Commun. Korean Math. Soc. **27** (2012), No. 1, pp. 15–22
http://dx.doi.org/10.4134/CKMS.2012.27.1.015

THE ESSENCE OF SUBTRACTION ALGEBRAS BASED ON $\mathcal{N}$-STRUCTURES

Kyoung Ja Lee and Young Bae Jun

Abstract. Using $\mathcal{N}$-structures, the notion of an $\mathcal{N}$-essence in a subtraction algebra is introduced, and related properties are investigated. Relations among an $\mathcal{N}$-ideal, an $\mathcal{N}$-subalgebra and an $\mathcal{N}$-essence are investigated.

1. Introduction

A (crisp) set A in a universe X can be defined in the form of its characteristic function $\mu_A : X \to \{0, 1\}$ yielding the value 1 for elements belonging to the set A and the value 0 for elements excluded from the set A. So far most of the generalization of the crisp set have been conducted on the unit interval $[0, 1]$ and they are consistent with the asymmetry observation. In other words, the generalization of the crisp set to fuzzy sets relied on spreading positive information that fit the crisp point $\{1\}$ into the interval $[0, 1]$. Because no negative meaning of information is suggested, we now feel a need to deal with negative information. To do so, we also feel a need to supply mathematical tool. To attain such object, Jun et al. [6] introduced a new function which is called negative-valued function, and constructed $\mathcal{N}$-structures. They discussed $\mathcal{N}$-subalgebras and $\mathcal{N}$-ideals in BCK/BCI-algebras. Schein [8] considered systems of the form $(\Phi; \circ, \setminus)$, where Φ is a set of functions closed under the composition "$\circ$" of functions (and hence $(\Phi; \circ)$ is a function semigroup) and the set theoretic subtraction "$\setminus$" (and hence $(\Phi; \setminus)$ is a subtraction algebra in the sense of [1]). He proved that every subtraction semigroup is isomorphic to a difference semigroup of invertible functions. Zelinka [9] discussed a problem proposed by Schein concerning the structure of multiplication in a subtraction semigroup. He solved the problem for subtraction algebras of a special type, called the atomic subtraction algebras. Jun et al. [3, 5] introduced the notion of ideals in subtraction algebras and discussed characterization of ideals. Jun et al. [7] provided conditions for an ideal to be irreducible. They introduced the notion

Received August 20, 2010.
2010 *Mathematics Subject Classification.* 06F35, 03G25.
Key words and phrases. subtraction algebra, $\mathcal{N}$-subalgebra, $\mathcal{N}$-ideal, $\mathcal{N}$-essence.

of an order system in a subtraction algebra, and investigated related properties. They provided relations between ideals and order systems, and dealt with the concept of a fixed map in a subtraction algebra, and investigate related properties. In [2], Jun et al. introduced the notion of a (created) $\mathcal{N}$-ideal of subtraction algebras, and investigated several characterizations of $\mathcal{N}$-ideals. They discussed how to make a created $\mathcal{N}$-ideal of an $\mathcal{N}$-structure (X, f).

In this paper, we introduced the notion of an $\mathcal{N}$-essence of a subtraction algebra, and investigate related properties. We consider relations among an $\mathcal{N}$-ideal, an $\mathcal{N}$-subalgebra and an $\mathcal{N}$-essence. We show that the union (resp. intersection) of $\mathcal{N}$-essences is also an $\mathcal{N}$-essence.

2. Preliminaries

By a *subtraction algebra* we mean an algebra $(X; -)$ with a single binary operation "$-$" that satisfies the following identities: for any $x, y, z \in X$,

(S1) $x - (y - x) = x$;
(S2) $x - (x - y) = y - (y - x)$;
(S3) $(x - y) - z = (x - z) - y$.

