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ON COMMUTING GRAPHS OF GROUP RING Z_nQ_8

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Abstract. The commuting graph of an arbitrary ring R, denoted by $\Gamma(R)$, is a graph whose vertices are all non-central elements of R, and two distinct vertices a and b are adjacent if and only if $ab = ba$. In this paper, we investigate the connectivity, the diameter, the maximum degree and the minimum degree of the commuting graph of group ring Z_nQ_8. The main result is that $\Gamma(Z_nQ_8)$ is connected if and only if n is not a prime. If $\Gamma(Z_nQ_8)$ is connected, then diam(Z_nQ_8)= 3, while $\Gamma(Z_nQ_8)$ is disconnected then every connected component of $\Gamma(Z_nQ_8)$ must be a complete graph with a same size. Further, we obtain the degree of every vertex in $\Gamma(Z_nQ_8)$, the maximum degree and the minimum degree of $\Gamma(Z_nQ_8)$.

1. Introduction

Let G be a group and R a ring. We denote RG by the set of all formal linear combinations of the forms $\alpha = \sum_{g\in G} a_g g$, where $a_g \in R$ and $a_g = 0$ almost everywhere, that is, only a finite number of coefficients are different from 0 in each of these sums. Notice that it follows from our definition that given two elements, $\alpha = \sum_{g\in G} a_g g$ and $\beta = \sum_{g\in G} b_g g \in RG$, we have that $\alpha = \beta$ if and only if $a_g = b_g$, $\forall\ g \in G$. We define the sum of two elements in RG componentwise:

$$\left(\sum_{g\in G} a_g g\right) + \left(\sum_{g\in G} b_g g\right) = \sum_{g\in G} (a_g + b_g)g.$$

Also, given two elements $\alpha = \sum_{g\in G} a_g g$ and $\beta = \sum_{h\in G} b_h h \in RG$ we define their product by

$$\alpha\beta = \sum_{g,\ h\in G} a_g b_h gh.$$

The commuting graph of an arbitrary ring R denoted by $\Gamma(R)$, is a graph with vertex set $R \setminus \mathcal{Z}(R)$, where $\mathcal{Z}(R)$ is the center of R, and two distinct

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vertices a and b are adjacent if and only if $ab = ba$. In 2004, the notion of commuting graph of a ring was first introduced by Akbari, Ghandehari, Hadian and Mohammadian in [2]. The commuting graphs of semisimple rings have been studied in [1, 2, 4, 3]. And in this paper, we investigate some properties of $\Gamma(Z_nQ_8)$, where $Z_nQ_8 = \{x_1+x_2a+x_3a^2+x_4a^3+x_5b+x_6ab+x_7a^2b+x_8a^3b \mid x_i \in Z_n, i = 1, 2, \ldots, 8\}$ and $Z_n = \{0, 1, \ldots, n-1\}$ is the module n residue class ring, $Q_8 = \langle a, b | a^4 = 1, b^2 = 1, ab = ba^{-1}\rangle = \{1, a, a^2, a^3, b, ab, a^2b, a^3b\}$ is the quaternion group.

Let R be a ring and $R^* = R\backslash\{0\}$. Given integers a and b, we denote by (a, b) the greatest common divisor of a and b. If p is a prime and t is a nonnegative integer, then we use the notation $p^t \| a$ to mean that $p^t | a$ and $p^{t+1} \nmid a$. The ring of n by n full matrices over a ring R is denoted by $M_n(R)$.

In this paper, all graphs are simple and undirected and $|G|$ denotes the number of vertices of the graph G. In a graph G, the degree of a vertex v is denoted by $d(v)$. And the minimum degree and maximum degree of G are denoted by $\delta(G)$ and $\triangle(G)$, respectively. A *path* of length r from a vertex x to another vertex y in G is a sequence of $r + 1$ distinct vertices starting with x and ending with y such that consecutive vertices are adjacent. For a connected graph H, the diameter of H is denoted by $\mathrm{diam}(H)$. An induced subgraph of G that is maximal, subject to being connected, is called a *connected component* of G.

In this paper, we investigate the connectivity, the diameter, the maximum degree and the minimum degree of the commuting graph of group ring Z_nQ_8. In Section 2, we show that $\Gamma(Z_nQ_8)$ is connected if and only if n is not a prime. If $\Gamma(Z_nQ_8)$ is connected, then $\mathrm{diam}(Z_nQ_8)= 3$, while $\Gamma(Z_nQ_8)$ is disconnected then every connected component of $\Gamma(Z_nQ_8)$ must be a complete graph with a same size. In Section 3, we obtain the degree of every vertex in $\Gamma(Z_nQ_8)$, the maximum degree and the minimum degree of $\Gamma(Z_nQ_8)$.

2. The connectivity and diameter of $\Gamma(Z_nQ_8)$

Lemma 2.1 ([2, Theorem 2]). *If F is a finite field, then $\Gamma(M_2(F))$ is a graph with $|F|^2 + |F| + 1$ connected components of size $|F|^2 - |F|$ which each of them is a complete graph.*

Lemma 2.2. *Let n be an arbitrary positive integer. Then $\mathcal{Z}(Z_nQ_8) = \{\alpha = x_1+x_2a+x_3a^2+x_2a^3+x_5b+x_6ab+x_5a^2b+x_6a^3b \mid x_1,\ x_2,\ x_3,\ x_5,\ x_6 \in Z_n\}$, $|\mathcal{Z}(Z_nQ_8)| = n^5$ and $|\Gamma(Z_nQ_8)| = n^8 - n^5$, where $\mathcal{Z}(Z_nQ_8)$ denotes the center of the group ring Z_nQ_8.*

Proof. $\forall\ \alpha = x_1 + x_2a + x_3a^2 + x_4a^3 + x_5b + x_6ab + x_7a^2b + x_8a^3b,\ \beta = y_1 + y_2a + y_3a^2 + y_4a^3 + y_5b + y_6ab + y_7a^2b + y_8a^3b \in \Gamma(Z_nQ_8)$, we have

$\alpha\beta = \beta\alpha$ if and only if the following system of congruence equations $(*)$ holds.

$$(*)\begin{cases} (x_6-x_8)y_5-(x_5-x_7)y_6-(x_6-x_8)y_7+(x_5-x_7)y_8 \equiv 0 \pmod n \ (1) \\ (x_6-x_8)y_2-(x_6-x_8)y_4-(x_2-x_4)y_6+(x_2-x_4)y_8 \equiv 0 \pmod n \ (2) \\ (x_5-x_7)y_2-(x_5-x_7)y_4-(x_2-x_4)y_5+(x_2-x_4)y_7 \equiv 0 \pmod n \ (3) \end{cases}$$

Suppose that $\alpha = x_1 + x_2a + x_3a^2 + x_4a^3 + x_5b + x_6ab + x_7a^2b + x_8a^3b \in \mathcal{Z}(Z_nQ_8)$, then it is clear that $a\alpha = \alpha a$. Thus by the system $(*)$, it follows that

$$\begin{cases} x_6 - x_8 \equiv 0 \pmod n \\ x_5 - x_7 \equiv 0 \pmod n \end{cases}$$

i.e., $x_6 \equiv x_8 \pmod n$, and $x_5 \equiv x_7 \pmod n$.

