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DUALS OF ANN-CATEGORIES

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Abstract. Dual monoidal category $\mathcal{C}^*$ of a monoidal functor $F : \mathcal{C} \to \mathcal{V}$ has been constructed by S. Majid. In this paper, we extend the construction of dual structures for an Ann-functor $F : \mathcal{B} \to \mathcal{A}$. In particular, when $F = \mathrm{id}_{\mathcal{A}}$, then the dual category $\mathcal{A}^*$ is indeed the center of $\mathcal{A}$ and this is a braided Ann-category.

1. Introduction

Categories with quasi-symmetry appeared under the heading "braided monoidal categories" in a connection with low dimensional topology [5], as well as in the context of quantum groups [6].

The concept "*dual of monoidal category*" appeared in [8] in the following case. The Hopf algebra can be built via a monoidal category $\mathcal{C}$ and a functor $F : \mathcal{C} \to \mathbf{Vec}$. This event can be generalized as **Vec** is replaced by a monoidal category $\mathcal{V}$. Now, if F is a monoidal functor, then $\mathcal{C}$ is called *functored* on $\mathcal{V}$, or $(\mathcal{C}, F)$ is called a $\mathcal{V}$-*category* in A. Grothendieck's terminology [4]. In this situation, S. Majid built the monoidal category $(\mathcal{C}, F)^* = (\mathcal{C}^*, F^*)$, named "*full dual category*" of $(\mathcal{C}, F)$. The objects of $(\mathcal{C}, F)^*$ are pairs (V, u_V), consisting of $V \in \mathcal{C}$ and a natural transformation $u_V = (u_{V,X} : V \otimes FX \to FX \otimes V)$ satisfying the compatition with the monoidal functor $(F, \widetilde{F})$. The full subcategory $(\mathcal{C}, F)^\circ$ consists of objects (V, u_V) where $u_{V,X}$ are isomorphisms. It is interesting when $\mathcal{V} = \mathcal{C}$ and $F = \mathrm{id}$, then $(\mathcal{C}, F)^\circ$ is a braided monoidal category, called the *center* $Z(\mathcal{C})$ of the monoidal category $\mathcal{C}$.

The notion of the center of a monoidal category appeared first in [5], [8]. It was a construction of a braided tensor category from an arbitrary tensor category. Then, the center of a category appears as a tool to study categorical groups [1] and graded categorical groups [3].

The detail proofs of the construction of $(\mathcal{C}, F)^*$ have showed in [9]. Concurrently, in [9], S. Majid enriched the results of the dual categories and established links between dual categories and braided groups.

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Monoidal categories were considered in a more general situation due to M. Laplaza with the name *distributivity category* [7]. After, A. Fröhllich and C. T. C. Wall [2] presented the concept of *ring-like category*. These two concepts are categorifications of the concept of commutative rings, as well as a generalization of the category of modules over a commutative ring R.

In order to have descriptions of structures, and a cohomological classification, N. T. Quang [11] has introduced the concept of *Ann-categories*, as a categorification of the concept of rings, with requirements of invertibility of objects and morphisms of the under-lying category, similar to those of categorical groups (see [1, 3]). The braiding of an Ann-category is considered in [13]. The relationship between (braided) Ann-categories and the categories mentioned above is presented in [13] ([10]). It was proved in [13] that for each Ann-category $\mathcal{A}$, one may construct a braided Ann-category, which is called the center of $\mathcal{A}$. This result extends the construction of the center of a monoidal category, given by A. Joyal and R. Street [5]. This motivates the paper, which is to construct the dual Ann-category of an arbitrary Ann-functor (see Section 3), extending Majid's construction. It is a new construction, differing from the construction of an Ann-category for a regular homomorphism in ring extension problem. Moreover, since the center of an Ann-category $\mathcal{A}$ is a dual in $\mathcal{A}$, there always exist braided Ann-categories in dual categories of an Ann-category.

In this paper, we sometimes denote by XY the tensor product of two objects X, Y instead of $X \otimes Y$.

2. Some basic definitions

Definition 2.1 ([11]). An *Ann-category* consists of:

(i) Category $\mathcal{A}$ together with two bifunctors $\oplus, \otimes : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$.

(ii) A fixed object $O \in \mathcal{A}$ together with naturality constraints a^+, c^+, g, d such that $(\mathcal{A}, \oplus, a^+, c^+, (O, g, d))$ is a symmetric categorical group.

(iii) A fixed object $I \in \mathcal{A}$ together with naturality constraints a, l, r such that $(\mathcal{A}, \otimes, a, (I, l, r))$ is a monoidal A-category.

(iv) Natural isomorphisms $\mathfrak{L}, \mathfrak{R}$

$$\begin{array}{llll} \mathfrak{L}_{A,X,Y}: & A \otimes (X \oplus Y) & \to & (A \otimes X) \oplus (A \otimes Y), \\ \mathfrak{R}_{X,Y,A}: & (X \oplus Y) \otimes A & \to & (X \otimes A) \oplus (Y \otimes A), \end{array}$$

such that the following conditions are satisfied:

(Ann-1) For each $A \in \mathcal{A}$, the pairs $(L^A, \breve{L}^A), (R^A, \breve{R}^A)$ defined by relations:

$$\begin{array}{ll} L^A = A \otimes -, & R^A = - \otimes A, \\ \breve{L}^A_{X,Y} = \mathfrak{L}_{A,X,Y}, & \breve{R}^A_{X,Y} = \mathfrak{R}_{X,Y,A} \end{array}$$

are $\oplus$-functors which are compatible with a^+ and c^+.