The last identity permits us to omit parentheses in expressions of the form $(x - y) - z$. The subtraction determines an order relation on X: $a \leq b \Leftrightarrow a - b = 0$, where $0 = a - a$ is an element that does not depend on the choice of $a \in X$. The ordered set $(X; \leq)$ is a semi-Boolean algebra in the sense of [1], that is, it is a meet semilattice with zero 0 in which every interval $[0, a]$ is a Boolean algebra with respect to the induced order. Here $a \wedge b = a - (a - b)$; the complement of an element $b \in [0, a]$ is $a - b$; and if $b, c \in [0, a]$, then

$$\begin{aligned} b \vee c &= (b' \wedge c')' = a - ((a - b) \wedge (a - c)) \\ &= a - ((a - b) - ((a - b) - (a - c))). \end{aligned}$$

In a subtraction algebra, the following are true (see [5]):

(a1) $(x - y) - y = x - y$.
(a2) $x - 0 = x$ and $0 - x = 0$.
(a3) $(x - y) - x = 0$.
(a4) $x - (x - y) \leq y$.
(a5) $(x - y) - (y - x) = x - y$.
(a6) $x - (x - (x - y)) = x - y$.
(a7) $(x - y) - (z - y) \leq x - z$.
(a8) $x \leq y$ if and only if $x = y - w$ for some $w \in X$.
(a9) $x \leq y$ implies $x - z \leq y - z$ and $z - y \leq z - x$ for all $z \in X$.
(a10) $x, y \leq z$ implies $x - y = x \wedge (z - y)$.
(a11) $(x \wedge y) - (x \wedge z) \leq x \wedge (y - z)$.

Definition 2.1 ([5]). A nonempty subset A of a subtraction algebra X is called an *ideal* of X, denoted by $A \lhd X$, if it satisfies:

(b1) $a - x \in A$ for all $a \in A$ and $x \in X$.

(b2) for all $a, b \in A$, whenever $a \vee b$ exists in X then $a \vee b \in A$.

Proposition 2.2 ([5])**.** *A nonempty subset A of a subtraction algebra X is an ideal of X if and only if it satisfies*:

(b3) $0 \in A$,
(b4) $(\forall x \in X)(\forall y \in A)(x - y \in A \Rightarrow x \in A)$.

Proposition 2.3. *An ideal A of a subtraction algebra X has the following property*:

$$(\forall x \in X)\ (\forall y \in A)\ \big(x \leq y \ \Rightarrow\ x \in A\big).$$

Proposition 2.4 ([5])**.** *Let X be a subtraction algebra and let $x, y \in X$. If $w \in X$ is an upper bound for x and y, then the element*

$$x \vee y := w - ((w - y) - x)$$

is a least upper bound for x and y.

3. $\mathcal{N}$-essences of subtraction algebras

Denote by $\mathcal{F}(X, [-1, 0])$ the collection of functions from a set X to $[-1, 0]$. We say that an element of $\mathcal{F}(X, [-1, 0])$ is a *negative-valued function* from X to $[-1, 0]$ (briefly, *$\mathcal{N}$-function* on X). By an *$\mathcal{N}$-structure* we mean an ordered pair (X, f) of X and an $\mathcal{N}$-function f on X. In what follows, let X denote a subtraction algebra and f an $\mathcal{N}$-function on X unless otherwise specified.

For any $\mathcal{N}$-structure (X, f) and $t \in [-1, 0)$, the set

$$C(f; t) := \{x \in X \mid f(x) \leq t\}$$

is called a *closed (f, t)-cut* of (X, f).

Definition 3.1 ([4])**.** If a nonempty subset G of X satisfies $G - X = G$, then we say that G is an *essence* of X, where $G - X := \{a - x \mid a \in G,\ x \in X\}$.

Definition 3.2. By an *essence* of X based on $\mathcal{N}$-function f (briefly, *$\mathcal{N}$-essence* of X), we mean an $\mathcal{N}$-structure (X, f) in which every nonempty closed (f, t)-cut of (X, f) is an essence of X for all $t \in [-1, 0)$.

Example 3.3. Let (X, f) be an $\mathcal{N}$-structure in which f is given by

$$f(x) = \begin{cases} t_1 & \text{if } x = 0, \\ t_2 & \text{otherwise} \end{cases}$$

for all $x \in X$ and $t_1, t_2 \in [-1, 0)$ with $t_1 < t_2$. Then

$$C(f; r) = \begin{cases} \emptyset & \text{if } r \in [-1, t_1), \\ \{0\} & \text{if } r \in [t_1, t_2), \\ X & \text{if } r \in [t_2, 0). \end{cases}$$

Thus if $r \in [t_1, t_2)$, then $C(f; r) - X = \{0\} - X = \{0\} = C(f; r)$. If $r \in [t_2, 0)$, then $C(f; r) - X = X - X = X = C(f; r)$. Hence (X, f) is an $\mathcal{N}$-essence of X.

