ON THE PLURIGENUS OF A CANONICAL THREEFOLD

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ABSTRACT. It is well known that plurigenus does not vanish for a sufficiently large multiple on a canonical threefold over $\mathbb{C}$. There is Reid-Fletcher formula for plurigenus. But, unlike in the case of surface of general type, it is not easy to compute plurigenus. In this paper, we induce a different version of Reid-Fletcher formula and show that the constant term in the induced formula has periodic properties. Using these properties we have an application to nonvanishing of plurigenus.

Throughout this paper X is assumed to be a projective threefold with only canonical singularities and an ample canonical divisor K_X over the complex number field $\mathbb{C}$, i.e., a canonical threefold.

It is well known that $H^0(X, \mathcal{O}_X(mK_X))$ does not vanish for a sufficiently large integer m. In the case of surface of general type and an algebraic curve, such integer m is well known. In a case of threefold, when $\chi(\mathcal{O}_X) \leq 0$, it is easy to have such integer m (see Fletcher [1]); however, when $\chi(\mathcal{O}_X) > 0$, it is not easy even to obtain an integer m such that $\dim H^0(X, \mathcal{O}_X(mK_X)) \geq 1$. There are some results about $\dim H^0(X, \mathcal{O}_X(mK_X))$, i.e., plurigenus. In Fletcher [1], A. R. Fletcher showed that

$$\dim H^0(X, \mathcal{O}_X(12K_X)) \geq 1 \text{ and } \dim H^0(X, \mathcal{O}_X(24K_X)) \geq 2$$

when $\chi(\mathcal{O}_X) = 1$. In Shin [7], Shin improved results of Fletcher. In Hanamura [2], Hanamura induced a formula for plurigenus and computed plurigenus according to 'global index'. For detailed matters, see Fletcher [1], Reid [6], Hanamura [2], Shin [7]. In this paper, we induce a different version of Reid-Fletcher formula and show that the constant term in the induced formula has periodic properties. Using these properties we have an application to nonvanishing of plurigenus.

M. Reid and A. R. Fletcher described the formula for $\chi(\mathcal{O}_X(nK_X))$. Combining the formula for $\chi(\mathcal{O}_X(nK_X))$ with a vanishing theorem, it is possible to compute $\dim H^0(X, \mathcal{O}_X(nK_X))$.

Reid-Flethcer formula for $\chi(\mathcal{O}_X(nK_X))$ is described as follows:

$$\chi(\mathcal{O}_X(nK_X)) = \frac{n(n-1)(2n-1)}{12}K_X^3 + (1-2n)\chi(\mathcal{O}_X) + \sum_q l(q,n),$$

where the summation is over a basket of singularities. Although singularities in a basket are not necessarily singularities in X, the singularities in X make the contribution as if they were in a basket. For detailed explanations about a basket of singularities, see Reid [6], Fletcher [1] or Kawamata [3]. The exact formula for $l(q,n)$ is as follows:

$$l(q,n) = \sum_{k=1}^{n-1} \frac{\overline{kb_q}(r_q - \overline{kb_q})}{2r_q},$$

where q is a singularity of type $\frac{1}{r_q}(1, -1, b_q)$, r_q and b_q are relatively prime, and $\overline{ib_q}$ is the nonnegative least residue of ib_q modulo r_q. To distinguish each point in a basket of singularities, we keep using a notation r_q, but for the sake of simplicity, we use a notation b instead of b_q without causing troubles.

We denote l.c.m of r_q in the baskets of singularities by r. Let's denote $\frac{n(n-1)(2n-1)}{12}K_X^3$ by K_n for the sake of simplicity.

The following proposition is a standard application of the Kawamata-Viehweg Vanishing Theorem.