(Ann-2) The following diagrams commute for all objects $A, B, X, Y \in \mathcal{A}$:

$$\begin{array}{ccccc}
(AB)(X\oplus Y) & \xleftarrow{a_{A,B,X\oplus Y}} & A(B(X\oplus Y)) & \xrightarrow{\mathrm{id}_A\otimes \breve{L}^B} & A(BX\oplus BY)\\
\Big\downarrow{\scriptstyle \breve{L}^{AB}} & & & & \Big\downarrow{\scriptstyle \breve{L}^{A}}\\
(AB)X\oplus (AB)Y & & \xleftarrow{\quad a_{A,B,X}\oplus a_{A,B,Y}\quad} & & A(BX)\oplus A(BY)
\end{array}$$

$$\begin{array}{ccccc}
(X\oplus Y)(BA) & \xrightarrow{a_{X\oplus Y,B,A}} & ((X\oplus Y)B)A & \xrightarrow{\breve{R}^B\otimes \mathrm{id}_A} & (XB\oplus YB)A\\
\Big\downarrow{\scriptstyle \breve{R}^{BA}} & & & & \Big\downarrow{\scriptstyle \breve{R}^{A}}\\
X(BA)\oplus Y(BA) & & \xrightarrow{\quad a_{X,B,A}\oplus a_{Y,B,A}\quad} & & (XB)A\oplus (YB)A
\end{array}$$

$$\begin{array}{ccccc}
(A(X\oplus Y))B & \xleftarrow{a_{A,X\oplus Y,B}} & A((X\oplus Y)B) & \xrightarrow{\mathrm{id}_A\otimes \breve{R}^B} & A(XB\oplus YB)\\
\Big\downarrow{\scriptstyle \breve{L}^{A}\otimes \mathrm{id}_B} & & & & \Big\downarrow{\scriptstyle \breve{L}^{A}}\\
(AX\oplus AY)B & \xrightarrow{\breve{R}^B} & (AX)B\oplus (AY)B & \xleftarrow{a\,\oplus\, a} & A(XB)\oplus A(YB)
\end{array}$$

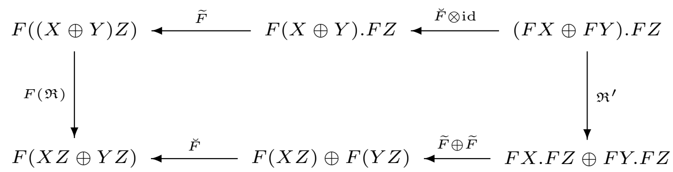
$$\begin{array}{ccccc}
(A\oplus B)X\oplus (A\oplus B)Y & \xleftarrow{\breve{L}} & (A\oplus B)(X\oplus Y) & \xrightarrow{\breve{R}} & A(X\oplus Y)\oplus B(X\oplus Y)\\
\Big\downarrow{\scriptstyle \breve{R}^{X}\oplus \breve{R}^{Y}} & & & & \Big\downarrow{\scriptstyle \breve{L}^{A}\oplus \breve{L}^{B}}\\
(AX\oplus BX)\oplus (AY\oplus BY) & & \xrightarrow{\quad \mathrm{v}\quad} & & (AX\oplus AY)\oplus (BX\oplus BY)
\end{array}$$

where $v = v_{U,V,Z,T} : (U \oplus V) \oplus (Z \oplus T) \to (U \oplus Z) \oplus (V \oplus T)$ is the unique morphism built from a^+, c^+, id in the symmetric monoidal category $(\mathcal{A}, \oplus)$.
(Ann-3) For the unit object $I \in \mathcal{A}$ of the operation $\otimes$, we have the following relations for all objects $X, Y \in \mathcal{A}$:

$$l_{X\oplus Y} = (l_X \oplus l_Y) \circ \breve{L}^I_{X,Y}, \quad r_{X\oplus Y} = (r_X \oplus r_Y) \circ \breve{R}^I_{X,Y}.$$

Definition 2.2. Let $\mathcal{A}$ and $\mathcal{A}'$ be Ann-categories. An *Ann-functor* from $\mathcal{A}$ to $\mathcal{A}'$ is a triple $(F, \breve{F}, \widetilde{F})$, where $(F, \breve{F})$ is a symmetric monoidal functor respect to the operation $\oplus$, $(F, \widetilde{F})$ is an A-functor (i.e., an associativity functor) respect to the operation $\otimes$, satisfying the two following commutative diagrams for all $X, Y, Z \in Ob(\mathcal{A})$:

$$\begin{array}{ccccc}
F(X(Y\oplus Z)) & \xleftarrow{\widetilde{F}} & FX.F(Y\oplus Z) & \xleftarrow{\mathrm{id}\otimes \breve{F}} & FX(FY\oplus FZ)\\
\Big\downarrow{\scriptstyle F(\mathfrak{L})} & & & & \Big\downarrow{\scriptstyle \mathfrak{L}'}\\
F(XY\oplus XZ) & \xleftarrow{\breve{F}} & F(XY)\oplus F(XZ) & \xleftarrow{\widetilde{F}\oplus\widetilde{F}} & FX.FY\oplus FX.FZ
\end{array}$$



Definition 2.3. A *braided Ann-category* $\mathcal{A}$ is an Ann-category $\mathcal{A}$ together with a braid c such that $(\mathcal{A}, \otimes, a, c, (I, l, r))$ is a braided tensor category, concurrently c satisfies the following relation:

$$(c_{A,X} \oplus c_{A,Y}) \circ \breve{L}^A_{X,Y} = \breve{R}^A_{X,Y} \circ c_{A,X\oplus Y},$$

and the condition $c_{O,O} = \mathrm{id}$.

Let us recall a result which has been known of an Ann-category.

Proposition 2.4 ([11, Proposition 3.1]). *In the Ann-category $\mathcal{A}$, there exist uniquely the isomorphisms*:

$$\hat{L}^A : A \otimes O \to A, \qquad \hat{R}^A : O \otimes A \to A$$

such that $(L^A, \breve{L}^A, \hat{L}^A)$, $(R^A, \breve{R}^A, \hat{R}^A)$ are the functors which are compatible with the unit constraints of the operator $\oplus$ (also called U-functors).

