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DISCRETE DUALITY FOR *TSH*-ALGEBRAS

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Abstract. In this article, we continue the study of tense symmetric Heyting algebras (or TSH-algebras). These algebras constitute a generalization of tense algebras. In particular, we describe a discrete duality for TSH-algebras bearing in mind the results indicated by Orłowska and Rewitzky in [E. Orłowska and I. Rewitzky, *Discrete Dualities for Heyting Algebras with Operators*, Fund. Inform. **81** (2007), no. 1-3, 275–295] for Heyting algebras. In addition, we introduce a propositional calculus and prove this calculus has TSH-algebras as algebraic counterpart. Finally, the duality mentioned above allowed us to show the completeness theorem for this calculus.

1. Introduction

Propositional logics usually do not incorporate the dimension of time. To obtain a tense logic, we enrich a propositional logic by adding new unary operators (or connectives) which are usually denoted by G, H, F and P. We can define F and P by means of G and H as follows: $\mathrm{F}(x) = \neg G(\neg x)$ and $\mathrm{P}(x) = \neg H(\neg x)$, where $\neg x$ denotes negation of the proposition x.

It is worth saying that tense operators were firstly introduced for the classical propositional logic (see [3]). Tense algebras are algebraic structures corresponding to the propositional tense logic [3, 13]. Recall that an algebra $\langle W, \vee, \wedge, \neg, G, H, 0, 1\rangle$ is a tense algebra if $\langle W, \vee, \wedge, \neg, 0, 1\rangle$ is a Boolean algebra and G, H are unary operators on W satisfying the axioms

$$G(1) = 1,\ H(1) = 1,$$
$$G(x \wedge y) = G(x) \wedge G(y),\ H(x \wedge y) = H(x) \wedge H(y),$$
$$x \leq G\mathrm{P}(x),\ x \leq H\mathrm{F}(x),$$

where $\mathrm{P}(x) = \neg H(\neg x)$ and $\mathrm{F}(x) = \neg G(\neg x)$.

In the last few years tense operators have been considered by different authors for varied classes of algebras. Some contributions in this area have been

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the papers of Diaconescu and Georgescu [7], Chiriţă [5, 6], Figallo and Pelaitay [10, 9], Chajda [4], and Botur et al. [2].

In 1942, Gr. C. Moisil [14] introduced the modal symmetric propositional calculus as an extension of the positive calculus of Hilbert-Bernays obtained by adding a new negation connective, $\sim$, the axiom schemata

$$\alpha \to \sim\sim \alpha, \qquad \sim\sim \alpha \to \alpha$$

and the contraposition rule

$$\text{if } \alpha \to \beta, \text{ then } \sim \beta \to \sim \alpha.$$

This propositional calculus has symmetric Heyting algebras as the algebraic counterpart. These algebras were investigated by Monteiro [15] and also by Iturrioz [12] and Sankappanavar [19]. Recall that an algebra $\langle W, \vee, \wedge, \to, \sim, 0, 1\rangle$ is a symmetric Heyting algebra (see [15]) if $\langle W, \vee, \wedge, \sim, 0, 1\rangle$ is a De Morgan algebra and $\langle W, \vee, \wedge, \to, 0, 1\rangle$ is a Heyting algebra.

On the other hand, a discrete duality (see [16, 17, 8]) is a duality where a class of abstract systems is a dual counterpart to a class of algebras. These relational systems are referred to as frames following the terminology of non–classical logics.

A topology is not involved in the construction of these frames and hence they may be thought of as having a discrete topology.

Establishing discrete duality involves the following steps. Given a class **Alg** of algebras (resp. a class **Frm** of frames) we define a class **Frm** of frames (resp. a class **Alg** of algebras). Next, for an algebra $W \in$ **Alg** we define its canonical frame $\mathcal{X}(W)$ and for each frame $X \in$ **Frm** we define its complex algebra $\mathcal{C}(X)$. Then we prove that $\mathcal{X}(W) \in$ **Frm** and $\mathcal{C}(X) \in$ **Alg**. A duality between **Alg** and **Frm** holds provided that the following facts are proved:

• Every algebra $W \in$ **Alg** is embeddable into the complex algebra $\mathcal{C}(\mathcal{X}(W))$ of its canonical frame.