TABLE 1. Cayley table

$-$	0	a	b	c
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
c	c	b	a	0

Example 3.4. Let $X = \{0, a, b, c\}$ be a subtraction algebra with the Cayley table which is given in Table 1 (see [4]). Let (X, f) be an $\mathcal{N}$-structure in which f is given by

$$f = \begin{pmatrix} 0 & a & b & c \\ -0.7 & -0.7 & -0.4 & -0.4 \end{pmatrix}.$$

It is easy to check that (X, f) is an $\mathcal{N}$-essence of X. But, if we consider an $\mathcal{N}$-structure (X, g) in which g is given by

$$g = \begin{pmatrix} 0 & a & b & c \\ -0.7 & -0.4 & -0.4 & -0.7 \end{pmatrix},$$

then

$$C(g; t) = \begin{cases} X & \text{if } -0.4 \leq t < 0, \\ \{0, c\} & \text{if } -0.7 \leq t < -0.4, \\ \emptyset & \text{if } -1 \leq t < -0.7. \end{cases}$$

If $-0.7 \leq t < -0.4$, then $C(g; t) - X = \{0, a, b, c\} \neq \{0, c\} = C(g; t)$. Hence (X, g) is not an $\mathcal{N}$-essence of X.

Proposition 3.5. *If an $\mathcal{N}$-structure (X, f) is an $\mathcal{N}$-essence of X, then*

$$(\forall t \in [-1, 0)) \left(C(f; t) = \{x \in X \mid x \leq e \textit{ for some } e \in C(f; t)\} \right). \tag{3.1}$$

Proof. Let $E := \{x \in X \mid x \leq e \text{ for some } e \in C(f; t)\}$. If $x \in E$, then $x \leq e$, i.e., $x - e = 0$, for some $e \in C(f; t)$, and so

$$x = x - 0 = x - (x - e) = e - (e - x) \in C(f; t) - X = C(f; t)$$

by using (a2) and (S2). Hence $E \subseteq C(f; t)$. Now let $x \in C(f; t)$. Then $x = e - y \leq e$ for some $e \in C(f; t)$ and $y \in X$. Thus $x \in E$, and therefore $C(f; t) \subseteq E$. □

The following example shows that if (X, f) is not an $\mathcal{N}$-essence of X in Proposition 3.5, then (3.1) is not valid, i.e., there exists $t \in [-1, 0)$ such that

$$C(f; t) \neq \{x \in X \mid x \leq e \text{ for some } e \in C(f; t)\}.$$

Example 3.6. Note that the $\mathcal{N}$-structure (X, g) in Example 3.4 is not an $\mathcal{N}$-essence of X. If we take $t \in [-0.7, -0.4)$, then

$$C(g; t) = \{0, c\} \neq X = \{x \in X \mid x \leq c\}.$$

Lemma 3.7 ([4]). *Let E be an essence of X. Then*

$$(\forall x \in X)\ (\forall a \in E)\ \big(x \leq a \ \Rightarrow\ x \in E\big). \tag{3.2}$$

Lemma 3.8 ([4]). *For any subset H of X with $0 \in H$, we have*

$$(\forall G \subseteq X)\ \big(G \subseteq G - H\big).$$

Theorem 3.9. *Given an essence E of X and $a \in X$, let (X, f_a) be an $\mathcal{N}$-structure in which f_a is given by*

$$f_a(x) = \begin{cases} \alpha & if\ x \in \{y \in X \mid y - a \in E\}, \\ \beta & otherwise \end{cases}$$

for all $x \in X$ and $\alpha, \beta \in [-1, 0)$ with $\alpha < \beta$. Then (X, f_a) is an $\mathcal{N}$-essence of X.