3. Duals of Ann-categories

In this section, we shall build *duals of Ann-categories* based on the construction of duals of monoidal categories by S. Majid [8].

Let $\mathcal{A}$ be an Ann-category. An Ann-category $\mathcal{B}$ is *functored* over $\mathcal{A}$ if there is an Ann-functor $F : \mathcal{B} \to \mathcal{A}$.

First, let us recall that an Ann-category is called *almost strict* if all its natural constraints, except for the commutativity constraint and the left distributivity constraint, are identities. Each Ann-category is Ann-equivalent to an almost strict Ann-category of the type (R, M) (see [12]). In this category, for each $A \in Ob(\mathcal{A})$, there exists an object $A' \in Ob(\mathcal{A})$ such that

$$A \oplus A' = O. \tag{1}$$

So, hereafter, we always assume that $\mathcal{A}$ is an almost strict Ann-category and satisfies the condition (1) and the Ann-functor $F : \mathcal{B} \to \mathcal{A}$ satisfies the conditions $F(O) = O, F(I) = I$.

Definition 3.1. Let $\mathcal{A}$ be an Ann-category. Let $(\mathcal{B}, F)$ be a functored Ann-category over $\mathcal{A}$. A right $(\mathcal{B}, F)$-module is a pair (A, u_A) consisting of an object A in $\mathcal{A}$ and a natural transformation $u_{A,X} : A \otimes F(X) \to F(X) \otimes A$ such that $u_{A,I} = \mathrm{id}$ and the following diagrams commute:

$$(2)\qquad \begin{array}{ccccc} A \otimes (FX \oplus FY) & \xrightarrow{\breve{L}^{A}_{FX,FY}} & (A \otimes FX) \oplus (A \otimes FY) & \xrightarrow{u_{A,X} \oplus u_{A,Y}} & (FX \otimes A) \oplus (FY \otimes A) \\ \downarrow{\scriptstyle \mathrm{id} \otimes \breve{F}} & & & & \downarrow{\scriptstyle \mathrm{id}} \\ A \otimes F(X \oplus Y) & \xrightarrow{u_{A,X \oplus Y}} & F(X \oplus Y) \otimes A & \xleftarrow{\breve{F} \otimes \mathrm{id}} & (FX \oplus FY) \otimes A \end{array}$$

$$(3)\qquad \begin{array}{ccccc} A \otimes (FX \otimes FY) & \xrightarrow{u_{A,X} \otimes \mathrm{id}} & FX \otimes A \otimes FY & \xrightarrow{\mathrm{id} \otimes u_{A,Y}} & FX \otimes FY \otimes A \\ \downarrow{\scriptstyle \mathrm{id} \otimes \widetilde{F}} & & & & \downarrow{\scriptstyle \widetilde{F} \otimes \mathrm{id}} \\ A \otimes F(X \otimes Y) & & \xrightarrow{\qquad u_{A,X \otimes Y} \qquad} & & F(X \otimes Y) \otimes A \end{array}$$

A morphism $f : (A, u_A) \to (B, u_B)$ between right $(\mathcal{B}, F)$-modules is a morphism $f : A \to B$ in $\mathcal{A}$ such that the following diagram commutes for all $X \in \mathcal{B}$:

$$(4)\qquad \begin{array}{ccc} A \otimes FX & \xrightarrow{u_{A,X}} & FX \otimes A \\ \downarrow{\scriptstyle f \otimes \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id} \otimes f} \\ B \otimes FX & \xrightarrow{u_{B,X}} & FX \otimes B \end{array}$$

Let $(\mathcal{B}, F)$ be a functored Ann-category over $\mathcal{A}$. We consider the category $\mathcal{B}^* = (\mathcal{B}, F)^*$ defined as follows. The objects of $\mathcal{B}^*$ are right $(\mathcal{B}, F)$-modules. The morphisms of $\mathcal{B}^*$ are morphisms between right $(\mathcal{B}, F)$-modules.

Now, we shall equip the operators and the structures for $\mathcal{B}^*$ so that $\mathcal{B}^*$ becomes an Ann-category.

Lemma 3.2. *For any two objects $(A, u_A), (B, u_B)$ in $\mathcal{B}^*$, $(A \oplus B, u_{A \oplus B})$ is an object of $\mathcal{B}^*$, where $u_{A \oplus B}$ is defined by:*

$$u_{A \oplus B, X} = \mathfrak{L}^{-1}_{FX,A,B} \circ (u_{A,X} \oplus u_{B,X}) \text{ for all } X \in \mathcal{A}.$$

Proof. Since $u_{A,I} = \mathrm{id}$, $u_{B,I} = \mathrm{id}$, $\mathfrak{L}_{FI,A,B} = \mathfrak{L}_{I,A,B} = \mathrm{id}$, we have $u_{A \oplus B, I} = \mathrm{id}$.

To prove that $u_{A \oplus B}$ satisfies the diagram (2), we consider the diagram (5) (see back page 33). In the diagram (5), the regions (I), (VII) commute thanks to the determination of $u_{A \oplus B}$, the region (II) commutes thanks to the naturality of $\mathfrak{R} = \mathrm{id}$, the regions (III), (VI) commute since $\mathcal{A}$ is an Ann-category, the region (V) commutes thanks to the naturality of $\mathfrak{L}$, the region (VIII) commutes thanks to the naturality of v, the perimeter commutes since $(A, u_A), (B, u_B)$ satisfy the diagram (2). Therefore, the region (IV) commutes, i.e., $(A \oplus B, u_{A \oplus B})$ satisfies the diagram (2).