Proposition 1. *For all $n \geq 2$,*

$$p_n \overset{\text{def}}{:=} \dim H^0(X, \mathcal{O}_X(nK_X)) = K_n + (1-2n)\chi(\mathcal{O}_X) + \sum_q l(q,n).$$

Lemma 1. *Let $(r,b) = 1$ and $0 < b < r$. Then*

$$\sum_{k=1}^{r} \overline{bk}(r - \overline{bk}) = \sum_{k=1}^{r-1} k(r-k) = \frac{r(r^2-1)}{6}.$$

Proof. The set $\{\overline{b}, \overline{2b}, \dots, \overline{rb}\}$ is the same as $\{0, 1, 2, \dots, r-1\}$ since $(r,b) = 1$. $\qquad\square$

In Reid [6] or Fletcher [1], R. Barlow's work is given as follows:

$$\rho^* K_X \cdot c_2(Y) = \sum \frac{r_q^2 - 1}{r_q} - 24\chi(\mathcal{O}_X),$$

where $\rho : Y \to X$ is a resolution of singularities of X. For the sake of simplicity, let's denote $\rho^* K_X \cdot c_2(Y)$ by $K_X \cdot c_2(X)$.

In [2], Hanamura induced the formula of the same type as in the following theorem, but our method is different and is a key to find the periodic behavior of the constant term in the formula.

Theorem 1. *Let X be a canonical threefold and let K_X be a canonical divisor of X. Then for $n \geq 2$,*

$$p_n = K_n + \frac{2n-1}{24} K_X \cdot c_2(X) \; + \; ct_s,$$

where $n = mr + s$, $0 \leq s < r$ and ct_s is a constant depending on s.

Proof. By Proposition 1 and R. Barlow's work, p_n is given as follows:

$$p_n = K_n + (1-2n)\chi(X, \mathcal{O}_X) + \sum_q l(q, n)$$

$$= K_n + (1-2n)\left[\frac{1}{24}\sum_q \frac{r_q^2 - 1}{r_q} - \frac{1}{24} K_X \cdot c_2(X)\right] + \sum_q l(q, n)$$

$$= K_n + \frac{2n-1}{24} K_X \cdot c_2(X) + \sum_q l(q, n) - \frac{2n-1}{24}\sum_q \frac{r_q^2 - 1}{r_q}.$$

The term $\sum_q l(q, n) - \frac{2n-1}{24}\sum_q \frac{r_q^2-1}{r_q}$ can be computed easily as follows:

$$\sum_q l(q, n) - \frac{2n-1}{24}\sum_q \frac{r_q^2 - 1}{r_q}$$

$$= \sum_q \sum_{k=1}^{n-1} \frac{\overline{bk}(r_q - \overline{bk})}{2r_q} - \frac{2n-1}{24}\sum_q \frac{r_q^2 - 1}{r_q}$$

$$= \sum_q \frac{1}{2r_q}\left[\sum_{k=1}^{n-1} \overline{bk}(r_q - \overline{bk}) - \frac{(2n-1)(r_q^2 - 1)}{12}\right].$$

Since r is a multiple of r_q, we let $r = h_q r_q$ and $n = mr + s = mh_q r_q + s$. When $s = 0$ or 1, we have

$$\sum_{k=1}^{n-1} \overline{bk}(r_q - \overline{bk}) - \frac{(2n-1)(r_q^2 - 1)}{12}$$

$$= \sum_{k=1}^{mh_q r_q} \overline{bk}(r_q - \overline{bk}) - \frac{2mh_q r_q(r_q^2 - 1)}{12} - \frac{(2s-1)(r_q^2 - 1)}{12}$$

$$= mh_q\left[\sum_{k=1}^{r_q - 1} \overline{bk}(r_q - \overline{bk}) - \frac{r_q(r_q^2 - 1)}{6}\right] - \frac{(2s-1)(r_q^2 - 1)}{12}.$$