Proof. Let $\gamma \in [-1, 0)$. If $\gamma < \alpha$, then $C(f_a; \gamma) = \emptyset$. If $\alpha \leq \gamma < \beta$, then $C(f_a; \gamma) = \{y \in X \mid y - a \in E\}$. Let $z \in C(f_a; \gamma)$ and $x \in X$. Then $(z - x) - a = (z - a) - x \leq z - a$. Since $z - a \in E$ and E is an essence, it follows from Lemma 3.7 that $(z - x) - a \in E$ so that $z - x \in C(f_a; \gamma)$. This shows that $C(f_a; \gamma) - X = C(f_a; \gamma)$. The reverse inclusion follows from Lemma 3.8. Hence $C(f_a; \gamma) - X = C(f_a; \gamma)$. If $\gamma \geq \beta$, then $C(f_a; \gamma) = X$ and thus $C(f_a; \gamma) - X = C(f_a; \gamma)$. Therefore f_a is an $\mathcal{N}$-essence of X. □

Proposition 3.10. *Every $\mathcal{N}$-essence (X, f) of X satisfies the following inequality:*

$$(\forall x \in X)\ \big(f(0) \leq f(x)\big). \tag{3.3}$$

Proof. Let (X, f) be an $\mathcal{N}$-essence of X. Then $C(f; \alpha) - X = C(f; \alpha)$ for all $\alpha \in \mathrm{Im}(f)$. Since $C(f; \alpha) \neq \emptyset$, there exists $x \in C(f; \alpha)$ and so

$$0 = x - x \in C(f; \alpha) - X = C(f; \alpha).$$

It follows $f(0) \leq f(x)$ for all $x \in X$. □

Theorem 3.11. *For any $a \in X$, let (X, f) be an $\mathcal{N}$-structure in which f is given by*

$$f(x) = \begin{cases} \alpha & if\ x \leq a, \\ \beta & otherwise \end{cases}$$

for all $x \in X$ and $\alpha, \beta \in [-1, 0)$ with $\alpha < \beta$. Then (X, f) is an $\mathcal{N}$-essence of X.

Proof. Let $\gamma \in [-1, 0)$. If $\gamma < \alpha$, then $C(f; \gamma) = \emptyset$. If $\alpha \leq \gamma < \beta$, then $C(f; \gamma) = \{x \in X \mid x \leq a\}$. Let $x \in C(f; \gamma)$ and $y \in X$. Then $x \leq a$, and so $x - y \leq a - y \leq a$. Hence $x - y \in C(f; \gamma)$, which shows that $C(f; \gamma) - X \subseteq C(f; \gamma)$. The reverse inclusion follows from Lemma 3.8. Hence $C(f; \gamma) - X = C(f, \gamma)$. If $\gamma \geq \beta$, then clearly $C(f; \gamma) - X = C(f, \gamma)$. Thus f is an $\mathcal{N}$-essence of X. □

Definition 3.12 ([2])**.** By an *ideal* (resp. *subalgebra*) of X based on $\mathcal{N}$-function f (briefly, *$\mathcal{N}$-ideal* (resp. *$\mathcal{N}$-subalgebra*) of X), we mean an $\mathcal{N}$-structure (X, f) in which every nonempty closed (f, t)-cut of (X, f) is an ideal (resp. subalgebra) of X for all $t \in [-1, 0)$.

Theorem 3.13. *Every $\mathcal{N}$-ideal is an $\mathcal{N}$-essence.*

Proof. Let (X, f) be an $\mathcal{N}$-ideal of X. Assume that $C(f; \alpha) \neq \emptyset$ for all $\alpha \in [-1, 0)$. Let $x \in X$ and $y \in C(f; \alpha)$. Since $y - x \leq y$ and $C(f; \alpha)$ is an ideal, it follows from Lemma 2.3 that $y - x \in C(f; \alpha)$. This shows that

$$C(f; \alpha) - X \subseteq C(f; \alpha). \tag{3.4}$$

Combining (3.4) and Lemma 3.8, we have $C(f; \alpha) - X = C(f; \alpha)$. Hence (X, f) is an $\mathcal{N}$-essence of X. □

The converse of Theorem 3.13 is not true in general as seen in the following example.