In addition, we also have $b\alpha = \alpha b$, hence we have that

$$\begin{cases} x_6 - x_8 \equiv 0 \pmod n \\ x_2 - x_4 \equiv 0 \pmod n \end{cases}$$

i.e., $x_6 \equiv x_8 \pmod n$, and $x_2 \equiv x_4 \pmod n$.

Therefore, we have $x_2 \equiv x_4 \pmod n$, $x_5 \equiv x_7 \pmod n$ and $x_6 \equiv x_8 \pmod n$. Hence, $\alpha = x_1 + x_2a + x_3a^2 + x_2a^3 + x_5b + x_6ab + x_5a^2b + x_6a^3b$ and it is easy to verify that such α is in the center of Z_nQ_8.

Thus $\mathcal{Z}(Z_nQ_8) = \{\alpha = x_1+x_2a+x_3a^2+x_2a^3+x_5b+x_6ab+x_5a^2b+x_6a^3b \mid x_1,\ x_2,\ x_3,\ x_5,\ x_6\ \in Z_n\}$ and $\mid \mathcal{Z}(Z_nQ_8) \mid= n^5$, $\mid \Gamma(Z_nQ_8) \mid= n^8 - n^5$. □

Theorem 2.3. *Suppose $n = p^t$, where $p \geq 2$ is a prime and $t \geq 2$. Then $\Gamma(Z_nQ_8)$ is a connected graph and* $\mathrm{diam}(\Gamma(Z_nQ_8)) = 3$.

Proof. For $\alpha, \beta \in \Gamma(Z_nQ_8)$, let $\alpha = x_1 + x_2a + x_3a^2 + x_4a^3 + x_5b + x_6ab + x_7a^2b + x_8a^3b$ and $\beta = y_1 + y_2a + y_3a^2 + y_4a^3 + y_5b + y_6ab + y_7a^2b + y_8a^3b$.

Case 1 Assume that $p^i \mid (x_2, x_4, x_5, x_6, x_7, x_8)$, $p^j \mid (y_2, y_4, y_5, y_6, y_7, y_8)$ for some $i, j \in \{1, 2, \dots, t-1\}$. Hence, if $i + j \geq t$, then $\alpha - \beta$ is an edge of $\Gamma(Z_nQ_8)$. Otherwise, $\alpha - p^{t-j}\alpha - \beta$ is a path of $\Gamma(Z_nQ_8)$.

Case 2 Assume that $p \nmid (x_2, x_4, x_5, x_6, x_7, x_8)$, $p \mid (y_2, y_4, y_5, y_6, y_7, y_8)$. We know $p^{t-1}\alpha \notin \mathcal{Z}(Z_nQ_8)$. Then $\alpha - p^{t-1}\alpha - \beta$ is a path of $\Gamma(Z_nQ_8)$.

Case 3 Assume that $p \mid (x_2, x_4, x_5, x_6, x_7, x_8)$, $p \nmid (y_2, y_4, y_5, y_6, y_7, y_8)$. We know $p^{t-1}\beta \notin \mathcal{Z}(Z_nQ_8)$. Then $\alpha - p^{t-1}\beta - \beta$ is a path of $\Gamma(Z_nQ_8)$.

Case 4 Assume that $p \nmid (x_2, x_4, x_5, x_6, x_7, x_8)$, $p \nmid (y_2, y_4, y_5, y_6, y_7, y_8)$, then $p^{t-1}\alpha,\ p^{t-1}\beta \notin \mathcal{Z}(Z_nQ_8)$. Then $\alpha - p^{t-1}\alpha - p^{t-1}\beta - \beta$ is a path of $\Gamma(Z_nQ_8)$.

Therefore, $\Gamma(Z_nQ_8)$ is a connected graph and $\mathrm{diam}(\Gamma(Z_nQ_8)) \leq 3$. In addition, note that $a, b \in \Gamma(Z_nQ_8)$, suppose $\gamma = z_1 + z_2a + z_3a^2 + z_4a^3 + z_5b + z_6ab + z_7a^2b + z_8a^3b \in \Gamma(Z_nQ_8)$ such that $a\gamma = \gamma a$ and $b\gamma = \gamma b$. Since $a\gamma = \gamma a \Longleftrightarrow z_6 \equiv z_8 \pmod{p^t}$ and $z_5 \equiv z_7 \pmod{p^t}$ while $b\gamma = \gamma b \Longleftrightarrow z_2 \equiv z_4 \pmod{p^t}$ and $z_6 \equiv z_8 \pmod{p^t}$, we have $z_2 \equiv z_4 \pmod{p^t}$, $z_5 \equiv z_7 \pmod{p^t}$ and $z_6 \equiv z_8 \pmod{p^t}$. By Lemma 2.2, we know $\gamma \in \mathcal{Z}(Z_nQ_8)$. Hence, there does not exist a vertex γ of $\Gamma(Z_nQ_8)$ such that $a - \gamma - b$ is a path of $\Gamma(Z_nQ_8)$. Hence, $\mathrm{diam}(\Gamma(Z_nQ_8))=3$. □

Lemma 2.4 ([7, Lemma 7.4.9])**.** *Let F be a field of characteristic different from 2. Then*

$$FQ_8 \cong F \oplus F \oplus F \oplus F \oplus H(F).$$

Lemma 2.5 ([7, Lemma 7.4.6])**.** *Assume that* char$(F) \neq 2$*. Then the quaternion algebra $H(F)$ is either a division ring or is isomorphic to $M_2(F)$, the ring of 2×2 matrices over F. The last possibility arises if and only if the equation $X^2 + Y^2 = -1$ can be solved in F.*

Theorem 2.6. *Let $p \geq 3$ be a prime. Then $Z_pQ_8 \cong Z_p \oplus Z_p \oplus Z_p \oplus Z_p \oplus M_2(Z_p)$.*

Proof. We know that the equation $X^2 + Y^2 = -1$ can always be solved in Z_p. Owing to Lemma 2.4 and Lemma 2.5, the result follows. □

Theorem 2.7. *If $p \geq 3$ is a prime, then $\Gamma(Z_pQ_8)$ is a graph with $p^2 + p + 1$ connected components of size $p^4(p^2 - p)$ which each of them is a complete graph.*