To prove that $u_{A \oplus B}$ satisfies the diagram (3), we consider the diagram (6) (see back page 34). In the diagram (6), the regions (I), (II) commute thanks to the naturality of $\mathfrak{R} = \mathrm{id}$, the regions (III), (VI), (VIII) commute thanks to the determination of $u_{A\oplus B}$, the regions (IV), (X) commute since $\mathcal{A}$ is an Ann-category, the regions (VII), (IX) commute thanks to the naturality of $\mathfrak{L}$, the perimeter commutes thanks to u_A, u_B satisfy the diagram (3). Therefore, the region (V) commutes, i.e., $u_{A\oplus B}$ satisfies the diagram (3). So, $(A\oplus B, u_{A\oplus B})$ is an object of $\mathcal{B}^*$. □

By Lemma 3.2, we can determine the operator "+" of $\mathcal{B}^*$ where the sum of two objects is defined by

$$(A, u_A) + (B, u_B) = (A \oplus B, u_{A\oplus B}),$$

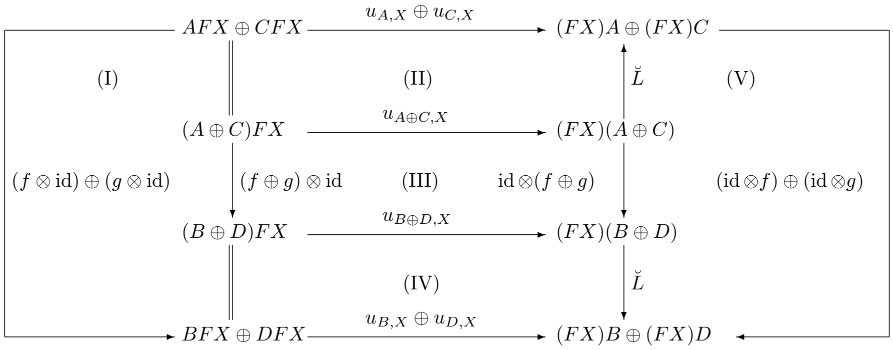
and the sum of two morphisms is the sum of morphisms in $\mathcal{A}$.

Proposition 3.3. *$\mathcal{B}^*$ is a symmetric categorical group where the associativity constraint is strict, the unit constraint is $((O, u_{O,X} = \hat{L}^{-1}_{FX}), \mathrm{id}, \mathrm{id})$, and the commutativity constraint is $c^+_{(A,u_A),(B,u_B)} = c^+_{A,B}$.*

Proof. Assume that $f : (A, u_A) \to (B, u_B)$ and $g : (C, u_C) \to (D, u_D)$ are two morphisms in the category $\mathcal{B}^*$. We shall prove that

$$f + g = f \oplus g$$

satisfies the diagram (4), so it is a morphism of $\mathcal{B}^*$. We consider the diagram:

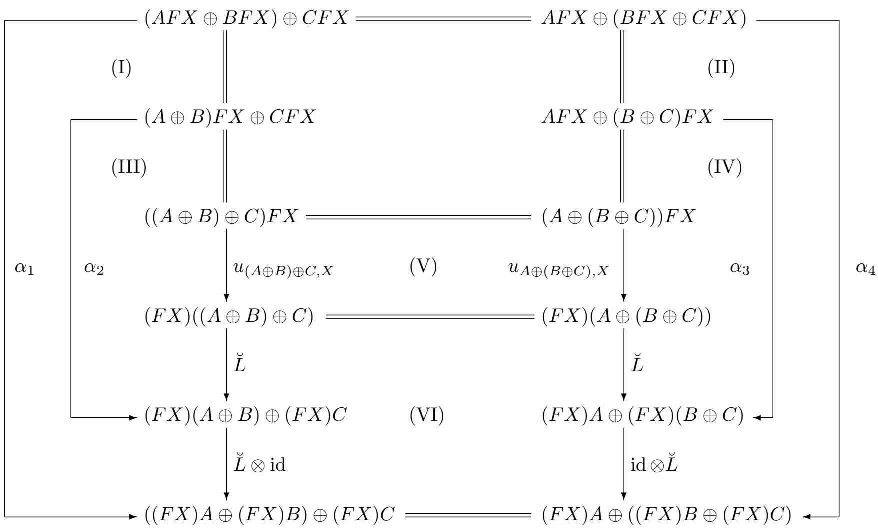


In this diagram, the region (I) commutes thanks to the naturality of $\mathfrak{R} = \mathrm{id}$, the region (II) commutes thanks to the determination of $u_{A\oplus C}$, the region (IV) commutes thanks to the determination of $u_{B\oplus D}$, the region (V) commutes thanks to the naturality of $\mathfrak{L}$; each component of the perimeter commutes since f and g are morphisms of $\mathcal{B}^*$. So, the perimeter commutes. Therefore, the region (III) commutes, i.e., $f + g = f \oplus g$ is a morphism of $\mathcal{B}^*$.

Next, we prove that $a^+ = \mathrm{id}$ is a morphism

$$((A, u_A) + (B, u_B)) + (C, u_C) \to (A, u_A) + ((B, u_B) + (C, u_C))$$

in $\mathcal{B}^*$. We consider the following diagram:



where $\alpha_1 = (u_{A,X} \oplus u_{B,X}) \oplus u_{C,X}$ $\qquad \alpha_2 = u_{A\oplus B,X} \oplus u_{C,X}$

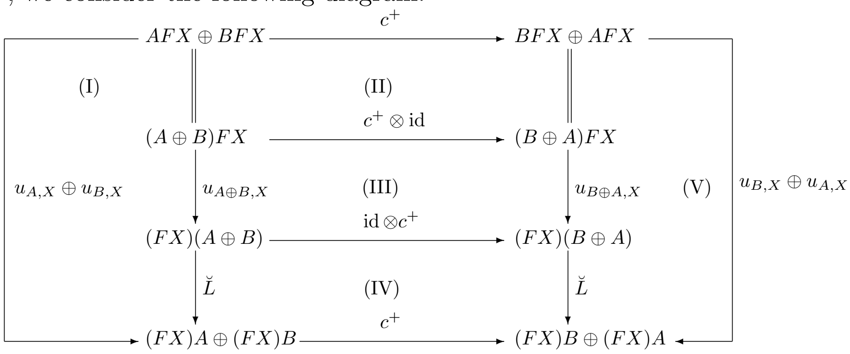
$\alpha_3 = u_{A,X} \oplus u_{B\oplus C,X}$ $\qquad \alpha_4 = u_{A,X} \oplus (u_{B,X} \oplus u_{C,X})$