Example 3.14. Consider the subtraction algebra $X = \{0, a, b, c\}$ which is established in Example 3.4. Let (X, f) be an $\mathcal{N}$-structure in which f is given by

$$f = \begin{pmatrix} 0 & a & b & c \\ -0.7 & -0.7 & -0.7 & -0.5 \end{pmatrix}.$$

It is easy to check that (X, f) is an $\mathcal{N}$-essence of X. But it is not an $\mathcal{N}$-ideal of X since $C(f; t) = \{0, a, b\}$ is not an ideal of X for $t \in [-0.7, -0.5)$.

Theorem 3.15. *Every $\mathcal{N}$-essence is an $\mathcal{N}$-subalgebra.*

Proof. Let (X, f) be an $\mathcal{N}$-essence of X. Assume that $C(f; \alpha) \neq \emptyset$ for all $\alpha \in [-1, 0)$. For any $x, y \in C(f; \alpha)$, we have $x - y \in C(f; \alpha) - C(f; \alpha) \subseteq C(f; \alpha) - X = C(f; \alpha)$. Thus $C(f; \alpha)$ is a subalgebra of X, and so (X, f) is an $\mathcal{N}$-subalgebra of X. □

The converse of Theorem 3.15 is not true in general as seen in the following example.

Example 3.16. Consider the $\mathcal{N}$-structure (X, g) which is given in Example 3.4. Then it is an $\mathcal{N}$-subalgebra of X, but not an $\mathcal{N}$-essence of X.

Combining Theorems 3.13 and 3.15, we have the following corollary.

Corollary 3.17. *Every $\mathcal{N}$-ideal is an $\mathcal{N}$-subalgebra.*

For any family $\{a_i \mid i \in \Lambda\}$ of real numbers, we define

$$\vee\{a_i \mid i \in \Lambda\} := \begin{cases} \max\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \sup\{a_i \mid i \in \Lambda\} & \text{otherwise.} \end{cases}$$

$$\wedge\{a_i \mid i \in \Lambda\} := \begin{cases} \min\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \inf\{a_i \mid i \in \Lambda\} & \text{otherwise.} \end{cases}$$

Given $\mathcal{N}$-structures (X, f) and (X, g), we consider new two $\mathcal{N}$-structures $(X, f \cap g)$ and $(X, f \cup g)$ in which $f \cap g$ and $f \cup g$ are given by

$$(f \cap g)(x) = \wedge\{f(x), g(x)\} \text{ and } (f \cup g)(x) = \vee\{f(x), g(x)\},$$

respectively, for all $x \in X$. Note that $C(f \cup g; \alpha) = C(f; \alpha) \cap C(g; \alpha)$ and $C(f \cap g; \alpha) = C(f; \alpha) \cup C(g; \alpha)$.

Lemma 3.18 ([4])**.** *For any subsets A, B and E of X, we have*

(1) $A \subseteq B \Rightarrow A - E \subseteq B - E, E - A \subseteq E - B.$
(2) $(A \cap B) - E \subseteq (A - E) \cap (B - E).$
(3) $E - (A \cap B) \subseteq (E - A) \cap (E - B).$
(4) $(A \cup B) - E = (A - E) \cup (B - E).$
(5) $E - (A \cup B) = (E - A) \cup (E - B).$

Theorem 3.19. *If two $\mathcal{N}$-structures (X, f) and (X, g) are $\mathcal{N}$-essences of X, then so are $(X, f \cup g)$ and $(X, f \cap g)$.*

Proof. Let $\alpha \in [-1, 0)$ be such that $C(f \cup g; \alpha) \neq \emptyset$. Then there exists $x \in C(f \cup g; \alpha)$, and so $(f \cup g)(x) = \vee\{f(x), g(x)\} \leq \alpha$. It follows that $f(x) \leq \alpha$ and $g(x) \leq \alpha$, that is, $x \in C(f; \alpha)$ and $x \in C(g; \alpha)$ so that $C(f; \alpha) - X = C(f; \alpha)$ and $C(g; \alpha) - X = C(g; \alpha)$. Using Lemma 3.18(2), we have

$$\begin{aligned} C(f \cup g; \alpha) - X &= \big(C(f; \alpha) \cap C(g; \alpha)\big) - X \\ &\subseteq \big(C(f; \alpha) - X\big) \cap \big(C(g; \alpha) - X\big) \\ &= C(f; \alpha) \cap C(g; \alpha) = C(f \cup g; \alpha). \end{aligned}$$