Proof. By Lemma 2.4, we have $Z_pQ_8 \cong Z_p \oplus Z_p \oplus Z_p \oplus Z_p \oplus M_2(Z_p)$. For $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$, $\beta = (\beta_1, \beta_2, \beta_3, \beta_4, \beta_5) \in \Gamma(Z_pQ_8)$, $\alpha_i, \beta_i \in Z_p$, $i = 1, 2, 3, 4$, and $\alpha_5, \beta_5 \in M_2(Z_p)$, we can easily conclude that $\alpha_5 \neq 0$, $\beta_5 \neq 0$. If α_5 and β_5 are not in the same connected component of $M_2(Z_p)$, then there is no edge between α and β. By Lemma 2.1, we know that $\Gamma(M_2(Z_p))$ is a graph with $p^2 + p + 1$ connected components of size $p^2 - p$ which each of them is a complete graph. Hence, $\Gamma(Z_pQ_8)$ is a graph with $p^2 + p + 1$ connected components of size $p^4(p^2 - p)$ which each of them is a complete graph. □

Theorem 2.8. *$\Gamma(Z_2Q_8)$ is a graph with 7 connected components of size 32 which each of them is a complete graph.*

Proof. First, we construct 7 subsets of $\Gamma(Z_2Q_8)$ as below:

$A_1 = \{\alpha = x_1 + x_2a + x_3a^2 + x_4a^3 + x_5b + x_6ab + x_7a^2b + x_8a^3b \in \Gamma(Z_2Q_8) \mid x_2 \equiv x_4 \pmod 2, x_5 \equiv x_7 \pmod 2,\ x_i \in Z_2\}$.

$A_2 = \{\alpha = x_1 + x_2a + x_3a^2 + x_4a^3 + x_5b + x_6ab + x_7a^2b + x_8a^3b \in \Gamma(Z_2Q_8) \mid x_2 \equiv x_4 \pmod 2,\ x_6 \equiv x_8 \pmod 2,\ x_i \in Z_2\}$.

$A_3 = \{\alpha = x_1 + x_2a + x_3a^2 + x_4a^3 + x_5b + x_6ab + x_7a^2b + x_8a^3b \in \Gamma(Z_2Q_8) \mid x_5 \equiv x_7 \pmod 2,\ x_6 \equiv x_8 \pmod 2,\ x_i \in Z_2\}$.

$A_4 = \{\alpha = x_1 + x_2a + x_3a^2 + x_4a^3 + x_5b + x_6ab + x_7a^2b + x_8a^3b \in \Gamma(Z_2Q_8) \mid x_2 \equiv x_4 \pmod 2,\ x_5 + x_6 + x_7 + x_8 \equiv 0 \pmod 2,\ x_i \in Z_2\}$.

$A_5 = \{\alpha = x_1 + x_2a + x_3a^2 + x_4a^3 + x_5b + x_6ab + x_7a^2b + x_8a^3b \in \Gamma(Z_2Q_8) \mid x_5 \equiv x_7 \pmod 2,\ x_2 + x_4 + x_6 + x_8 \equiv 0 \pmod 2,\ x_i \in Z_2\}$.

$A_6 = \{\alpha = x_1 + x_2a + x_3a^2 + x_4a^3 + x_5b + x_6ab + x_7a^2b + x_8a^3b \in \Gamma(Z_2Q_8) \mid x_6 \equiv x_8 \pmod 2,\ x_2 + x_4 + x_5 + x_7 \equiv 0 \pmod 2,\ x_i \in Z_2\}$.

$A_7 = \{\alpha = x_1 + x_2a + x_3a^2 + x_4a^3 + x_5b + x_6ab + x_7a^2b + x_8a^3b \in \Gamma(Z_2Q_8) \mid x_5 + x_6 + x_7 + x_8 \equiv 0 \pmod 2,\ x_2 + x_4 + x_6 + x_8 \equiv 0 \pmod 2,\ x_2 + x_4 + x_5 + x_7 \equiv 0 \pmod 2,\ x_i \in Z_2\}$.

Clearly, $A_1 \cup A_2 \cup \cdots \cup A_7 = Z_2Q_8 \setminus \mathcal{Z}(Z_2Q_8)$, $A_i \cap A_j = \emptyset$, $\forall\ i \neq j$ and $\mid A_1 \mid = \mid A_2 \mid = \cdots = \mid A_7 \mid = 32$.

Second, $\forall\ \alpha = x_1 + x_2a + x_3a^2 + x_4a^3 + x_5b + x_6ab + x_7a^2b + x_8a^3b \in A_i$ and $\forall\ \beta = y_1 + y_2a + y_3a^2 + y_4a^3 + y_5b + y_6ab + y_7a^2b + y_8a^3b \in \Gamma(Z_2Q_8)$, we can conclude that $\alpha\beta = \beta\alpha \Longleftrightarrow \beta \in A_i$. Moreover, we can conclude that each connected component A_i $(i = 1, 2, \ldots, 7)$ is a complete graph. This completes our proof. □

Lemma 2.9 ([9, Proportion 8.1.20])**.** *Let R be a commutative Noetherian ring and let G be an arbitrary group. Then there exist finitely many indecomposable rings $R_1, R_2, \ldots, R_n$ such that $RG \cong R_1G \times R_2G \times \cdots \times R_nG$. In particular, $\mathscr{U}(RG) \cong \mathscr{U}(R_1G) \times \mathscr{U}(R_2G) \times \cdots \times \mathscr{U}(R_nG)$.*

Theorem 2.10. *Let p be a prime. Then $\Gamma(Z_{2p}Q_8)$ is a connected graph and* $\mathrm{diam}(\Gamma(Z_{2p}Q_8)) = 3$.

Proof. (1) If $p = 2$, by Theorem 2.3, the result follows.

(2) If $p \geq 3$, by Lemma 2.6 and Lemma 2.9, we have $Z_{2p}Q_8 \cong Z_2Q_8 \oplus Z_p \oplus Z_p \oplus Z_p \oplus Z_p \oplus M_2(Z_p)$. Then $\forall\ \alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) \in Z_{2p}Q_8$ and $\beta = (\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6) \in Z_{2p}Q_8$, where $\alpha_1, \beta_1 \in Z_2Q_8, \alpha_2, \beta_2, \alpha_3, \beta_3, \alpha_4, \beta_4, \alpha_5, \beta_5 \in Z_p, \alpha_6, \beta_6 \in M_2(Z_p)$. By symmetry, we have the following cases to consider.