By Lemma 1, $\sum_{k=1}^{r_q - 1} \overline{bk}(r_q - \overline{bk}) - \frac{r_q(r_q^2-1)}{6} = 0$. Thus,

$$\sum_{k=1}^{n-1} \overline{bk}(r_q - \overline{bk}) - \frac{(2n-1)(r_q^2 - 1)}{12} = -\frac{(2s-1)(r_q^2 - 1)}{12}.$$

When $s > 1$, we have

$$\sum_{k=1}^{n-1} \overline{bk}(r_q - \overline{bk}) - \frac{(2n-1)(r_q^2-1)}{12}$$

$$= \sum_{k=1}^{mh_q r_q} \overline{bk}(r_q - \overline{bk}) - \frac{2mh_q r_q(r_q^2-1)}{12} + \sum_{k=mh_q r_q+1}^{mh_q r_q+s-1} \overline{bk}(r_q - \overline{bk}) - \frac{(2s-1)(r_q^2-1)}{12}$$

$$= mh_q \left[\sum_{k=1}^{r_q-1} \overline{bk}(r_q - \overline{bk}) - \frac{r_q(r_q^2-1)}{6} \right] + \sum_{k=1}^{s-1} \overline{bk}(r_q - \overline{bk}) - \frac{(2s-1)(r_q^2-1)}{12}.$$

By Lemma 1, $\sum_{k=1}^{r_q-1} \overline{bk}(r_q - \overline{bk}) - \frac{r_q(r_q^2-1)}{6} = 0$. Thus,

$$\sum_{k=1}^{n-1} \overline{bk}(r_q - \overline{bk}) - \frac{(2n-1)(r_q^2-1)}{12} = \sum_{k=1}^{s-1} \overline{bk}(r_q - \overline{bk}) - \frac{(2s-1)(r_q^2-1)}{12}.$$

For the sake of simplicity, we may denote that the notation of the sum means a zero if $j < i$ in the summation notation $\sum_{k=i}^{j}$. Hence

$$\sum_q l(q,n) - \frac{2n-1}{24} \sum_q \frac{r_q^2-1}{r_q} = \sum_q \frac{1}{2r_q} \left[\sum_{k=1}^{s-1} \overline{bk}(r_q - \overline{bk}) - \frac{(2s-1)(r_q^2-1)}{12} \right].$$

Then we denote

$$\sum_q \frac{1}{2r_q} \left[\sum_{k=1}^{s-1} \overline{bk}(r_q - \overline{bk}) - \frac{(2s-1)(r_q^2-1)}{12} \right]$$

by ct_s.

Hence we have

$$p_n = K_n + \frac{2n-1}{24} K_X \cdot c_2(X) + ct_s,$$

where the term ct_s depends on s. $\square$

Corollary 1 (Formula for ct_s). *Under the same conditions and notations in Theorem 1, let $s = m_q r_q + s_q$, $(0 \leq s_q < r_q)$ for each point in a basket of singularities. Then we have*

$$ct_s = \sum_q \frac{1}{2r_q} \left[\sum_{k=1}^{s_q-1} \overline{bk}(r_q - \overline{bk}) - \frac{(2s_q-1)(r_q^2-1)}{12} \right].$$

Proof. As in the proof of Theorem 1, we apply the same procedure and notation to the term in ct_s which is computed as follows:

$$\sum_{k=1}^{s-1} \overline{bk}(r_q - \overline{bk}) - \frac{(2s-1)(r_q^2-1)}{12}$$

$$= \sum_{k=1}^{m_q r_q} \overline{bk}(r_q - \overline{bk}) - \frac{2m_q r_q(r_q^2 - 1)}{12} + \sum_{k=m_q r_q + 1}^{m_q r_q + s_q - 1} \overline{bk}(r_q - \overline{bk}) - \frac{(2s_q - 1)(r_q^2 - 1)}{12}$$

$$= m_q \left[\sum_{k=1}^{r_q - 1} \overline{bk}(r_q - \overline{bk}) - \frac{r_q(r_q^2 - 1)}{6} \right] + \sum_{k=1}^{s_q - 1} \overline{bk}(r_q - \overline{bk}) - \frac{(2s_q - 1)(r_q^2 - 1)}{12}$$

$$= \sum_{k=1}^{s_q - 1} \overline{bk}(r_q - \overline{bk}) - \frac{(2s_q - 1)(r_q^2 - 1)}{12}.$$