• Every frame $X \in$ **Frm** is embeddable into the canonical frame $\mathcal{X}(\mathcal{C}(X))$ of its complex algebra.

An important application of discrete duality is that it provides a Kripke semantics (resp. an algebraic semantics) once an algebraic semantics (resp. a Kripke semantics) for a formal language is given (see [17]).

In this paper we apply the methodology of discrete duality to tense symmetric Heyting algebras (or TSH-algebras, for short) [11]. In addition, we introduce a propositional calculus and prove this calculus has TSH-algebras as algebraic counterpart. Finally, the duality mentioned above allowed us to show the completeness theorem for this calculus.

2. Preliminaries

In this paper we take for granted the concepts and results on Heyting algebras. To obtain more information on this topics, we direct the reader to the bibliography indicated in [1]. However, in order to simplify reading, in this section we summarize the fundamental concepts we use.

Let T be a binary relation on a set X and let A be a subset of X. In what follows we will denote by $[T]A$ the set $\{x \in X : \text{for all } y, x\,T\,y \text{ implies } y \in A\}$.

In [16], Orlowska and Rewitzky introduced the notion of Heyting frame (or H-frame, for short) as a pair $(X, \leq)$ where X is a non-empty set and $\leq$ is a quasi-order on X. These authors proved that if $\langle W, \vee, \wedge, \rightarrow, 0, 1\rangle$ is a Heyting algebra, then its canonical frame is $(\mathcal{X}(W), \leq^c)$, where $\mathcal{X}(W)$ is the set of all prime filters of W and $\leq^c$ is $\subseteq$. It is easy to see that this canonical frame is an H-frame. On the other hand, given an H-frame $(X, \leq)$, they show that its complex algebra is $\langle \mathcal{C}(X), \vee^c, \wedge^c, \rightarrow^c, 0^c, 1^c\rangle$, where $\mathcal{C}(X) = \{A \subseteq X : [\leq]A = A\}$, $0^c = \emptyset$, $1^c = X$, $A \vee^c B = A \cup B$, $A \wedge^c B = A \cap B$ and $A \rightarrow^c B = [\leq]((X \setminus A) \cup B)$ for all $A, B \in \mathcal{C}(X)$.

These results allowed them to obtain a discrete duality for Heyting algebras by defining the embeddings as follows:

$$h : W \rightarrow \mathcal{C}(\mathcal{X}(W)),\ h(a) = \{F \in \mathcal{X}(W) : a \in F\},$$
$$k : X \rightarrow \mathcal{X}(\mathcal{C}(X)),\ k(x) = \{A \in \mathcal{C}(X) : x \in A\}.$$

3. Tense symmetric Heyting algebras

In this section we shall recall some definitions and basic results on tense symmetric Heyting algebras from [11].

Definition 1. A tense symmetric Heyting algebra (or TSH-algebra, for short) is an algebra $\langle W, \vee, \wedge, \rightarrow, \sim, G, H, 0, 1\rangle$, where the reduct $\langle W, \vee, \wedge, \rightarrow, \sim, 0, 1\rangle$ is a symmetric Heyting algebra and G, H are unary operators on W verifying the following conditions,

(T1) $G(1) = 1$, $H(1) = 1$,
(T2) $G(x \wedge y) = G(x) \wedge G(y)$, $H(x \wedge y) = H(x) \wedge H(y)$,
(T3) $x \leq G(\sim H(\sim x))$, $x \leq H(\sim G(\sim x))$.

In what follows, we will denote these algebras by (W, G, H) or simply by W where no confusion may arise.