First, let $A_1, A_2, \ldots, A_7$ are the sets of vertices of the connected components of $\Gamma(Z_2Q_8)$. By Lemma 2.1, we know that there are $p^2 + p + 1$ connected components in $\Gamma(M_2(Z_p))$ and we denotes them as B_i, $i = 1, 2, \ldots, p^2 + p + 1$.

Case 1 Assume that $\alpha_1 \in \mathcal{Z}(Z_2Q_8),\ \beta_1 \in \Gamma(Z_2Q_8),\ \alpha_6 \in \Gamma(M_2(Z_p)),\ \beta_6 \in \mathcal{Z}(M_2(Z_p))$, then α-β is an edge of $\Gamma(Z_{2p}Q_8)$.

Case 2 Assume that $\alpha_1,\ \beta_1 \in \mathcal{Z}(Z_2Q_8),\ \alpha_6,\ \beta_6 \in \Gamma(M_2(Z_p))$. If $\alpha_6,\ \beta_6 \in B_i$ for some i, then α-β is an edge of $\Gamma(Z_{2p}Q_8)$. Otherwise, $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6)$-$(0, 0, 0, 0, 0, \alpha_6')$-$(\beta_1, 0, 0, 0, 0, 0)$-$(\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6)$ is a path of $\Gamma(Z_{2p}Q_8)$, where $\alpha_6,\ \alpha_6' \in B_i$.

Case 3 Assume that $\alpha_1 \in \mathcal{Z}(Z_2Q_8),\ \beta_1 \in \Gamma(Z_2Q_8),\ \alpha_6,\ \beta_6 \in \Gamma(M_2(Z_p))$. By similar argument above, we have the same results.

Case 4 Let $\alpha_1,\ \beta_1 \in \Gamma(Z_2Q_8),\ \alpha_6,\ \beta_6 \in \Gamma(M_2(Z_p))$.

Subcase 4.1 Suppose that $\alpha_1,\ \beta_1 \in A_i,\ \alpha_6,\ \beta_6 \in B_j$ for some $i,\ j$, then $\alpha - \beta$ is an edge of $\Gamma(Z_{2p}Q_8)$.

Subcase 4.2 Suppose that $\alpha_1, \beta_1 \in A_i,\ \alpha_6 \in B_j, \beta_6 \in B_k$ for some i, j, k, $j \neq k$, then $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6)$-$(\alpha_1, 0, 0, 0, 0, 0)$-$(\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6)$ is a path of $\Gamma(Z_{2p}Q_8)$.

Subcase 4.3 Suppose that $\alpha_1 \in A_i, \beta_1 \in A_j,\ \alpha_6 \in B_t, \beta_6 \in B_k$ for some i, j, k, t and $i \neq j,\ t \neq k$, then $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6)$-$(\alpha_1', 0, 0, 0, 0, 0)$-$(0, 0, 0, 0, 0, \beta_6')$-$(\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6)$ is a path of $\Gamma(Z_{2p}Q_8)$, where $\alpha_1' \in A_i,\ \beta_6' \in B_k$.

Therefore, $\Gamma(Z_{2p}Q_8)$ is a connected graph and $\mathrm{diam}(\Gamma(Z_{2p}Q_8))= 3$. □

Theorem 2.11. *If $n(> 1)$ is not a prime, then $\Gamma(Z_nQ_8)$ is a connected graph and* $\mathrm{diam}(\Gamma(Z_nQ_8)) = 3$.

Proof. Let $n = p_1^{t_1}p_2^{t_2}\cdots p_m^{t_m}$ with $m \geq 2$ and $t_1, t_2, \ldots, t_m \geq 1$, $p_1, p_2, \ldots, p_m$ are distinct primes and $2 \leq p_1 \leq p_2 \leq \cdots \leq p_m$.

(1) When $m = 1,\ n = p_1^{t_1},\ t_1 > 1$, by Theorem 2.3, the result follows.

(2) If $n = 2p,\ p$ is a prime, by Theorem 2.10, the result follows.

(3) We suppose $m > 1, n \neq 2p$. Let R_i denotes $Z_{p_i^{t_i}}Q_8$, then by Lemma 2.9, we have $Z_nQ_8 \cong R_1 \oplus R_2 \oplus \cdots \oplus R_m \triangleq R$. Note that $\forall\ \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in R, \alpha \in \mathcal{Z}(R)$ if and only if $\alpha_i \in \mathcal{Z}(R_i)$, $\forall i = 1, 2, \ldots, m$. So $\forall\ \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in \Gamma(R), \beta = (\beta_1, \beta_2, \ldots, \beta_m) \in \Gamma(R)$, we should consider the following three cases:

Case 1 Assume that $\forall\ \ i = 1, 2, \ldots, m,\ \alpha_i \in \mathcal{Z}(R_i)$ or $\beta_i \in \mathcal{Z}(R_i)$, then $\alpha - \beta$ is an edge of $\Gamma(R)$.

Case 2 Assume that there exists $i \in \{1, 2, \ldots, m\}$ such that $\alpha_i \in \mathcal{Z}(R_i)$ or $\beta_i \in \mathcal{Z}(R_i)$. Without loss of generality, we can assume that $\alpha_i \in \mathcal{Z}(R_i)$, and take $\gamma_i \in R_i \setminus \mathcal{Z}(R_i)$ such that $\beta_i\gamma_i = \gamma_i\beta_i$, where $\gamma_i \neq \beta_i$. Set $\gamma = (0, 0, \ldots, \gamma_i, 0, \ldots, 0) \in R$, then $\gamma \in \mathcal{Z}(R)$ and $\gamma \neq \alpha,\ \beta$. So $\alpha - \gamma - \beta$ is an path of $\Gamma(R)$.

Case 3 Assume that $\forall\ i = 1, 2, \ldots, m$, neither α_i nor β_i belongs to $\mathcal{Z}(R_i)$. If there exists $\gamma_i \in R_i \setminus \mathcal{Z}(R_i)$, where $i = 1,\ 2,\ \ldots,\ m$, such that $\alpha_i - \gamma_i - \beta_i$ is a path of $\Gamma(R_i)$, then we put $\gamma = (0, 0, \ldots, \gamma_i, 0, \ldots, 0) \in R$. It is obvious that $\alpha - \gamma - \beta$ is an path of $\Gamma(R)$. Otherwise, taking $\gamma' = (\alpha'_1, 0, \ldots, 0) \in \Gamma(R)$ with $\alpha_1\alpha'_1 = \alpha'_1\alpha_1$ and $\gamma'' = (0, \ldots, 0, \beta'_m) \in \Gamma(R)$ with $\beta_m\beta'_m = \beta'_m\beta_m$, then $\alpha - \gamma' - \gamma'' - \beta$ is an path of $\Gamma(R)$.