Since $0 \in X$, the reverse inclusion follows from Lemma 3.8. Hence we have $C(f \cup g; \alpha) - X = C(f \cup g; \alpha)$, and so $(X, f \cup g)$ is an $\mathcal{N}$-essence of X. Now assume that $C(f \cap g; \beta) \neq \emptyset$. Then there exists $y \in C(f \cap g; \beta)$, and thus $(f \cap g)(y) = \wedge\{f(y), g(y)\} \leq \beta$. It follows that $f(y) \leq \beta$ or $g(y) \leq \beta$. We may assume that $f(y) \leq \beta$ without loss of generality. Then $y \in C(f; \beta)$, and so $C(f; \beta) - X = C(f; \beta)$. If $C(g; \beta) = \emptyset$, then $C(f \cap g; \beta) - X = \big(C(f; \beta) \cup C(g; \beta)\big) - X = C(f; \beta) - X = C(f; \beta) = C(f; \beta) \cup C(g; \beta) = C(f \cap g; \beta)$. If $C(g; \beta) \neq \emptyset$, then $C(g; \beta) - X = C(g; \beta)$. Using Lemma 3.18(4), we have $C(f \cap g; \beta) - X = \big(C(f; \beta) \cup C(g; \beta)\big) - X = \big(C(f; \beta) - X\big) \cup \big(C(g; \beta) - X\big) = C(f; \beta) \cup C(g; \beta) = C(f \cap g; \beta)$. Therefore $(X, f \cap g)$ is an $\mathcal{N}$-essence of X. □

Generally, we have the following assertion.

Theorem 3.20. *If $\{(X, f_i) \mid i \in \Lambda \subseteq \mathbb{N}\}$ is a family of $\mathcal{N}$-essences of X, then so are $\Big(X, \bigcup_{i \in \Lambda} f_i\Big)$ and $\Big(X, \bigcap_{i \in \Lambda} f_i\Big)$, where $\Big(\bigcup_{i \in \Lambda} f_i\Big)(x) = \vee\{f_i(x) \mid i \in \Lambda\}$ and $\Big(\bigcap_{i \in \Lambda} f_i\Big)(x) = \wedge\{f_i(x) \mid i \in \Lambda\}$.*

References

[1] J. C. Abbott, *Sets, Lattices and Boolean Algebras*, Allyn and Bacon, Inc., Boston, Mass. 1969.
[2] Y. B. Jun, J. Kavikumar, and K. S. So, *$\mathcal{N}$-ideals of subtraction algebras*, Commun. Korean Math. Soc. **25** (2010), no. 2, 173–184.
[3] Y. B. Jun and H. S. Kim, *On ideals in subtraction algebras*, Sci. Math. Jpn. **65** (2007), no. 1, 129–134.
[4] Y. B. Jun, H. S. Kim, and K. J. Lee, *The essence of subtraction algebras*, Sci. Math. Jpn. **64** (2006), no. 3, 601–606.
[5] Y. B. Jun, H. S. Kim, and E. H. Roh, *Ideal theory of subtraction algebras*, Sci. Math. Jpn. **61** (2005), no. 3, 459–464.
[6] Y. B. Jun, K. J. Lee, and S. Z. Song, *$\mathcal{N}$-ideals of BCK/BCI-algebras*, J. Chungcheong Math. Soc. **22** (2009), 417–437.
[7] Y. B. Jun, C. H. Park, and E. H. Roh, *Order systems, ideals and right fixed maps of subtraction algebras*, Commun. Korean Math. Soc. **23** (2008), no. 1, 1–10.
[8] B. M. Schein, *Difference semigroups*, Comm. Algebra **20** (1992), no. 8, 2153–2169.
[9] B. Zelinka, *Subtraction semigroups*, Math. Bohem. **120** (1995), no. 4, 445–447.

Kyoung Ja Lee
Department of Mathematics Education
Hannam University
Daejeon 306-791, Korea
E-mail address: kjlee@hnu.kr

Young Bae Jun
Department of Mathematics Education (and RINS)
Gyeongsang National University
Chinju 660-701, Korea
E-mail address: skywine@gmail.com