Remark 3.1. If $\langle W, \vee, \wedge, \rightarrow, \sim, G, H, 0, 1\rangle$ is a TSH-algebra in which every element of W is Boolean, then $\langle W, \vee, \wedge, \sim, G, H, 0, 1\rangle$ is a tense algebra.

Definition 2. For any TSH-algebra (W, G, H), let us considerer the unary operations P, F defined by $\mathrm{P}(x) = \sim H(\sim x)$ and $\mathrm{F}(x) = \sim G(\sim x)$.

Lemma 3.2. *The following properties hold in any TSH-algebra (W, G, H):*

(i) $x \leq y$ *implies* $G(x) \leq G(y)$ *and* $H(x) \leq H(y)$,
(ii) $x \leq y$ *implies* $\mathrm{P}(x) \leq \mathrm{P}(y)$ *and* $\mathrm{F}(x) \leq \mathrm{F}(y)$,
(iii) $\mathrm{P}(0) = 0$ *and* $\mathrm{F}(0) = 0$,
(iv) $\mathrm{P}(x \vee y) = \mathrm{P}(x) \vee \mathrm{P}(y)$ *and* $\mathrm{F}(x \vee y) = \mathrm{F}(x) \vee \mathrm{F}(y)$,
(v) $\mathrm{F}H(x) \leq x$ *and* $\mathrm{P}G(x) \leq x$.

Proof. It is routine. □

Lemma 3.3. *Let G, H be two unary operations on a symmetric Heyting algebra $\langle W, \vee, \wedge, \rightarrow, \sim, 0, 1\rangle$ such that $G(1) = 1$, $H(1) = 1$. Then condition* (T2) *is equivalent to the following one*:

$$G(x \rightarrow y) \leq G(x) \rightarrow G(y), \;\; H(x \rightarrow y) \leq H(x) \rightarrow H(y). \tag{1}$$

Proof. We will only prove the equivalence between (T2) and (1) in the case of G. From (T2), and (i) in Lemma 3.2, we have that $G(x) \wedge G(x \rightarrow y) = G(x \wedge (x \rightarrow y)) = G(x \wedge y) \leq G(y)$. Therefore, $G(x \rightarrow y) \leq G(x) \rightarrow G(y)$. Conversely, let $x, y \in W$ be such that $x \leq y$. Then, $x \rightarrow y = 1$ and so, from (1) and the hypothesis, we obtain that $1 = G(x \rightarrow y) \leq G(x) \rightarrow G(y)$. Hence, $G(x) \leq G(y)$ from which we get that G is increasing. This last assertion and (1) we infer that $G(x) \leq G(y \rightarrow (x \wedge y)) \leq G(y) \rightarrow G(x \wedge y)$. Thus, $G(x) \wedge G(y) \leq G(x \wedge y)$. From this statement and taking into account that G is increasing we conclude that $G(x) \wedge G(y) = G(x \wedge y)$. □

Thus, if we replace in Definition 1 the axiom (T2) with the condition (1), we obtain an equivalent definition of TSH-algebra.

Lemma 3.4. *Let (W, G, H) be a TSH-algebra. If F is a filter of W, then $G^{-1}(F)$ and $H^{-1}(F)$ are also filters of W.*

Proof. The proof is a direct consecuence of (T1) and (T2). □

4. A discrete duality for TSH-algebras

In this section, we describe a discrete duality for TSH-algebras taking into account the one indicated above for Heyting algebras. To this end, we introduce the following:

Definition 3. A TSH-frame is a structure $(X, \leq, g, R, Q)$ where $(X, \leq)$ is a H-frame, $g : X \rightarrow X$ is a function, R, Q are binary relations on X and the following conditions are satisfied:

(K1) if $x \leq y$, then $g(y) \leq g(x)$ for $x, y \in X$,
(K2) $g(g(x)) = x$ for $x \in X$,
(K3) $(\leq \circ R \circ \leq) \subseteq R$,
(K4) $(\leq \circ Q \circ \leq) \subseteq Q$,
(K5) $x \, R \, g(y)$ if and only if $y \, Q \, g(x)$ for $x, y \in X$.