Consequently, we must have $\Gamma(R)$ is a connected graph and $\text{diam}(\Gamma(R)) \leq 3$. Furthermore, note that there must exist an odd prime q such that $q \neq p_i,\ \forall\ i = 1, 2, \ldots, m$, we have $qa,\ qb \in \Gamma(R)$, then by the similar argument of Theorem 2.3, we can conclude that there doesn't exist a vertex α of $\Gamma(R)$ such that $qa - \alpha - qb$ is a path of $\Gamma(R)$. Thus $\text{diam}(\Gamma(R)) = 3$. This completes the proof. □

3. The maximum degree and the minimum degree of $\Gamma(Z_nQ_8)$

Lemma 3.1 ([8, Exercise 12]). *The number of solutions of congruence equation in $x_1, x_2, \ldots, x_k$: $a_1x_1 + a_2x_2 + \cdots + a_kx_k \equiv b \pmod m$ which $a_1, a_2, \ldots, a_k, b, m$ are integers and $m \geq 1$, is equal to $m^{k-1}(a_1, a_2, \ldots, a_k, m)$, if $(a_1, a_2, \ldots, a_k, m) \mid b$.*

Lemma 3.2. *Assume that $n = p^t$, $x_2, x_4, x_5, x_6, x_7, x_8 \in \{0, 1, 2, \ldots, p^t - 1\}$, where $t \geq 2, p \geq 2$ is a prime.*

(1) *Suppose $p \nmid (x_2 - x_4, x_5 - x_7, x_6 - x_8)$. Then the number of solutions of congruence system $(*)$ in $y_2, y_4, y_5, y_6, y_7, y_8$ is p^{4t}.*

(2) *Suppose $p^\tau \mid\mid (x_2 - x_4, x_5 - x_7, x_6 - x_8)$, where $1 \leq \tau \leq t - 1$. Then the number of solutions of congruence system $(*)$ in $y_2, y_4, y_5, y_6, y_7, y_8$ is $p^{4t+2\tau}$.*

Proof. (1) First of all, since $p \nmid (x_2 - x_4, x_5 - x_7, x_6 - x_8)$, without loss of generality, we can suppose $p \nmid (x_2 - x_4)$.

Case 1.1 Assume that $x_5 - x_7 \not\equiv 0 \pmod{p^t}$, $x_6 - x_8 \not\equiv 0 \pmod{p^t}$. Since $(x_2 - x_4, x_6 - x_8, p^t) = 1$, so by Lemma 3.1, we know that the number of solutions of the equation (2) is p^{3t}. Suppose $y_2 \equiv y_s^{(2)} \pmod{p^t}$, $y_4 \equiv y_s^{(4)} \pmod{p^t}$, $y_6 \equiv y_s^{(6)} \pmod{p^t}$, $y_8 \equiv y_s^{(8)} \pmod{p^t}$ are the solutions of the equation (2), $s = 1, 2, \ldots, p^{3t}$. So we have

(4)
$$(x_6 - x_8)y_s^{(2)} + (x_8 - x_6)y_s^{(4)} + (x_4 - x_2)y_s^{(6)} + (x_2 - x_4)y_s^{(8)} \equiv 0 \pmod{p^t}.$$

Substituting $y_2 \equiv y_s^{(2)} \pmod{p^t}$, $y_4 \equiv y_s^{(4)} \pmod{p^t}$ into the equation (3), and note that $(x_2 - x_4, p^t) = 1$, thus the equation (3) in y_5, y_7 has p^t solutions, denoted by $y_5 \equiv y_m^{(5)} \pmod{p^t}$, $y_7 \equiv y_m^{(7)} \pmod{p^t}$, $m = 1, 2, \ldots, p^t$. So we have

(5)
$$(x_5 - x_7)y_s^{(2)} + (x_7 - x_5)y_s^{(4)} + (x_4 - x_2)y_m^{(5)} + (x_2 - x_4)y_m^{(7)} \equiv 0 \pmod{p^t}.$$

In addition, note that $x_5 - x_7 \not\equiv 0 \pmod{p^t}$, $x_6 - x_8 \not\equiv 0 \pmod{p^t}$, so by equations (4) and (5), we have

$$\begin{aligned}(6)\qquad & (x_5 - x_7)(x_6 - x_8)y_s^{(2)} + (x_5 - x_7)(x_8 - x_6)y_s^{(4)} \\ & + (x_5 - x_7)(x_4 - x_2)y_s^{(6)} + (x_5 - x_7)(x_2 - x_4)y_s^{(8)} \equiv 0 \pmod{p^t},\end{aligned}$$

$$\begin{aligned}(7)\qquad & (x_6 - x_8)(x_5 - x_7)y_s^{(2)} + (x_6 - x_8)(x_7 - x_5)y_s^{(4)} \\ & + (x_6 - x_8)(x_4 - x_2)y_m^{(5)} + (x_6 - x_8)(x_2 - x_4)y_m^{(7)} \equiv 0 \pmod{p^t}.\end{aligned}$$

From the above equations (6) and (7), we get

$$\begin{aligned}& (x_6 - x_8)(x_4 - x_2)y_m^{(5)} - (x_5 - x_7)(x_4 - x_2)y_s^{(6)} \\ & + (x_6 - x_8)(x_2 - x_4)y_m^{(7)} + (x_5 - x_7)(x_2 - x_4)y_s^{(8)} \equiv 0 \pmod{p^t}.\end{aligned}$$

Since $p \nmid (x_2 - x_4)$, thus we have

$$(x_6 - x_8)y_m^{(5)} + (x_7 - x_5)y_s^{(6)} + (x_8 - x_6)y_m^{(7)} + (x_5 - x_7)y_s^{(8)} \equiv 0 \pmod{p^t}.$$

It follows that $y_5 \equiv y_m^{(5)} \pmod{p^t}$, $y_6 \equiv y_s^{(6)} \pmod{p^t}$,$y_7 \equiv y_m^{(7)} \pmod{p^t}$, $y_8 \equiv y_s^{(8)} \pmod{p^t}$ satisfy the equation (1). Consequently,

$$\begin{aligned} y_2 &\equiv y_s^{(2)} \pmod{p^t},\ y_4 \equiv y_s^{(4)} \pmod{p^t},\ y_5 \equiv y_m^{(5)} \ (v\ p^t), \\ y_6 &\equiv y_s^{(6)} \pmod{p^t},\ y_7 \equiv y_m^{(7)} \pmod{p^t},\ y_8 \equiv y_s^{(8)} \pmod{p^t}\end{aligned}$$

are solutions of the system $(*)$.

Therefore, the number of solutions of the system $(*)$ is $p^{3t} \times p^t = p^{4t}$.