In the above diagram, the region (I) commutes thanks to the determination of $u_{A\oplus B}$, the region (II) commutes thanks to the determination of $u_{B\oplus C}$, the region (III) commutes thanks to the determination of $u_{(A\oplus B)\oplus C}$, the region (IV) commutes thanks to the determination of $u_{A\oplus(B\oplus C)}$, the region (VI) commutes since $\mathcal{A}$ is an Ann-category, the perimeter commutes thanks to the naturality of $a^+ = \mathrm{id}$. Therefore, the region (V) commutes, i.e., $a^+ = \mathrm{id}$ is a morphism of $\mathcal{B}^*$.

To prove that c^+ is the morphism

$$(A, u_A) + (B, u_B) \to (B, u_B) + (A, u_A)$$

in $\mathcal{B}^*$, we consider the following diagram.



In this diagram, the region (I) commutes thanks to the determination of $u_{A\oplus B}$, the regions (II), (IV) commute since $\mathcal{A}$ is an Ann-category, the region (V) commutes thanks to the determination of $u_{B\oplus A}$, the perimeter commutes thanks to the naturality of c^+. Therefore, the region (III) commutes, i.e., c^+ is a morphism in $\mathcal{B}^*$.

One can verify that $((O, u_{O,X} = \hat{L}^{-1}_{FX}), \mathrm{id}, \mathrm{id})$ is the unit constraint of $\mathcal{B}^*$. Finally, we shall prove that each object of $\mathcal{B}^*$ is invertible.

Let (A, u_A) be an object of $\mathcal{B}^*$. By the condition (1), there exsits an object $A' \in Ob(\mathcal{A})$ such that

$$A \oplus A' = O.$$

The family of natural transformations $u_{A',X} : A' \otimes FX \to FX \otimes A'$ is defined by:

$$u_{A,X} \oplus u_{A',X} = \mathfrak{L}_{FX,A,A'} \circ u_{O,X}.$$

One can prove that $(A', u_{A'})$ is the invertible object of the object (A, u_A) in the category $\mathcal{B}^*$. □

Lemma 3.4. *For any two objects $(A, u_A), (B, u_B)$ of $\mathcal{B}^*$, $(A \otimes B, u_{A\otimes B})$ is an object of $\mathcal{B}^*$, where $u_{A\otimes B}$ is defined by:*

$$u_{A\otimes B,X} = (u_{A,X} \otimes \mathrm{id}_B) \circ (\mathrm{id}_A \otimes u_{B,X}) \textit{ for all } X \in \mathcal{A}.$$

Proof. Let $(A, u_A), (B, u_B)$ be two objects of $\mathcal{B}^*$. Since $u_{A,I} = \mathrm{id}$ and $u_{B,I} = \mathrm{id}$, we have $u_{A\otimes B,I} = \mathrm{id}$. Moreover, by [8, Theorem 3.3], $u_{A\otimes B}$ satisfies the diagram (3).

Finally, to prove that $u_{A\otimes B}$ satisfies the diagram (2), we consider the diagram (7) (see back page 35). In the diagram (7), the region (I) commutes since (B, u_B) satisfies the diagram (2), the regions (II), (VII) and (IX) commute thanks to the naturality of $a^+ = \mathrm{id}$, the region (III) commutes thanks to the naturality of $\mathfrak{L}$, the regions (IV), (XI) and the perimeter commutes since $\mathcal{A}$ is an Ann-category, the regions (VI), (VIII) commute thanks to the determination of u_{AB}, the region (X) commutes since (A, u_A) satisfies the diagram (2), the region (XII) commutes thanks to the naturality of $\mathfrak{R} = \mathrm{id}$. Therefore, the region (V) commutes, i.e., (AB, u_{AB}) satisfies the diagram (2). So $(A \otimes B, u_{A\otimes B})$ is an object of $\mathcal{B}^*$. □

By Lemma 3.4, we can determine the operator "×" of $\mathcal{B}^*$ where the product of two objects is defined by

$$(A, u_A) \times (B, u_B) = (A \otimes B, u_{A\otimes B}),$$

and the tensor product of two morphisms is the tensor product of two morphisms in $\mathcal{A}$.

Proposition 3.5. *$\mathcal{B}^*$ is a strict monoidal category.*

Proof. Assume that $f : (A, u_A) \to (B, u_B)$ and $g : (C, u_C) \to (D, u_D)$ are two morphisms in the category $\mathcal{B}^*$. By [8, Theorem 3.3], the morphism

$$f \times g = f \otimes g : (A, u_A) \times (C, u_C) \to (B, u_B) \times (D, u_D)$$

satisfies the diagram (4), i.e., $f \times g$ is a morphism in $\mathcal{B}^*$.