By Lemma 1, we have a formula for ct_s.   □

Remark 1. Recall that $K_X \cdot c_2(X)$ is positive (For a reference, see Miyaoka [4]). Thus, in order for p_n to be 0, ct_s must be negative.

By the periodic property (Theorem 1), we consider ct_{r+i} as the same as ct_i. Denote by $[x]$ the largest integer less than or equal to x.

Theorem 2 (Properties of ct_s). *Under the same conditions and notations in Theorem 1, we have the following*:

(1) $ct_0 = \sum \frac{r_q^2 - 1}{24 r_q}$. *In particular*, $ct_0 \geq 0$.
(2) $ct_0 = -ct_1$.
(3) $ct_s = -ct_{r-s+1}$ *when* $s \geq 2$.
(4) $ct_{[r/2]+1} = 0$ *when* r *is odd* ≥ 3.
(5) $ct_{r-1} \geq 0$ *if* r *is odd or even without the type* $\frac{1}{2}(1, -1, 1)$.
(6) $\chi(\mathcal{O}_X) < ct_0$.

Proof. Since r is the l.c.m of r_q, the cases (1) and (2) both come directly from a formula of ct_s in the proof of Theorem 1. Thus, we may assume $s \geq 2$. Recall that $s = m_q r_q + s_q$ $(0 \leq s_q < r_q)$.

The constant ct_s is given as follows:

$$ct_s = \sum_q \frac{1}{2r_q} \left[\sum_{k=1}^{s_q - 1} \overline{bk}(r_q - \overline{bk}) - \frac{(2s_q - 1)(r_q^2 - 1)}{12} \right].$$

Let's denote $\sum_{k=1}^{s_q-1} \overline{bk}(r_q - \overline{bk}) - \frac{(2s_q-1)(r_q^2-1)}{12}$ by $ct_{s,q}$. Then

$$ct_s = \sum_q \frac{1}{2r_q} ct_{s,q}.$$

If we prove $ct_{s,q} = -ct_{r-s+1,q}$, then our proof for (3) is complete.

If $s_q = 0$ or 1, then $ct_{s,q} = -ct_{r-s+1,q}$ clearly. Thus we may assume $s_q \geq 2$. Then $r_q - s_q + 1$ is the nonnegative least residue of $r - s + 1$ modulo r_q.

$$ct_{r-s+1,q} = \sum_{k=1}^{r_q - s_q} \overline{bk}(r_q - \overline{bk}) - \frac{(2r_q - 2s_q + 2 - 1)(r_q^2 - 1)}{12}$$

$$= \sum_{k=1}^{r_q-1} \overline{bk}(r_q - \overline{bk}) - \sum_{r_q-s_q+1}^{r_q-1} \overline{bk}(r_q - \overline{bk}) - \frac{(2r_q - 2s_q + 1)(r_q^2 - 1)}{12}$$

$$= \frac{r_q(r_q^2 - 1)}{6} - \sum_{r_q-s_q+1}^{r_q-1} \overline{bk}(r_q - \overline{bk}) - \frac{(2r_q - 2s_q + 1)(r_q^2 - 1)}{12}$$

$$= - \sum_{r_q-s_q+1}^{r_q-1} \overline{bk}(r_q - \overline{bk}) + \frac{(2s_q - 1)(r_q^2 - 1)}{12}.$$