Case 1.2 Assume that $x_5 - x_7 \equiv 0 \pmod{p^t}$, $x_6 - x_8 \not\equiv 0 \pmod{p^t}$. With the same argument of Case 1.1, we know that the equation (2) has p^{3t} solutions. Moreover, note that $p \nmid (x_2 - x_4)$, so the equation (3) has and only has p_t solutions, i.e., $y_5 \equiv y_7 \equiv 0, 1, \ldots, p^t - 1 \pmod{p^t}$, and all of them satisfy the equation (1). Hence, the system $(*)$ has and only has $p^{3t} \times p^t = p^{4t}$ solutions.

Similarly, if $x_5 - x_7 \not\equiv 0 \pmod{p^t}$, $x_6 - x_8 \equiv 0 \pmod{p^t}$, we also have the same result.

Case 1.3 Assume that $x_5 - x_7 \equiv 0 \pmod{p^t}$, $x_6 - x_8 \equiv 0 \pmod{p^t}$. Notice that $p \nmid (x_2 - x_4)$, thus $y_6 \equiv y_8 \equiv 0, 1, \dots, p^t - 1 \pmod{p^t}$ and $y_2 \equiv y_4 \equiv 0, 1, \dots, p^t - 1 \pmod{p^t}$ satisfy the equation (2). And $y_5 \equiv y_7 \equiv 0, 1, \dots, p^t - 1 \pmod{p^t}$ and $y_2 \equiv y_4 \equiv 0, 1, \dots, p^t - 1 \pmod{p^t}$ satisfy the equation (3). Thus $y_2 \equiv y_4 \equiv 0, 1, \dots, p^t - 1 \pmod{p^t}$, $y_5 \equiv y_7 \equiv 0, 1, \dots, p^t - 1 \pmod{p^t}$, $y_6 \equiv y_8 \equiv 0, 1, \dots, p^t - 1 \pmod{p^t}$ satisfy the equation $(*)$. Hence, the system $(*)$ has $p^{2t} \times p^t \times p^t = p^{4t}$ solutions.

(2) We will consider it from two cases:

Case 2.1 Suppose $x_2 - x_4, x_5 - x_7, x_6 - x_8 \not\equiv 0 \pmod{p^t}$. Since $p^\tau \mid\mid (x_2 - x_4, x_5 - x_7, x_6 - x_8)$, without loss of generality, we assume that $x_2 - x_4 = p^\tau u$, $x_5 - x_7 = p^\lambda v$, $x_6 - x_8 = p^\sigma w$, where $p \nmid u,\ v,\ w$ and $t - 1 \geq \sigma \geq \lambda \geq \tau \geq 1$. Since $(x_2 - x_4, x_6 - x_8, p^t) = (p^\tau u, p^\sigma w, p^t) = p^\tau$, thus by Lemma 3.1, the total number of solutions of the equation (2) is $p^{3t} \times p^\tau = p^{3t+\tau}$. Suppose $y_2 \equiv y_s^{(2)} \pmod{p^t}$, $y_4 \equiv y_s^{(4)} \pmod{p^t}$, $y_6 \equiv y_s^{(6)} \pmod{p^t}$, $y_8 \equiv y_s^{(8)} \pmod{p^t}$ are the solutions of the equation (2), $s = 1, 2, \dots, p^{3t+\tau}$. So we have

$$p^\sigma w y_s^{(2)} - p^\sigma w y_s^{(4)} - p^\tau u y_s^{(6)} + p^\tau u y_s^{(8)} \equiv 0 \pmod{p^t}. \tag{8}$$

Substituting $y_2 \equiv y_s^{(2)} \pmod{p^t}$, $y_4 \equiv y_s^{(4)} \pmod{p^t}$ into the equation (3), then we will conclude the following equation:

$$p^\tau u y_5 - p^\tau u y_7 \equiv p^\lambda v y_s^{(2)} + p^\lambda v y_s^{(4)} \pmod{p^t}. \tag{9}$$

Since $(p^\tau, p^t) = p^\tau$ and $p^\tau \mid p^\lambda$, thus the equation (3) in y_5, y_7 has $p^t \times p^\tau = p^{t+\tau}$ solutions. And we denote the solutions as $y_5 \equiv y_m^{(5)} \pmod{p^t}$, $y_7 \equiv y_m^{(7)} \pmod{p^t}$, where $m = 1, 2, \dots, p^{t+\tau}$. So we have

$$p^\lambda v y_s^{(2)} - p^\lambda v y_s^{(4)} + p^\tau u y_m^{(5)} + p^\tau u y_m^{(7)} \equiv 0 \pmod{p^t}. \tag{10}$$

Moreover, notice that $v,\ w \neq 0$, so by the equations (8) and (10), we have

$$p^\sigma wv y_s^{(2)} - p^\sigma wv y_s^{(4)} - p^\tau uv y_s^{(6)} + p^\tau uv y_s^{(8)} \equiv 0 \pmod{p^t}, \tag{11}$$

$$p^\lambda vw y_s^{(2)} - p^\lambda vw y_s^{(4)} + p^\tau uw y_m^{(5)} + p^\tau uw y_m^{(7)} \equiv 0 \pmod{p^t}. \tag{12}$$

Furthermore, we can get

$$p^{\lambda-\tau+\sigma} wv y_s^{(2)} - p^{\lambda-\tau+\sigma} wv y_s^{(4)} - p^\lambda uv y_s^{(6)} + p^\lambda uv y_s^{(8)} \equiv 0 \pmod{p^t}, \tag{13}$$

$$p^{\sigma-\tau+\lambda} vw y_s^{(2)} - p^{\sigma-\tau+\lambda} vw y_s^{(4)} + p^\sigma uw y_m^{(5)} + p^\sigma uw y_m^{(7)} \equiv 0 \pmod{p^t}. \tag{14}$$

So by the equations (13) and (14), we have

$$p^\sigma uw y_m^{(5)} - p^\lambda uv y_s^{(6)} - p^\sigma uw y_m^{(7)} + p^\lambda uv y_s^{(8)} \equiv 0 \pmod{p^t}.$$

Notice that $p \nmid u$, we get

$$p^\sigma w y_m^{(5)} - p^\lambda v y_s^{(6)} - p^\sigma w y_m^{(7)} + p^\lambda v y_s^{(8)} \equiv 0 \pmod{p^t}.$$

Consequently, $y_5 \equiv y_m^{(5)} \pmod{p^t}$, $y_6 \equiv y_s^{(6)} \pmod{p^t}$, $y_7 \equiv y_m^{(7)} \pmod{p^t}$, $y_8 \equiv y_s^{(8)} \pmod{p^t}$ satisfy the equation (1). Thus,

$$y_2 \equiv y_s^{(2)} \pmod{p^t},\ y_4 \equiv y_s^{(4)} \pmod{p^t},\ y_5 \equiv y_m^{(5)} \pmod{p^t},$$

$$y_6 \equiv y_s^{(6)} \pmod{p^t},\ y_7 \equiv y_m^{(7)} \pmod{p^t},\ y_8 \equiv y_s^{(8)} \pmod{p^t}$$

are solutions of the system $(*)$.