The composition of two morphisms in $\mathcal{B}^*$ is the normal composition. By [8, Theorem 3.3], $\mathcal{B}^*$ has the associativity constraint be strict. One can easily prove that (I, id) is an object in $\mathcal{B}^*$ and it together with the strict constraints $l = \mathrm{id}, r = \mathrm{id}$ is the unit constraint of the operator $\times$ in $\mathcal{B}^*$. □

Theorem 3.6. *$\mathcal{B}^*$ is an Ann-category with the distributivity constraints are given by*

$$\mathfrak{L}_{(A,u_A),(B,u_B),(C,u_C)} = \mathfrak{L}_{A,B,C},\ \mathfrak{R}_{(A,u_A),(B,u_B),(C,u_C)} = \mathrm{id}\,.$$

Proof. By Proposition 3.3, $(\mathcal{B}^*, +)$ is a symmetric categorical group. By Proposition 3.5, $(\mathcal{B}^*, \times)$ is a monoidal category. One can prove that

$$\mathfrak{L} : (A, u_A) \times ((B, u_B) + (C, u_C)) \to (A, u_A) \times (B, u_B) + (A, u_A) \times (C, u_C),$$

$$\mathfrak{R} = \mathrm{id} : ((A, u_A + (B, u_B)) \times (C, u_C) \to (A, u_A) \times (C, u_C) + (B, u_B) \times (C, u_C)$$

are morphisms in $\mathcal{B}^*$.

Moreover, the constraints $a^+ = \mathrm{id}, c^+, a = \mathrm{id}, \mathfrak{L}, \mathfrak{R} = \mathrm{id}$ of the Ann-category $\mathcal{A}$ satisfy the conditions (Ann-1), (Ann-2), (Ann-3), so, in the category $\mathcal{B}^*$, they also satisfy these conditions. Thus $\mathcal{B}^*$ is an Ann-category. □

The following proposition is obvious.

Proposition 3.7. *$\mathcal{B}^*$ is functored over $\mathcal{A}$ with the forgetful Ann-functor*

$$F^* : \mathcal{B}^* \to \mathcal{A}.$$

Example 1. The center of an Ann-category $\mathcal{A}$

Let $\mathcal{A}$ be an Ann-category. Let $\mathcal{B} = \mathcal{A}$ and $F = \mathrm{id}$. Then $\mathcal{B}^* = \mathcal{C}_{\mathcal{A}}$, where $\mathcal{C}_{\mathcal{A}}$ is the center of the Ann-category $\mathcal{A}$ which is built in [13]. This is a braided Ann-category with the quasi-symmetric

$$c_{(A,u_A),(B,u_B)} = u_{A,B} : A \otimes B \to B \otimes A.$$

Next, we shall apply above results to build the dual Ann-category of the pair $(\mathcal{B}, F)$, where $\mathcal{B} = (R', M', f')$, $\mathcal{A} = (R, M, f)$ are Ann-categories.

Example 2. Duals of an Ann-category of the type (R, M)

Let R be a ring and M be a R-bimodule. An Ann-category of the type (R, M) is a category $\mathcal{I}$ whose objects are elements of R, and whose morphisms are automorphisms, $(x, a) : x \to x,\ \forall a \in M$. The composition of morphisms is the addition in M. The two operators $\oplus$ and $\otimes$ of $\mathcal{I}$ are given by

$$x \oplus y = x + y, \quad (x, a) \oplus (y, b) = (x + y, a + b),$$

$$x \otimes y = x.y, \quad (x, a) \otimes (y, b) = (xy, xb + ay).$$

All constraints of $\mathcal{I}$ are strict, except for the left distributivity constraint and the commutativity constraint given by

$$\begin{aligned} \mathfrak{L}_{x,y,z} &= (\bullet, \lambda(x, y, z)) : x(y + z) \to xy + xz, \\ c^+_{x,y} &= (\bullet, \eta(x, y)) : x + y \to y + x, \end{aligned}$$

where $\lambda : R^3 \to M, \eta : R^2 \to M$ are functions satisfying the some certain coherence conditions (for detail, see [12]).

Let $\mathcal{A}$ be an almost strict Ann-category of the type (R, M) and $\mathcal{B}$ be an almost strict Ann-category of the type (R', M'). Let $(F, \breve{F}, \widetilde{F}) : \mathcal{B} \to \mathcal{A}$ be an Ann-functor. Then, by [14, Theorem 4.3], F is a functor of the type (p, q), i.e.,

$$F(x) = p(x), F(x, a) = (p(x), q(a)),$$

where $p : R' \to R$ is a ring homomorphism and $q : M' \to M$ is a group homomorphism and

$$q(xa) = p(x)q(a), \quad q(ax) = q(a)p(x) \text{ for all } x \in R, a \in M.$$

Moreover, $\breve{F}, \widetilde{F}$ are associated, respectively, to μ, ν which satisfy some certain coherence conditions (for detail, see [14, Theorem 4.4]).

According to the above steps, each object of $\mathcal{B}^*$ is a pair (r, u_r), where r is in the centerization of Im$p = p(R')$ in the ring R, (i.e., $rp(x) = p(x)r \ \forall x \in R'$) and $u_r : R' \to M$ is a function satisfying the condition $u_{r,1} = 0$ and the two following conditions for all $x, y \in R'$:

$$\begin{aligned} u(r, x) - u(r, x + y) + u(r, y) &= \mu(x, y)r + r\mu(x, y) - \lambda(r, px, py), \\ xu(r, y) - u(r, xy) + u(r, x)y &= r\nu(x, y) - \nu(x, y)r. \end{aligned}$$

We now describe a morphism $f : (r, u_r) \to (s, u_s)$ of $\mathcal{B}^*$. Since $f : r \to s$ is a morphism in the Ann-category $\mathcal{A}$, $s = r$, and $f = (r, a)$ with $a \in M$.

From the commutation of the diagram (4), we have

$$p(x)a = ap(x) \text{ for all } x \in R'.$$

Now, $\mathcal{B}^*$ is an Ann-category with the two operators given by

$$\begin{aligned} (r, u_r) + (s, u_s) &= (r + s, u_{r+s}), \\ (r, u_r) \times (s, u_s) &= (rs, u_{rs}), \end{aligned}$$

where

$$\begin{aligned} u_{r+s,x} &= u_{r,x} + u_{s,x} - \lambda(px, r, s), \\ u_{rs,x} &= u_{r,x}s + r.u_{s,x}, \end{aligned}$$

and $f + g = f \oplus g$, $f \times g = f \otimes g$ where $f : (r, u_r) \to (r, u_r), g : (s, u_s) \to (s, u_s)$.