The sum $\sum_{r_q-s_q+1}^{r_q-1} \overline{bk}(r_q - \overline{bk})$ can be computed as follows:

$$\sum_{r_q-s_q+1}^{r_q-1} \overline{bk}(r_q - \overline{bk})$$

$$= \overline{b(r_q - s_q + 1)}(r_q - \overline{b(r_q - s_q + 1)}) + \cdots + \overline{b(r_q - 1)}(r_q - \overline{b(r_q - 1)})$$

$$= \overline{-b(s_q - 1)}(r_q - \overline{-b(s_q - 1)}) + \cdots + \overline{-b}(r_q - \overline{-b})$$

$$= (r_q - \overline{b(s_q - 1)})(r_q - (r_q - \overline{b(s_q - 1)})) + \cdots + (r_q - \overline{b})(r_q - (r_q - \overline{b}))$$

$$= \overline{b(s_q - 1)}(r_q - \overline{b(s_q - 1)}) + \cdots + \overline{b}(r_q - \overline{b})$$

$$= \sum_{k=1}^{s_q-1} \overline{bk}(r_q - \overline{bk}).$$

Therefore,

$$ct_{r-s+1,q} = - \sum_{r_q-s_q+1}^{r_q-1} \overline{bk}(r_q - \overline{bk}) + \frac{(2s_q - 1)(r_q^2 - 1)}{12}$$

$$= - \sum_{k=1}^{s_q-1} \overline{bk}(r_q - \overline{bk}) + \frac{(2s_q - 1)(r_q^2 - 1)}{12}$$

$$= -ct_{s,q}.$$

So the property (3) is proved.

For a proof of the property (4), $ct_{[r/2]+1} = -ct_{r-[r/2]}$ by the property (3). Since r is odd, $[r/2] + 1 = r - [r/2]$. Hence $ct_{[r/2]+1} = 0$.

For a proof of the property (5), $ct_{r-1} = -ct_2$ by the property (3). It is enough to prove $ct_2 \leq 0$.

$$ct_2 = \sum_q \frac{1}{2r_q} \left(b(r_q - b) - \frac{r_q^2 - 1}{4} \right)$$

$$= \sum_q \frac{1}{2r_q} \left(\frac{4br_q - 4b^2 - r_q^2 + 1}{4} \right)$$

$$= \sum_q \frac{1}{2r_q} \left(\frac{1 - (r_q - 2b)^2}{4} \right).$$

$1 - (r - 2b)^2 \leq 0$ since r is odd or even without the type $\frac{1}{2}(1, -1, 1)$.

For a proof of the property (6), as in the proof of Theorem 1,

$$\chi(X, \mathcal{O}_X) = \frac{1}{24} \sum_q \frac{r_q^2 - 1}{r_q} - \frac{1}{24} K_X \cdot c_2(X).$$

Since $\frac{1}{24} K_X \cdot c_2(X) > 0$ by Miyaoka inequality and $ct_0 = \sum_q \frac{r_q^2 - 1}{24 r_q}$, we have $\chi(X, \mathcal{O}_X) < ct_0$ (For a reference, see Miyaoka [4]). □

In the next theorem, we show an application of these periodic properties of constant terms ct_s. Some of the following results may be already known, but we may have the same results very easily using these periodic properties.

Theorem 3. *Under the same conditions and notations in Theorem 1, we have the following*:

  (1) $p_{[r/2]+1} \geq 1$ *when r is odd ≥ 3.*
  (2) $p_{r-1} \geq 1$ *if r is odd or even without the type $\frac{1}{2}(1, -1, 1)$.*
  (3) *For $r \geq 3$ we have the following*:

$$1 \leq p_{r-[r/2]} + p_{[r/2]+1} < p_{r-[r/2]-1} + p_{[r/2]+2} < \cdots < p_2 + p_{r-1}.$$

   Thus, $p_n \geq 1$ for more than half of $\{2, \ldots, r-1\}$.
     Moreover, if $r \geq 7$, then $3 \leq p_{r-[r/2]} + p_{[r/2]+1}$. Thus, $p_n \geq 2$ for more than half of $\{2, \ldots, r-1\}$.
  (4) $p_r \geq 1$. *Moreover, $p_r \geq 2$ with the following possible exceptions*:

   (i) $K_X^3 = 1/2$, $K_X \cdot c_2 = 9/2$, $\chi(\mathcal{O}_X) = 0$, $r = 2$

$$\mathcal{B} = \left\{ \frac{1}{2}(1, -1, 1) \times 3 \right\}, \ p_2 = 1, \ p_n \geq 2 \ (n \geq 3),$$

   (ii) $K_X^3 = 1$, $K_X \cdot c_2 = 3$, $\chi(\mathcal{O}_X) = 0$, $r = 2$

$$\mathcal{B} = \left\{ \frac{1}{2}(1, -1, 1) \times 2 \right\}, \ p_2 = 1, \ p_n \geq 2 \ (n \geq 3),$$

   (iii) $K_X^3 = 3/2$, $K_X \cdot c_2 = 3/2$, $\chi(\mathcal{O}_X) = 0$, $r = 2$

$$\mathcal{B} = \left\{ \frac{1}{2}(1, -1, 1) \right\}, \ p_2 = 1, \ p_n \geq 2 \ (n \geq 3),$$

   where $\frac{1}{2}(1, -1, 1) \times n$ means n points of type $\frac{1}{2}(1, -1, 1)$.
  (5) $p_n \geq 2$ *for $n > r + 1$.*

Proof. For a proof of (1), $ct_{[r/2]+1} = 0$ by (4) of Theorem 2 when r is odd ≥ 3. $p_{[r/2]+1} \geq 1$ by Theorem 1 since $K_X \cdot c_2(X) > 0$.

For a proof of (2), recall $ct_{r-1} \geq 0$ if r is odd or even without type $\frac{1}{2}(1, -1, 1)$. Thus $p_{r-1} \geq 1$ by Theorem 1.

For a proof of (3), recall $ct_s = -ct_{r-s+1}$ for $s \geq 2$ by (3) of Theorem 2. For s ($[r/2] + 1 \leq s \leq r - 1$), add up $p_{r-s+1} + p_s$. Then we have

$$p_{r-s+1} + p_s = \frac{[(r+1)(2r+1) + 6s(s-r-1)]}{12} r K_X^3 + \frac{rK_X \cdot c_2(X)}{12}.$$

In the above expression, the minimum of term $6s(s - r - 1)$ occurs at $s = \frac{r+1}{2}$. Thus, $p_{r-s+1} + p_s$ is strictly increasing from $s = [r/2] + 1$ to $s = r - 1$. Now, we have

$$p_{r-s+1} + p_s > \frac{r^2 - 1}{24},$$

since $K_X \cdot c_2(X) > 0$ and rK_X^3 is a positive integer. $p_{r-s+1} + p_s \geq 1$ and moreover, greater than 2 if $r \geq 7$. Thus, a proof for (3) is complete.

For a proof of (4), $p_r \geq 1$ since $ct_0 \geq 0$. For the second part, we are going to consider the following three cases:

$$\text{(i) } \chi(\mathcal{O}_X) < 0, \text{ (ii) } \chi(\mathcal{O}_X) > 0, \text{ (iii) } \chi(\mathcal{O}_X) = 0.$$

If $\chi(\mathcal{O}_X) < 0$, then $p_n \geq 2$ for $n \geq 2$ by Reid-Fletcher formula.

If $\chi(\mathcal{O}_X) > 0$, then $p_r \geq 2$ since $ct_0 > \chi(\mathcal{O}_X) \geq 1$ by (6) of Theorem 2.

