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ANSWERS TO LEE AND PARK'S QUESTIONS

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Abstract. In [K. J. Lee and C. H. Park, *Some questions on fuzzifications of ideals in subtraction algebras*, Commun. Korean Math. Soc. **22** (2007), no. 3, 359–363], Lee and Park posed three questions. In this paper, the affirmative answers to their questions are provided, and characterizations of fuzzy ideals are investigated.

1. Introduction

B. M. Schein [8] considered systems of the form $(\Phi; \circ, \setminus)$, where Φ is a set of functions closed under the composition "$\circ$" of functions (and hence $(\Phi; \circ)$ is a function semigroup) and the set theoretic subtraction "$\setminus$" (and hence $(\Phi; \setminus)$ is a subtraction algebra in the sense of [1]). He proved that every subtraction semigroup is isomorphic to a difference semigroup of invertible functions. B. Zelinka [9] discussed a problem proposed by B. M. Schein concerning the structure of multiplication in a subtraction semigroup. He solved the problem for subtraction algebras of a special type, called the atomic subtraction algebras. Y. B. Jun et al. [3] introduced the notion of ideals in subtraction algebras and discussed characterization of ideals. In [2], Y. B. Jun and H. S. Kim established the ideal generated by a set, and discussed related results. Jun et al. discussed several notions and properties in subtraction algebras (see [4, 5, 6]). Lee and Park posed three questions in the paper [7]. The aim of this paper is to provide affirmative answers to these questions. We investigate several characterizations of fuzzy ideals in subtraction algebras.

2. Preliminaries

By a *subtraction algebra* we mean an algebra $(X; -)$ with a single binary operation "$-$" that satisfies the following identities: for any $x, y, z \in X$,

(S1) $x - (y - x) = x$;
(S2) $x - (x - y) = y - (y - x)$;
(S3) $(x - y) - z = (x - z) - y$.

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The last identity permits us to omit parentheses in expressions of the form $(x - y) - z$. The subtraction determines an order relation on X: $a \leq b \Leftrightarrow a - b = 0$, where $0 = a - a$ is an element that does not depend on the choice of $a \in X$. The ordered set $(X; \leq)$ is a semi-Boolean algebra in the sense of [1], that is, it is a meet semilattice with zero 0 in which every interval $[0, a]$ is a Boolean algebra with respect to the induced order. Here $a \wedge b = a - (a - b)$; the complement of an element $b \in [0, a]$ is $a - b$; and if $b, c \in [0, a]$, then

$$\begin{aligned} b \vee c &= (b' \wedge c')' = a - ((a - b) \wedge (a - c)) \\ &= a - ((a - b) - ((a - b) - (a - c))). \end{aligned}$$

In a subtraction algebra, the following are true (see [3]):

(a1) $(x - y) - y = x - y$.
(a2) $x - 0 = x$ and $0 - x = 0$.
(a3) $(x - y) - x = 0$.
(a4) $x - (x - y) \leq y$.
(a5) $(x - y) - (y - x) = x - y$.
(a6) $x - (x - (x - y)) = x - y$.
(a7) $(x - y) - (z - y) \leq x - z$.
(a8) $x \leq y$ if and only if $x = y - w$ for some $w \in X$.
(a9) $x \leq y$ implies $x - z \leq y - z$ and $z - y \leq z - x$ for all $z \in X$.
(a10) $x, y \leq z$ implies $x - y = x \wedge (z - y)$.
(a11) $(x \wedge y) - (x \wedge z) \leq x \wedge (y - z)$.

Definition 2.1 ([3])**.** A nonempty subset A of a subtraction algebra X is called an *ideal* of X, denoted by $A \lhd X$, if it satisfies:

(b1) $a - x \in A$ for all $a \in A$ and $x \in X$.
(b2) for all $a, b \in A$, whenever $a \vee b$ exists in X then $a \vee b \in A$.