All constraints of $\mathcal{B}^*$ are strict, except for the commutativity constraint and the left distributivity constraint given by

$$\begin{aligned} c^+_{(r,u_r),(s,u_s)} &= c^+_{r,s} = (\bullet, \eta(r, s)), \\ \mathfrak{L}_{(r,u_r),(s,u_s),(t,u_t)} &= \mathfrak{L}_{r,s,t} = (\bullet, \lambda(r, s, t)). \end{aligned}$$

The invertible object of the object (r, u_r) respect to the operator $+$ is $(-r, u_{-r})$, where $-r$ is the opposite element of r in the group $(R, +)$ and $u_{-r} : R' \to M$ is given by:

$$u_{-r,x} = \lambda(px, r, -r) - u_{r,x}.$$

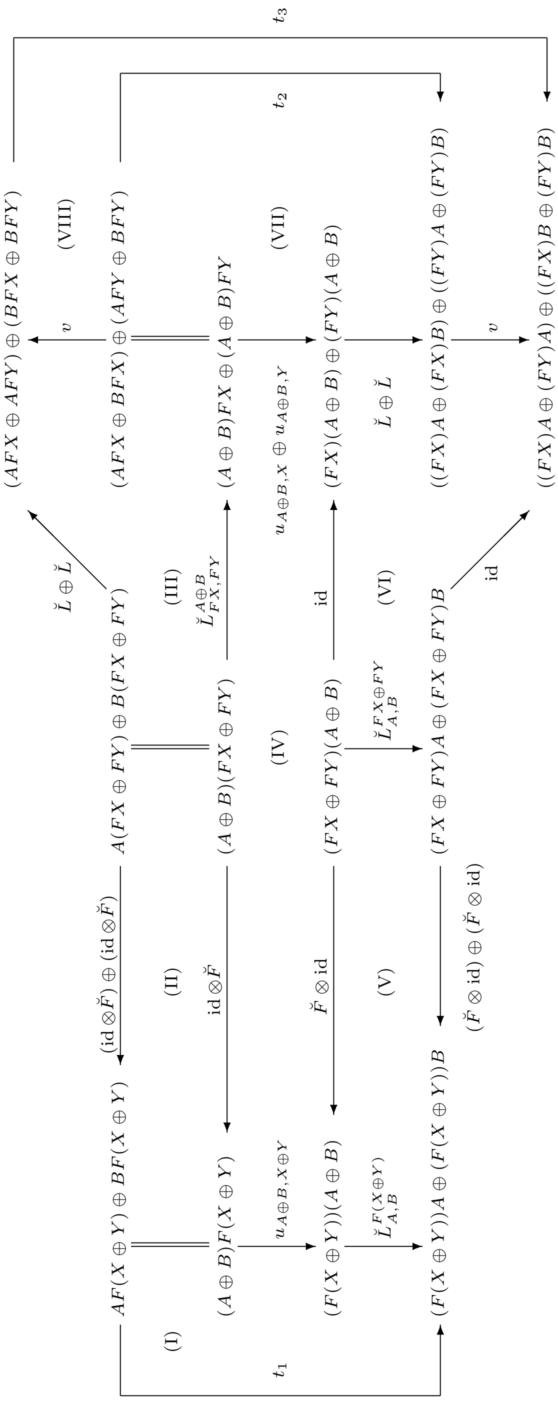


Diagram (5)

where $t_1 = u_{A,X\oplus Y} \oplus u_{B,X\oplus Y}$

$t_2 = (u_{A,X} \oplus u_{B,X}) \oplus (u_{A,Y} \oplus u_{B,Y})$

$t_3 = (u_{A,X} \oplus u_{A,Y}) \oplus (u_{B,X} \oplus u_{B,Y})$

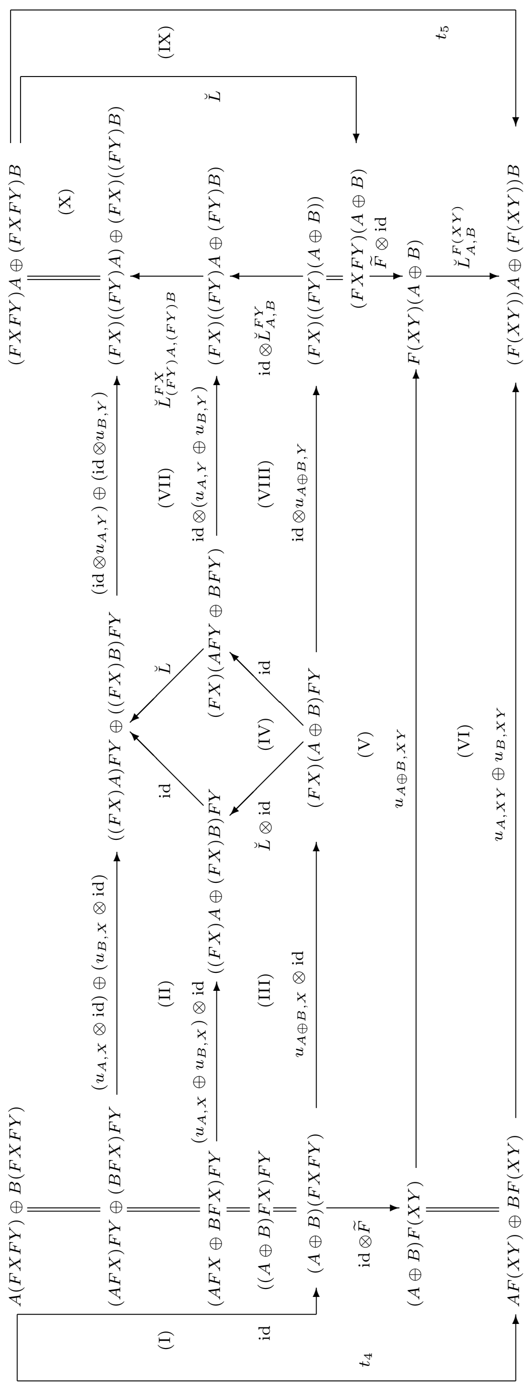


Diagram (6)

where $t_4 = (\mathrm{id}_A \otimes \widetilde{F}_{X,Y}) \oplus (\mathrm{id}_A \otimes \widetilde{F}_{X,Y})$