In what follows, TSH-frames will be denoted simply by X when no confusion may arise.

Definition 4. A canonical frame of a TSH-algebra (W, G, H) is a structure $(\mathcal{X}(W), \leq^c, g^c, R^c, Q^c)$, where $(\mathcal{X}(W), \leq^c)$ is the canonical frame associated with $\langle W, \vee, \wedge, \rightarrow, 0, 1\rangle$ and the following conditions are verified for $P, F \in \mathcal{X}(W)$:

(F1) $g^c(P) = \{a \in W : \sim a \notin P\}$,

(F2) PR^cF if and only if $G^{-1}(P) \subseteq F$,
(F3) PQ^cF if and only if $H^{-1}(P) \subseteq F$.

Lemma 4.1. *The canonical frame of a TSH-algebra is a TSH-frame.*

Proof. Taking into account the results established in [8, Lemma 11.1], we only have to prove (K3), (K4) and (K5).
(K3): Let $(P, F) \in \leq^c \circ R^c \circ \leq^c$. Then there exist $T, S \in \mathcal{X}(W)$ such that $P \subseteq T$, TR^cS and $S \subseteq F$. From the last two assertions we have that $G^{-1}(T) \subseteq F$. Therefore, since $P \subseteq T$ we infer that $P\,R^c\,F$.
(K4): It is proved in a similar way to (K3).
(K5): Let $F\,R^c g^c(P)$ and $a \in H^{-1}(P)$. Suppose that $\sim a \in F$. On the other hand, from (T3) we have that $\sim a \leq G(\sim H(a))$ and so, we get that $G(\sim H(a)) \in F$. From this last assertion and the fact that $G^{-1}(F) \subseteq g^c(P)$, we obtain $\sim H(a) \in g^c(P)$. Hence, $H(a) \notin P$ which is a contradiction. Therefore, $a \in g^c(F)$ from which we conclude that $PQ^c g^c(F)$. The converse is proved similarly. □

Definition 5. The complex algebra of a TSH-frame $(X, \leq, g, R, Q)$ is $\langle \mathcal{C}(X), \vee^c, \wedge^c, \to^c, \sim^c, G^c, H^c, 0^c, 1^c \rangle$, where $\langle \mathcal{C}(X), \vee^c, \wedge^c, \to^c, 0^c, 1^c \rangle$ is the complex algebra of the H-frame $(X, \leq)$, $\sim^c A = X \setminus g(A)$, $G^c(A) = [R]A$ and $H^c(A) = [Q]A$ for all $A \in \mathcal{C}(X)$.

Lemma 4.2. *The complex algebra of a TSH-frame is a TSH-algebra.*

Proof. From [8, 16], $\mathcal{C}(X)$ is closed under the lattice operations, $\sim^c$ and $\to^c$. Now, we show that it is also closed under G^c, i.e., $G^cA = [\leq]G^cA$. From the reflexivity of $\leq$, we have that $[\leq]G^cA \subseteq G^cA$. Assume that $x \in G^cA$. Let $y \in X$ be such that $x \leq y$ and take any $z \in X$ verifying yRz. Hence, from the reflexivity of $\leq$ and (K3) we infer that xRz. So, $z \in A$ and therefore, $x \in [\leq]G^cA$. Thus, $G^cA \subseteq [\leq]G^cA$. Similarly, it is proved that $H^cA = [\leq]H^cA$. On the other hand, clearly (T1) and (T2) are verified. Therefore, it only remains to prove (T3). Let $x \in A$ and suppose that $x \notin G^c(\sim^c H^c(\sim^c A))$. Then there is y such that xRy and $y \notin \sim^c H^c(\sim^c A)$. From this last statement, $y \in g(H^c(\sim^c A))$ and so, $y = g(z)$ for some $z \in H^c(\sim^c A)$. Hence, $xRg(z)$ and from (K5) we get that $zQg(x)$. This assertion and the fact that $z \in H^c(\sim^c A)$ enable us to infer that $g(x) \notin g(A)$, which is a contradiction. So, $A \subseteq G^c(\sim^c H^c(\sim^c A))$. Analogously, it is proved that $A \subseteq H^c(\sim^c G^c(\sim^c A))$. □

