multipliers (centralizers) since all results presented in this paper are also true for right α-multipliers because of left and right symmetry.

2. Main results

In [4, Theorems 2.1 and 2.2], the first author together with M. Ashraf proved that if a prime ring R admits a nonzero left multiplier H with $H(x) \neq x$ for all x in a nonzero ideal I of R such that $H([x,y]) = [x,y]$ for all $x, y \in I$ or $H([x,y]) + [x,y] = 0$ for all $x, y \in I$, then R is commutative. In the following theorem, we extend Theorems 2.1 and 2.2 of [4] for semiprime rings.

Theorem 2.1. *Let R be a semiprime ring and I a nonzero ideal of R. Suppose that α is an endomorphism of R and H is a left α-multiplier such that α is not the identity map on I. If one of the following conditions holds*:

(i) $H([x,y]) - [x,y] = 0$ *for all* $x, y \in I$,

(ii) $H([x,y]) + [x,y] = 0$ *for all* $x, y \in I$,

(iii) *for all* $x, y \in I$, *either* $H([x,y]) - [x,y] = 0$ *or* $H([x,y]) + [x,y] = 0$, *then* $I \subseteq Z(R)$.

Proof. (i) By the given hypothesis we have

$$H([x,y]) = [x,y] \quad \text{for all } x, y \in I. \tag{2.1}$$

If $H = 0$, then $[x,y] = 0$ for all $x, y \in I$. Replacing y by yr in this relation, we get $0 = [x,yr] = [x,y]r + y[x,r] = y[x,r]$ for all $x, y \in I$ and $r \in R$. In particular, both $yst[x,r] = 0$ and $syt[x,r] = 0$ for all $x, y \in I$ and $r, s, t \in R$. Combining the last two relations, we arrive at $[y,s]R[x,r] = \{0\}$ for all $x, y \in I$ and $r, s \in R$. In particular, $[x,r]R[x,r] = \{0\}$ for all $x \in I$ and $r \in R$. The semiprimeness of R forces that $[x,r] = 0$, hence $I \subseteq Z(R)$.

Now we assume that $H \neq 0$, replacing y by yx in (2.1) we find that $H([x,yx]) = [x,yx]$ for all $x, y \in I$. This can be rewritten as $H([x,y]x) = [x,y]x$, which implies that

$$H([x,y])\alpha(x) = [x,y]x \quad \text{for all } x, y \in I. \tag{2.2}$$

On combining (2.1) and (2.2), we obtain that

$$[x,y](\alpha(x) - x) = 0 \quad \text{for all } x, y \in I. \tag{2.3}$$

Replacing y by yz in (2.3), we get

$$[x,y]z(\alpha(x) - x) = 0 \quad \text{for all } x, y, z \in I. \tag{2.4}$$

That is

$$[x,y]I(\alpha(x) - x) = \{0\} \quad \text{for all } x, y \in I.$$

Since I is an ideal of R, we have $IR \subseteq I$ and hence

$$[x,y]IR(\alpha(x) - x) = \{0\} \quad \text{for all } x, y \in I. \tag{2.5}$$

Since R is semiprime, it must contain a family $\mathcal{P} = \{P_\alpha : \alpha \in \wedge\}$ of prime ideals such that $\bigcap_\alpha P_\alpha = (0)$ (see [3] for more details). Let P denote a fixed one of the P_α. Then from (2.5) it follows that for each $x \in I$, either

(a) $\alpha(x) - x \in P$; or

(b) $[x,y]I \subseteq P$ for all $y \in I$.

Define I_a to be the set of $x \in I$ for which (a) holds and I_b the set of $x \in I$ for which (b) holds. Note that both are additive subgroups of I and their union is equal to I. Thus either $I_a = I$ or $I_b = I$, and hence P satisfies one of the following:

$(a)^{'}$ $\alpha(x) - x \in P$ for all $x \in I$;
$(b)^{'}$ $[x, y]I \subseteq P$ for all $x, y \in I$.