In the case of $\chi(\mathcal{O}_X) = 0$, $ct_0 = \frac{1}{24} K_X \cdot c_2(X)$ since $\chi(X, \mathcal{O}_X) = ct_0 - \frac{1}{24} K_X \cdot c_2(X)$ by combining the Barlow's work and (1) of Theorem 2. Thus,

$$p_r = \frac{(r-1)(2r-1)}{12} r K_X^3 + 2rct_0.$$

If $r \geq 4$, then $\frac{(r-1)(2r-1)}{12} > 1$. Thus, $p_r \geq 2$ since $ct_0 > 0$.

If $r = 3$, then the basket of singularities must contain points of type $\frac{1}{3}(1, -1, b)$ only. Thus we have $ct_0 \geq \frac{1}{9}$ by (1) of Theorem 2 and $p_3 \geq \frac{5}{6} + \frac{2}{3}$ by the formula of p_r given above. Thus $p_3 \geq 2$.

Hence it is enough to consider the case $r = 2$. The basket of singularities must contain points of type $\frac{1}{2}(1, -1, 1)$ only. Thus, by (1) of Theorem 2,

$$ct_0 = \frac{1}{16} \#,$$

where $\#$ is the number of points in the basket of singularities. Then, by the formula of p_r given above, we have

$$p_2 = \frac{K_X^3}{2} + \frac{1}{4} \#.$$

If $\# \geq 4$, then $p_2 \geq 2$. The only remaining cases $p_2 = 1$ can occur when $\# \leq 3$. Recall $p_2 \geq 1$ since $ct_0 > 0$. By computing p_2 according to the cases $\# = 1, 2, 3$, we have three possible exceptions which are described in the statement of Theorem 3.

For a proof of (5), if X is one of the possible exceptions in (4), then p_n is positive and strictly increasing. Thus we may assume $p_r \geq 2$.

Let $n = qr + s$ $(0 \leq s < r)$. We consider the following three cases:

$$\text{(i)}\ s = 0, \quad \text{(ii)}\ s \geq 2, \quad \text{(iii)}\ s = 1.$$

If $s = 0$, then n is a multiple of r, i.e., $n = qr$ $(q \geq 2)$ since $n > r + 1$. Since $p_r \geq 2$, $p_n \geq 2$ clearly (For a reference, see [5, Theorem 3.4.15, p. 258]).

Let's prove the case $s \geq 2$. For this case, it is enough to prove $p_{r+s} \geq 2$. The reason is as follows: if $p_{r+s} \geq 2$ and $q \geq 2$, then $p_n \geq p_{(q-1)r} + p_{r+s} - 1 \geq 2$ (For a reference, see [5, Theorem 3.4.15, p. 258]).

We have proved $p_s + p_{r-s+1} \geq 1$ in (3). Compute the following:

$$p_{r+s} - p_s - (p_s + p_{r-s+1}) = r^2(s - 1/2)K_X^3 > 0.$$

Thus, $p_{r+s} > p_s + p_{r-s+1} \geq 1$. Hence we prove $p_{r+s} \geq 2$.

Now, we are going to prove the case $s = 1$. Similarly, for this case, it is enough to prove $p_{2r+1} \geq 2$. If $p_{2r+1} \geq 2$, then $p_{qr+1} \geq p_{2r+1}$ clearly for $q \geq 3$ since $p_{qr+1} \geq p_{(q-2)r} + p_{2r+1} - 1$.

Choose k such that $2 \leq r - k \leq r - 1$ and $p_{r-k} \geq 1$. This is possible because $p_t \neq 0$ for more than half of $t \in \{2, \ldots, r-1\}$. We proved $p_{r+k+1} \geq 2$ just before since $2 \leq k + 1 \leq r - 1$. Thus,

$$p_{2r+1} \geq p_{r-k} + p_{r+k+1} - 1 \geq 2.$$

A proof for (5) is complete.   □

Remark 2. In Theorem 3, we described the numerical data of possible exceptions. But it does not imply the existence of canonical threefolds with given numerical data. Those exceptional cases may or may not exist.

Remark 3. We have no example which shows $p_{r+1} = 0$ yet.

References

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