Theorem 4.3. *Each TSH-algebra W is embeddable into $\mathcal{C}(\mathcal{X}(W))$.*

Proof. Let us consider the function $h : W \to \mathcal{C}(\mathcal{X}(W))$ defined by $h(a) = \{P \in \mathcal{X}(W) : a \in P\}$ for all $a \in W$ (see [8, 16]). Let $F \in h(G(a))$; then $G(a) \in F$. Suppose that $P \in \mathcal{X}(W)$ verifies that FR^cP. Then from (F2), $G^{-1}(F) \subseteq P$ and so, $a \in P$. Therefore, $F \in G^c(h(a))$ from which we infer that $h(G(a)) \subseteq G^c(h(a))$. Conversely, assume that $F \in G^c(h(a))$. Then for every $P \in \mathcal{X}(W)$, FR^cP implies that $P \in h(a)$. Suppose that $G(a) \notin F$.

Then $G^{-1}(F)$ is a filter and $a \notin G^{-1}(F)$. Hence, there is $T \in \mathcal{X}(W)$ such that $a \notin T$ and $G^{-1}(F) \subseteq T$. This last assertion and (F2) allow us to conclude that FR^cT. From this statement we have that $T \in h(a)$ and so, $a \in T$, which is a contradiction. Therefore, $h(G(a)) = G^c(h(a))$. Similarly, it is shown that $h(H(a)) = H^c(h(a))$. Thus, by virtue of the results established in [8, 16] the proof is completed. □

Lemma 4.4 will show that the order-embedding $k : X \to \mathcal{X}(\mathcal{C}(X))$ defined by $k(x) = \{A \in \mathcal{C}(X) : x \in A\}$ for every $x \in X$ (see [8, 16]) preserves the relations R and Q.

Lemma 4.4. *Let $(X, \leq, g, R, Q)$ be a TSH-frame and let $x, y \in X$. Then*

(i) *xRy if and only if $k(x)R^ck(y)$,*
(ii) *xQy if and only if $k(x)Q^ck(y)$.*

Proof. We will only prove (i). Assume that xRy and suppose that $A \in \mathcal{C}(X)$ verifies $G^c(A) \in k(x)$. Then it is easy to see that $y \in A$ and so, $k(x)R^ck(y)$. Conversely, let $x, y \in X$ be such that $k(x)R^ck(y)$. Then $G^{c^{-1}}(k(x)) \subseteq k(y)$. On the other hand, note that $[\leq](X \setminus (y]) \in \mathcal{C}(X)$ and $y \notin [\leq](X \setminus (y])$. Thus, $[\leq](X \setminus (y]) \notin k(y)$ and so, $[\leq](X \setminus (y]) \notin G^{c^{-1}}(k(x))$. Therefore, $[R]([\leq](X \setminus (y])) \notin k(x)$ from which we infer that $x \notin [R]([\leq](X \setminus (y]))$. Then there is z such that xRz and $z \notin [\leq](X \setminus (y])$. From this last assertion there is w such that $z \leq w$ and $w \leq y$, which allow us to infer that $z \leq y$. Hence, by virtue of the reflexivity of $\leq$ and (K3), xRy as required. □

Lemma 4.4 and the results indicated in [8, 16] enable us to conclude:

Theorem 4.5. *Every TSH-frame X is embeddable into the canonical frame of its complex algebra $\mathcal{X}(\mathcal{C}(X))$.*

Theorems 4.3 and 4.5 enable us to obtain a discrete duality for TSH-algebras.

5. A propositional calculus based on TSH-algebras

In this section, we will describe a propositional calculus that has TSH-algebras as the algebraic counterpart. The terminology and symbols used here coincide in general with those used in [18].