Call a prime ideal in $\mathcal{P}$ a type-one prime if it satisfies $(a)^{'}$, and call all other members of $\mathcal{P}$ type-two primes. Define P_1 and P_2 respectively as the intersection of all type-one primes and the intersection of all type-two primes, and note that

$$P_1P_2 \;=\; P_2P_1 \;=\; P_1 \cap P_2 = \{0\}. \tag{2.6}$$

Now, suppose that for all $x \in I,\ \alpha(x) - x \in P$. This means that $\alpha(x) - x \in \bigcap_\alpha P_\alpha = \{0\}$, i.e., $\alpha(x) = x$ for all $x \in I$, a contradiction. Therefore, there exists an element $x \in I$ such that $\alpha(x) - x \notin P$, which implies that $[x, y]I \subseteq P$ for all $y \in I$. Let z be any element of I. On the one hand, if $\alpha(z+x)-(z+x) \notin P$, then $[z + x, y]I \subseteq P$ and hence $[z, y]I \subseteq P$ because of $[x, y]I \subseteq P$. On the other hand, if $\alpha(z + x) - (z + x) \in P$, then $\alpha(z) - z \notin P$ since $\alpha(x) - x \notin P$, and hence $[z, y]I \subseteq P$. Thus we conclude that $[z, y]I \subseteq P$ for all $y, z \in I$, hence $[z, y]I \subseteq \bigcap_\alpha P_\alpha = \{0\}$. This implies that $[z, y] \in l(I)$. Since I is an ideal of R, it is clear that $[z, y] \in I$ for all $y, z \in I$. Therefore, $[z, y] \in l(I) \bigcap I = \{0\}$, otherwise $l(I) \bigcap I$ is a nonzero ideal of R and $(l(I) \bigcap I)^2 \subseteq l(I)I = \{0\}$ contradicting the fact R having no nonzero nilpotent ideals. Therefore, the arguments in the beginning yields that $I \subseteq Z(R)$.

(ii) If H is a left α-multiplier satisfying the property $H([x, y])+[x, y] = 0$ for all $x, y \in I$, then the left α-multiplier $-H$ satisfies the condition: $(-H)([x, y])- [x, y] = 0$ for all $x, y \in I$, and hence $I \subseteq Z(R)$ by (i).

(iii) For each fixed $x \in I$, we set $I_x = \{y \in I \mid H([x, y]) - [x, y] = 0\}$ and ${I^*}_x = \{y \in I \mid H([x, y]) + [x, y] = 0\}$. Then, I_x and ${I^*}_x$ are both additive subgroups of I such that $I = I_x \cup {I^*}_x$. But a group can not be a union of two its proper subgroups, so we have either $I_x = I$ or ${I^*}_x = I$. Further, using similar arguments, we obtain $I = \{x \in I \mid I_x = I\}$ or $I = \{x \in I \mid {I^*}_x = I\}$. Now, we apply (i) and (ii). This completes the proof of the theorem. □

Theorem 2.2. *Let R be a semiprime ring and I a nonzero ideal of R. Suppose that α is an endomorphism of R and H is a left α-multiplier such that α is not the identity map on I. If one of the following conditions holds*:

(i) $H(xy) = xy$ *for all* $x, y \in I$,
(ii) $H(xy) = -xy$ *for all* $x, y \in I$,
(iii) *for all* $x, y \in I$, *either* $H(xy) = xy$ *or* $H(xy) = -xy$,

then $I \subseteq Z(R)$.

Proof. (i) Assume that $H(xy) = xy$ for all $x, y \in I$. Then we have $H(xy-yx) = H(xy)-H(yx) = xy-yx$ for all $x, y \in I$. This implies that $H([x, y])-[x, y] = 0$ for all $x, y \in I$, and hence $I \subseteq Z(R)$ by Theorem 2.1(i).