Therefore, the number of solutions of the system $(*)$ is $p^{3t+\tau} \times p^{t+\tau} = p^{4t+2\tau}$.

Case2.2 If at least one of $x_2 - x_4,\ x_5 - x_7,\ x_6 - x_8$ is 0 in Z_{p^t}, then the similar argument of Case 2.1 can be applied in here. □

Theorem 3.3. *Suppose $n = p^t$ where $p \geq 2$ is a prime and $t \geq 2$. $\forall\ \alpha = x_1 + x_2a + x_3a^2 + x_4a^3 + x_5b + x_6ab + x_7a^2b + x_8a^3b \in \Gamma(Z_nQ_8)$.*

(1) *If $p \nmid (x_2 - x_4,\ x_5 - x_7,\ x_6 - x_8)$, then $d(\alpha) = p^{6t} - p^{5t} - 1$;*

(2) *If $p^\tau \mid\mid (x_2 - x_4,\ x_5 - x_7,\ x_6 - x_8)$, where $1 \leq \tau \leq t - 1$, then $d(\alpha) = p^{6t+2\tau} - p^{5t} - 1$;*

(3) *The minimum degree $\delta(\Gamma(Z_nQ_8)) = p^{6t} - p^{5t} - 1$, while $d(\alpha = \delta(\Gamma(Z_nQ_8))$ if and only if $p \nmid (x_2 - x_4, x_5 - x_7, x_6 - x_8)$.*

(4) *The maximum degree $\triangle(\Gamma(Z_nQ_8)) = p^{8t-2} - p^{5t} - 1$, while $d(\alpha) = \triangle(\Gamma(Z_nQ_8))$ if and only if $p^{t-1} \mid\mid (x_2 - x_4,\ x_5 - x_7,\ x_6 - x_8)$.*

Proof. (1) Assume that $p \nmid (x_2 - x_4,\ x_5 - x_7,\ x_6 - x_8)$, then by Lemma 3.2, we have $d(\alpha) = p^{2t} \cdot p^{4t} - p^{5t} - 1 = p^{6t} - p^{5t} - 1$.

(2) Assume that $p^\tau \mid\mid (x_2 - x_4,\ x_5 - x_7,\ x_6 - x_8)$, then by Lemma 3.2, we have $d(\alpha) = p^{2t} \cdot p^{4t+2\tau} - p^{5t} - 1 = p^{6t+2\tau} - p^{5t} - 1$.

(3) and (4) follows directly by (1) and (2). □

Theorem 3.4. *Suppose $n = p$ where p is a prime. $\forall\ \alpha = x_1 + x_2a + x_3a^2 + x_4a^3 + x_5b + x_6ab + x_7a^2b + x_8a^3b \in \Gamma(Z_nQ_8)$.*

(1) *If $p = 2$, then $\triangle(\Gamma(Z_2Q_8)) = \delta(\Gamma(Z_2Q_8)) = 31$;*

(2) *If $p^\tau \mid\mid (x_2 - x_4,\ x_5 - x_7,\ x_6 - x_8)$, where $1 \leq \tau \leq t - 1$, then $\triangle(\Gamma(Z_nQ_8)) = \delta(\Gamma(Z_nQ_8)) = p^6 - p^5 - 1$.*

Proof. (1) Owing to Theorem 2.8, the results follows.

(2) By the condition (1) of Lemma 3.2 for $t = 1$, we can conclude that the number of solutions of congruence system $(*)$ in $y_2, y_4, y_5, y_6, y_7, y_8$ is p^4. Hence, $\triangle(\Gamma(Z_nQ_8)) = \delta(\Gamma(Z_nQ_8)) = d(\alpha) = p^2 \cdot p^4 - p^5 - 1 = p^6 - p^5 - 1$. □

Remark 3.5. Suppose $n > 1$ and n has unique normal decomposition $n = p_1^{t_1} p_2^{t_2} \cdots p_m^{t_m}$ with $m \geq 2,\ t_1, t_2, \ldots, t_m \geq 1$ and $2 \leq p_1 < p_2 < \cdots < p_m$ where $p_1, p_2, \ldots, p_m$ are distinct primes. By Lemma 2.9, we have

$$Z_nQ_8 \cong Z_{p_1^{t_1}}Q_8 \oplus Z_{p_2^{t_2}}Q_8 \oplus \cdots \oplus Z_{p_m^{t_m}}Q_8.$$

Moreover, we denote this isomorphism by ψ. $\forall\ \alpha = x_1 + x_2a + x_3a^2 + x_4a^3 + x_5b + x_6ab + x_7a^2b + x_8a^3b \in \Gamma(Z_2Q_8)$, let $f_i = x_i$ and let $f_{i1}, f_{i2}, \ldots, f_{im}$ are the remainder of f_i mod $p_1^{t_1}, p_2^{t_2}, \ldots, p_m^{t_m}$, respectively. Then $\varphi(\alpha) = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ where $\alpha_i = x_{i1} + x_{i2}a + x_{i3}a^2 + x_{i4}a^3 + x_{i5}b + x_{i6}ab + x_{i7}a^2b +$

$x_{i8}a^3b \in \Gamma(Z_{p_\lambda^{t_\lambda}}Q_8)$, $\lambda = 1, 2, \ldots, m$. By ([11], Remark 3.6), we have the following results:

(1) let q^σ denotes any term of $p_1^{t_1}, p_2^{t_2}, \ldots, p_m^{t_m}$, then we can claim that if there exists $1 \leq \tau \leq \sigma - 1$ such that $q^\tau \mid f_i$, then we must have $q^\tau \mid f_{is}$ where f_{is} is the remainder of f_i mod q^σ.

(2) If $q^\tau \mid\mid f_i$, then we also have $q^\tau \mid f_{is}$.

Corollary 3.6. *Suppose n has at least two distinct prime divisors and the normal decomposition of n and α have been given in Remark 3.5. Let $A_\lambda = \{\beta \in p_\lambda^{t_\lambda} \mid \alpha_\lambda\beta = \beta\alpha_\lambda\}, \lambda = 1, 2, \ldots, m$.*

(1) Assume that $t_\lambda = 1, p_\lambda = 2$. Then $|A_\lambda| = 32$.