Let $\mathcal{L} = (A^0, \mathrm{For}[V])$ be a formalized language of zero order, where in the alphabet $A^0 = (V, L_0, L_1, L_2, U)$ the set

- V of propositional variables is enumerable,
- L_0 is empty,
- L_1 contains three elements denoted by $\sim$, G and H called negation sign and tense operators signs, respectively,
- L_2 contains three elements denoted by $\vee$, $\wedge$, $\to$, called disjunction sign, conjunction sign and implication sign, respectively,
- U contains two elements denoted by (,).

For any α, β in the set For[V] of all formulas over A^0, instead of $(\alpha \to \beta) \wedge (\beta \to \alpha)$, $\sim G \sim \alpha$ and $\sim H \sim \alpha$ we will write for brevity $\alpha \leftrightarrow \beta$, F$\alpha$ and Pα, respectively.

We assume that the set $\mathcal{A}_l$ of logical axioms consists of all formulas of the following form, where α, β are any formulas in For[V]:

(M0) the axioms of the symmetric modal propositional calculus, i.e., the axioms (A1)-(A10) indicated in [15, page 60],
(M1) $G(\alpha \to \beta) \to (G\alpha \to G\beta),\ H(\alpha \to \beta) \to (H\alpha \to H\beta)$,
(M2) $\alpha \to G\mathrm{P}\alpha,\ \alpha \to H\mathrm{F}\alpha$.

The consequence operation $C_{\mathcal{L}}$ in $\mathcal{L}$ is determined by $\mathcal{A}_l$ and by the following rules of inference:

(R1) $\dfrac{\alpha,\ \ \alpha \to \beta}{\beta}$, (R3) $\dfrac{\alpha}{G\alpha}$,

(R2) $\dfrac{\alpha \to \beta}{\sim \beta \to \sim \alpha}$, (R4) $\dfrac{\alpha}{H\alpha}$.

The system $\mathcal{TMS} = (\mathcal{L}, C_{\mathcal{L}})$ thus obtained will be called the $\mathcal{TMS}$-propositional calculus. We will denote by $\mathcal{T}$ the set of all formulas derivable in $\mathcal{TMS}$. If α belongs to $\mathcal{T}$ we will write $\vdash \alpha$.

Let $\approx$ be the binary relation on For[V] defined by

$$\alpha \approx \beta \text{ if and only if } \vdash \alpha \leftrightarrow \beta.$$

Then it is easy to check that $\approx$ is a congruence relation on $\langle \mathrm{For}[V], \vee, \wedge, \to, \sim, G, H\rangle$ and $\mathcal{T}$ determines an equivalence class which we will denote by 1. Moreover, taking into account [15, page 62] it is straightforward to prove:

Theorem 5.1. *$\langle \mathrm{For}[V]/\approx, \vee, \wedge, \to, \sim, G, H, 0, 1\rangle$ is a TSH-algebra, being $0 = \sim 1$.*

Definition 6. A TSH-model based on a TSH-frame $K = (X, \leq, g, R, Q)$ is a system $M = (K, m)$ such that $m : V \to \mathcal{P}(X)$ is a meaning function that assigns subsets of states to propositional variables, i.e., satisfies the following condition:

(her) $x \leq y$ and $x \in m(p)$ imply $y \in m(p)$.

Definition 7. A TSH-model $M = ((X, \leq, g, R, Q); m)$ satisfies a formula α at the state x and we write $M \models_x \alpha$, if the following conditions are satisfied:

- $M \models_x p$ if and only if $x \in m(p)$ for $p \in V$,
- $M \models_x \alpha \vee \beta$ if and only if $M \models_x \alpha$ or $M \models_x \beta$,
- $M \models_x \alpha \wedge \beta$ if and only if $M \models_x \alpha$ and $M \models_x \beta$,
- $M \models_x \sim \alpha$ if and only if $M \not\models_{g(x)} \alpha$,
- $M \models_x \alpha \to \beta$ if and only if for all y, if $x \leq y$ and $M \models_y \alpha$, then $M \models_y \beta$,
- $M \models_x G\alpha$ if and only if for all y, if xRy, then $M \models_y \alpha$,
- $M \models_x H\alpha$ if and only if for all y, if xQy, then $M \models_y \alpha$.