(ii) and (iii) can be proved by using similar arguments in (i). □

Using similar techniques as used in proof of above theorem, we can prove the following:

Theorem 2.3. *Let R be a semiprime ring and I a nonzero ideal of R. Suppose that α is an endomorphism of R and H is a left α-multiplier such that α is not the identity map on I. If one of the following conditions holds*:

(i) *$H(xy) = yx$ for all $x, y \in I$,*
(ii) *$H(xy) = -yx$ for all $x, y \in I$,*
(iii) *for all $x, y \in I$, either $H(xy) = xy$ or $H(xy) = -yx$,*

then $I \subseteq Z(R)$.

As immediate consequences of the above theorems we have the following corollaries.

Corollary 2.1. *Let R be a prime ring and I a nonzero ideal of R. Suppose that α is an endomorphism of R and H is a left α-multiplier such that α is not the identity map on I. If one of the following conditions holds*:

(i) *$H([x, y]) - [x, y] = 0$ for all $x, y \in I$,*
(ii) *$H([x, y]) + [x, y] = 0$ for all $x, y \in I$,*
(iii) *for all $x, y \in I$, either $H([x, y]) - [x, y] = 0$ or $H([x, y]) + [x, y] = 0$,*

then R is commutative.

Corollary 2.2. *Let R be a prime ring and I a nonzero ideal of R. Suppose that α is an endomorphism of R and H is a left α-multiplier such that α is not the identity map on I. If one of the following conditions holds*:

(i) *$H(xy) = xy$ for all $x, y \in I$,*
(ii) *$H(xy) = -xy$ for all $x, y \in I$,*
(iii) *for all $x, y \in I$, either $H(xy) = xy$ or $H(xy) = -xy$,*

then R is commutative.

Corollary 2.3. *Let R be a prime ring and I a nonzero ideal of R. Suppose that α is an endomorphism of R and H is a left α-multiplier such that α is not the identity map on I. If one of the following conditions holds*:

(i) *$H(xy) = yx$ for all $x, y \in I$,*
(ii) *$H(xy) = -yx$ for all $x, y \in I$,*
(iii) *for all $x, y \in I$, either $H(xy) = yx$ or $H(xy) = -yx$,*

then R is commutative.

The following theorem is motivated by [4, Theorems 2.3 and 2.4].

Theorem 2.4. *Let R be a semiprime ring and I a nonzero ideal of R. Suppose that α is an endomorphism of R and H is a left α-multiplier such that α is not the identity map on I. If one of the following conditions holds*:

(i) $H(x \circ y) - x \circ y = 0$ *for all* $x, y \in I$,

(ii) $H(x \circ y) + x \circ y = 0$ *for all* $x, y \in I$,

(iii) *for all* $x, y \in I$, *either* $H(x \circ y) - x \circ y = 0$ *or* $H(x \circ y) + x \circ y = 0$,

then $I \subseteq Z(R)$.

Proof. (i) By the assumption, we have

$$H(x \circ y) = x \circ y \text{ for all } x, y \in I. \tag{2.7}$$

If $H = 0$, then $x \circ y = 0$ for all $x, y \in I$. Replacing y by yz, we obtain $0 = x \circ (yz) = y(x \circ z) + [x, y]z = [x, y]z$ for all $x, y, z \in I$. It follows that $[x, y]I = \{0\}$ for all $x, y \in I$, and hence $[x, y] \in l(I) \bigcap I = \{0\}$. Using the same arguments as used in the beginning of Theorem 2.1(i), we get $I \subseteq Z(R)$.