(2) Assume that $t_\lambda = 1,\ p_\lambda \geq 3$.

Then $|A_\lambda| = \begin{cases} p_\lambda^6 & p_\lambda \nmid (x_2 - x_4,\ x_5 - x_7,\ x_6 - x_8) \\ p_\lambda^8 & p_\lambda \mid (x_2 - x_4,\ x_5 - x_7,\ x_6 - x_8). \end{cases}$

(3) Assume that $t_\lambda \geq 2,\ p_\lambda \nmid (x_2 - x_4,\ x_5 - x_7,\ x_6 - x_8)$. Then $|A_\lambda| = p_\lambda^{6t_\lambda}$.

(4) Assume that $t_\lambda \geq 2,\ p_\lambda^{\tau_\lambda} \mid\mid (x_2 - x_4,\ x_5 - x_7,\ x_6 - x_8),\ 1 \leq \tau_\lambda \leq t_\lambda - 1$. Then $|A_\lambda| = p_\lambda^{6t_\lambda + 2\tau_\lambda}$.

(5) Assume that $t_\lambda \geq 2,\ p_\lambda^{t_\lambda} \mid (x_2 - x_4,\ x_5 - x_7,\ x_6 - x_8)$. Then $|A_\lambda| = p_\lambda^{8t_\lambda}$.

Theorem 3.7. *Suppose $n > 1$, $n \neq p^\lambda$ where $p \geq 2$ is a prime and $\lambda \geq 1$. The normal decomposition of n has been given in Remark 3.5. $\forall\ \alpha = x_1 + x_2a + x_3a^2 + x_4a^3 + x_5b + x_6ab + x_7a^2b + x_8a^3b \in \Gamma(Z_nQ_8)$ and we define two subsets $I_1,\ I_2$ of $I = \{1, 2, \ldots, m\}$ as following:*

$I_1 = \{\sigma \in I \mid \exists\tau_\sigma,\ 1 \leq \tau_\sigma \leq t_\sigma - 1,$ such that $p_\sigma^{\tau_\sigma} \mid\mid (x_2 - x_4,\ x_5 - x_7,\ x_6 - x_8)\}$,
$I_2 = \{\lambda \in I \mid p_\lambda^{\tau_\lambda} \mid (x_2 - x_4,\ x_5 - x_7,\ x_6 - x_8)\}$. Then

(1) Assume that $p_i \neq 2$. Then $d(\alpha) = n^6 \prod_{\sigma \in I_1 - \{1\}} p_\sigma^{2\tau_\sigma} \prod_{\lambda \in I_2 - \{1\}} p_\lambda^{2\tau_\lambda} - n^5 - 1$;

(2) Assume that $p_1 = 2, t_1 = 1$. Then $d(\alpha) = 32n^6 \prod_{\sigma \in I_1 - \{1\}} p_\sigma^{2\tau_\sigma} \prod_{\lambda \in I_2 - \{1\}} p_\lambda^{2\tau_\lambda} - n^5 - 1$;

(3) Assume that $p_1 = 2,\ t_1 \geq 2$. Then $d(\alpha) = n^6 \prod_{\sigma \in I_1} p_\sigma^{2\tau_\sigma} \prod_{\lambda \in I_2} p_\lambda^{2\tau_\lambda} - n^5 - 1$.

Proof. (1) $\forall\ \alpha,\ \beta \in Z_nQ_8$, $\varphi(\alpha),\ \varphi(\beta)$ are defined in Remark 3.5. Then

$$\begin{aligned} &\alpha\beta = \beta\alpha \\ \Longleftrightarrow\ &(\alpha_1, \alpha_2, \ldots, \alpha_m)(\beta_1, \beta_2, \ldots, \beta_m) = (\beta_1, \beta_2, \ldots, \beta_m)(\alpha_1, \alpha_2, \ldots, \alpha_m) \\ \Longleftrightarrow\ &\alpha_i\beta_i = \beta_i\alpha_i,\ i = 1, 2, \ldots, m. \end{aligned}$$

By Corollary 3.6, and note that $|\mathcal{Z}(Z_nQ_8)| = n^5$, we have $d(\alpha) = \prod_{\sigma \in I_1 - \{1\}} p_\sigma^{6t_\sigma + 2\tau_\sigma} \prod_{\lambda \in I_2 - \{1\}} p_\lambda^{8\tau_\lambda} \prod_{k \in I - I_1 - I_2 - \{1\}} p_k^{6t_k} - n^5 - 1 = n^6 \prod_{\sigma \in I_1 - \{1\}} p_\sigma^{2\tau_\sigma} \prod_{\lambda \in I_2 - \{1\}} p_\lambda^{2\tau_\lambda} - n^5 - 1$.

By the similar argument above, we can conclude that the formulas of (2) and (3). □

Theorem 3.8. *Suppose $n > 1$, $n \neq p^\lambda$ where $p \geq 2$ is a prime and $\lambda \geq 1$. The normal decomposition of n has been given in Remark 3.5. $\forall\ \alpha = x_1 + x_2a + x_3a^2 + x_4a^3 + x_5b + x_6ab + x_7a^2b + x_8a^3b \in \Gamma(Z_nQ_8)$.*

(1) *Assume that $p_i \neq 2$, then $\delta(\Gamma(Z_nQ_8)) = n^6 - n^5 - 1$ and $\triangle(\Gamma(Z_nQ_8)) = \frac{n^8}{p_2^2} - n^5 - 1$;*

(2) *Assume that $p_1 = 2$, $t_1 = 1$, then $\delta(\Gamma(Z_nQ_8)) = 32n^6 - n^5 - 1$ and $\triangle(\Gamma(Z_nQ_8)) = \frac{32n^8}{p_2^2} - n^5 - 1$;*

(3) *Assume that $p_1 = 2$, $t_1 \geq 2$, then $\delta(\Gamma(Z_nQ_8)) = n^6 - n^5 - 1$ and $\triangle(\Gamma(Z_nQ_8)) = \frac{n^8}{p_2^2} - n^5 - 1$.*

Proof. (1) By Theorem 3.7, we have $d(\alpha) = \delta(\Gamma(Z_nQ_8)) \iff I_1 = \emptyset$ and $I_2 = \emptyset$. Thus $\delta(\Gamma(Z_nQ_8)) = n^6 - n^5 - 1$. Moreover, if $t_2 = 1$, then $d(\alpha) = \triangle(\Gamma(Z_nQ_8)) \iff I_1 = \{2\}$ and $I_2 = \{3, 4, \dots, m\}$. So we derive that

$$\triangle(\Gamma(Z_nQ_8)) = \frac{n^8}{p_2^2} - n^5 - 1.$$

By the similar argument above, we can conclude that the formulas of (2) and (3). □

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