A formula α is *true in a* TSH*-model* M (denoted by $M \models \alpha$) if and only if for every $x \in W$, $M \models_x \alpha$. The formula α is *true in a* TSH*-frame* K (denoted by $K \models \alpha$) if and only if it is true in every TSH-model based on K. The formula α is TSH*-valid* if and only if it is true in every TSH-frame.

Proposition 5.2. *Given a* TSH*-model* $M = ((X, \leq, g, R, Q); m)$*, the meaning function* m *can be extended to all formulae by* $m(\alpha) = \{x \in X : M \models_x \alpha\}$*. For every* TSH*-model* M *and for every formula* α*, this extension has the property*

(her) if $x \leq y$ and $x \in m(\alpha)$, then $y \in m(\alpha)$.

Proof. The proof is by induction with respect to complexity of α. By way of an example we show (her) for formulas of the form $G\alpha$. Let (1) $x \leq y$ and (2) $M \models_x G(\alpha)$. Suppose that yRz, then by (1), (2) and (K3), we have $M \models_z \alpha$. □

Theorem 5.3 (Completeness Theorem)**.** *Let* α *be a formula in* $\mathcal{TMS}$*. Then the following conditions are equivalent*:

(i) α *is derivable in* $\mathcal{TMS}$;
(ii) α *is* TSH*-valid.*

Proof. (i) $\Rightarrow$ (ii): We proceed by induction on the complexity of the formula α. For example, we shall prove that the axiom (M2) is TSH-valid. Let $K = (X, \leq, g, R, Q)$ be a TSH-frame and M a TSH-model based on K.

(1) Let $x, y \in X$ be such that $x \leq y$, [hip.]
(2) $M \models_y \alpha$, [hip.]
(3) Let $z \in X$ be such that $y\,Q\,z$. [hip.]

Suppose that

(4) $M \models_{g(z)} G \sim \alpha$, [hip.]
(5) $g(z)\,R\,g(y)$, [(3),(K2),(K5)]
(6) $M \models_{g(y)} \sim \alpha$, [(4),(5)]
(7) $M \not\models_y \alpha$. [(6),(K2)]

(7) contradicts (2). Then

(8) $M \not\models_{g(z)} G \sim \alpha$, [(4),(7)]
(9) $M \models_z \sim G \sim \alpha$, [(8)]
(10) $M \models_y H \sim G \sim \alpha$, [(3),(9)]
(11) $M \models_x \alpha \to H \sim G \sim \alpha$. [(1),(2),(10)]

(ii) $\Rightarrow$ (i): Assume that α is not derivable, i.e., $[\alpha]_{\approx} \neq 1$. We apply Theorem 4.3 to the TSH-algebra $\mathrm{For}[V]/\approx$, hence there exists a TSH-frame $\mathcal{X}(\mathrm{For}[V]/\approx)$ and an injective morphism of TSH-algebras $h : \mathrm{For}[V]/\approx \to \mathcal{C}(\mathcal{X}(\mathrm{For}[V]/\approx))$. Let us consider the function $m : \mathcal{TMS} \to \mathcal{C}(\mathcal{X}(\mathrm{For}[V]/\approx))$ defined by $m(\alpha) = h([\alpha]_{\approx})$ for all $\alpha \in \mathrm{For}[V]$. It is straightforward to prove that m is an meaning function. Since h is injective, $m(\alpha) = h([\alpha]_{\approx}) \neq \mathcal{X}(\mathrm{For}[V]/\approx)$, i.e., $(\mathcal{X}(\mathrm{For}[V]/\approx), m) \not\models_{x_o} \alpha$ for some $x_o \in \mathcal{X}(\mathrm{For}[V]/\approx)$. Thus α is not TSH-valid. □

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