Hence, onward we assume that $H \neq 0$. Replacing y by yx in (2.7) we get $H(x \circ (yx)) = x \circ (yx)$, which implies that $H((x \circ y)x) = (x \circ y)x$ for all $x, y \in I$. This can be rewritten as $H(x \circ y)\alpha(x) = (x \circ y)x$ for all $x, y \in I$. Application of (2.7) yields that

$$(x \circ y)(\alpha(x) - x) = 0 \text{ for all } x, y \in I. \tag{2.8}$$

Replacing y by yz in (2.8) we have $(y(x \circ z) + [x, y]z)(\alpha(x) - x) = 0$ for all $x, y, z \in I$. This reduces to

$$[x, y]z(\alpha(x) - x) = 0 \text{ for all } x, y, z \in I. \tag{2.9}$$

Thus, the equation (2.9) is same as (2.4) and henceforth the proof follows by the last paragraph of the proof of Theorem 2.1.

(ii) and (iii) can be proved by using the same technique as used in Theorem 2.1(ii) and (iii). □

Theorem 2.5. *Let R be a semiprime ring and I a nonzero ideal of R. Suppose that α is an endomorphism of R and H is a left α-multiplier such that α is not the identity map on I. If one of the following conditions holds*:

(i) $H(x^2) = x^2$ *for all* $x \in I$,

(ii) $H(x^2) = -x^2$ *for all* $x \in I$,

then $I \subseteq Z(R)$.

Proof. (i) We are given that $H(x^2) = x^2$ for all $x \in I$. For all $x, y \in I$, we have $H((x + y)^2) = (x + y)^2$, which implies that $H(x \circ y) = x \circ y$. Hence $I \subseteq Z(R)$ by Theorem 2.4(i).

Using similar arguments (ii) can be proved. □

Corollary 2.4. *Let R be a prime ring and I a nonzero ideal of R. Suppose that α is an endomorphism of R and H is a left α-multiplier such that α is not the identity map on I. If one of the following conditions holds*:

(i) $H(x \circ y) - x \circ y = 0$ *for all* $x, y \in I$,

(ii) $H(x \circ y) + x \circ y = 0$ *for all* $x, y \in I$,
(iii) *for all* $x, y \in I$, *either* $H(x \circ y) - x \circ y = 0$ *or* $H(x \circ y) + x \circ y = 0$,

then R *is commutative.*

Corollary 2.5. *Let* R *be a prime ring and* I *a nonzero ideal of* R. *Suppose that* α *is an endomorphism of* R *and* H *is a left* α*-multiplier such that* α *is not the identity map on* I. *If one of the following conditions holds*:
(i) $H(x^2) = x^2$ *for all* $x \in I$,
(ii) $H(x^2) = -x^2$ *for all* $x \in I$,

then R *is commutative.*

The following example shows that the above results are not true in the case of arbitrary rings.

Example 2.1. Suppose that S is any ring. Next, let $R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} | a, b, c \in S \right\}$ and $I = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} | a, b \in S \right\}$ be an ideal of R. Define maps $H, \alpha : R \longrightarrow R$ such that $H \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $\alpha \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -a & b \\ 0 & 0 & -c \\ 0 & 0 & 0 \end{pmatrix}$. Then, it is straightforward to see that H is a left α-multiplier of R, and satisfies: (i) $H([x,y]) - [x,y] = 0$, (ii) $H([x,y]) + [x,y] = 0$, (iii) $H(x \circ y) - x \circ y = 0$, (iv) $H(x \circ y) + x \circ y = 0$, (v) $H(xy) = xy$, (vi) $H(xy) = yx$, (vii) $H(x^2) = x^2$, (viii) $H(x^2) = -x^2$ for all $x, y \in I$, however, $I \nsubseteq Z(R)$.

Remark 2.1. In a 2-torsion-free semiprime ring, the above results are also true for Jordan left α-multiplier since every Jordan left α-multiplier is a left α-multiplier [2, Theorem 2.2].

Remark 2.2. Replacing the ideal I by a square closed Lie ideal U in above Theorems 2.1 and 2.5, it is interesting to find additional conditions such that $U \subseteq Z(R)$.

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