$t_5 = (\widetilde{F}_{X,Y} \otimes \mathrm{id}_A) \oplus (\widetilde{F}_{X,Y} \otimes \mathrm{id}_B)$

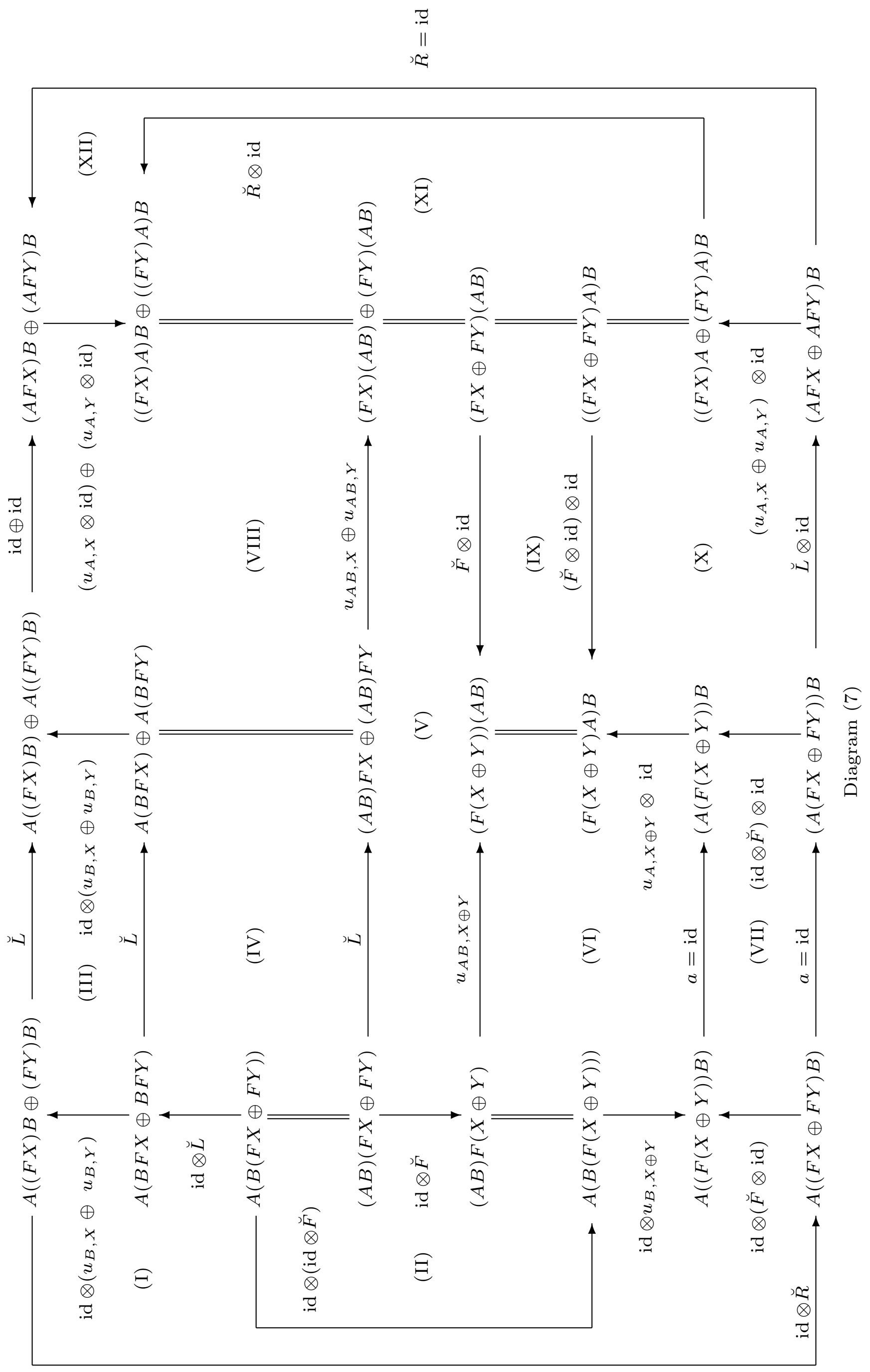

A((FX)B ⊕ (FY)B)
A(BFX ⊕ BFY)
A(B(FX ⊕ FY))
(AB)(FX ⊕ FY)
(AB)F(X ⊕ Y)
A(B(F(X ⊕ Y)))
A((F(X ⊕ Y))B)
A((FX ⊕ FY)B)
A((FX)B) ⊕ A((FY)B)
A(BFX) ⊕ A(BFY)
(AB)FX ⊕ (AB)FY
(F(X ⊕ Y))(AB)
(F(X ⊕ Y)A)B
(A(F(X ⊕ Y))B
(A(FX ⊕ FY))B
(AFX)B ⊕ (AFY)B
((FX)A)B ⊕ ((FY)A)B
(FX)(AB) ⊕ (FY)(AB)
(FX ⊕ FY)(AB)
((FX ⊕ FY)A)B
((FX)A ⊕ (FY)A)B
(AFX ⊕ AFY)B
(I)
(II)
(III)
(IV)
(V)
(VI)
(VII)
(VIII)
(IX)
(X)
(XI)
(XII)

Diagram (7)

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