Proposition 2.2 ([3])**.** *Let X be a subtraction algebra and let $x, y \in X$. If $w \in X$ is an upper bound for x and y, then the element*

$$x \vee y := w - ((w - y) - x)$$

is a least upper bound for x and y.

3. Answers to Lee and Park's questions

In what follows let X be a subtraction algebra unless otherwise specified.

Definition 3.1 ([7])**.** A fuzzy set μ in X is called a *fuzzy ideal* of X if it satisfies:

(c1) $(\forall x, y \in X)\ (\mu(x - y) \geq \mu(x))$,
(c2) $(\forall x, y \in X)\ (\exists x \vee y \Rightarrow \mu(x \vee y) \geq \min\{\mu(x), \mu(y)\})$.

Proposition 3.2 ([7])**.** *If a fuzzy set μ in X satisfies*

(c3) $(\forall x, a, b \in X)\ (\mu(x - ((x - a) - b)) \geq \min\{\mu(a), \mu(b)\})$,

then μ is a fuzzy ideal of X.

Proposition 3.3 ([7]). *Let μ be a fuzzy set in X such that*

(c4) $(\forall x \in X)(\mu(0) \geq \mu(x))$,

(c5) $(\forall x, y, z \in X)(\mu(x - z) \geq \min\{\mu((x - y) - z), \mu(y)\})$.

Then we have

$$(\forall a, x \in X)(x \leq a \Rightarrow \mu(x) \geq \mu(a)). \tag{3.1}$$

Theorem 3.4 ([7]). *If μ is a fuzzy ideal of X, then*

$$(\forall \alpha \in [0, 1])(U(\mu; \alpha) \neq \emptyset \Rightarrow U(\mu; \alpha) \lhd X), \tag{3.2}$$

where $U(\mu; \alpha) := \{x \in X \mid \mu(x) \geq \alpha\}$ which is called the level set of μ.

Lee and Park posed the following questions.

Question. 1. Does any fuzzy ideal of a subtraction algebra satisfy condition (c3)?

2. Does any fuzzy ideal of a subtraction algebra satisfy condition (c5)?

3. Does the converse of Theorem 3.4 hold?

We provide affirmative answers to these questions. We first answer to the third question.

Theorem 3.5. *Let μ be a fuzzy set in X for which* (3.2) *is valid. Then μ is a fuzzy ideal of X.*

Proof. If there are $a, b \in X$ such that $\mu(a - b) < \mu(a)$, then $\mu(a - b) < t_a \leq \mu(a)$ for some $t_a \in (0, 1]$. Thus $a \in U(\mu; t_a)$, but $a - b \notin U(\mu; t_a)$. This is a contradiction, and so $\mu(x - y) \geq \mu(x)$ for all $x, y \in X$. Assume that there exist $a, b \in X$ such that $a \vee b$ exists and $\mu(a \vee b) < \min\{\mu(a), \mu(b)\}$. Then $\mu(a \vee b) < t_0 \leq \min\{\mu(a), \mu(b)\}$ for some $t_0 \in (0, 1]$. It follows that $a, b \in U(\mu; t_0)$ and $a \vee b \notin U(\mu; t_0)$ which is a contradiction. Therefore (c2) is valid. Hence μ is a fuzzy ideal of X. □

Now we answer to the second question.

Theorem 3.6. *Every fuzzy ideal μ of X satisfies the inequality* (c5).

Proof. If we put $x = x - z$ in (a3), then $((x - z) - y) - (x - z) = 0$, i.e., $(x - z) - y \leq x - z$. If we put $x = y$ and $y = x - z$ in (a4), then $y - (y - (x - z)) \leq x - z$. Hence $x - z$ is an upper bound for $(x - z) - y$ and $y - (y - (x - z))$. It follows from Proposition 2.2, (S2), (S3) and (a2) that

$$\begin{aligned}
&((x - y) - z) \vee y \\
&= ((x - y) - z) \vee (y - 0) \\
&= ((x - y) - z) \vee (y - (y - y)) \\
&= ((x - y) - z) \vee (y - (y - (x - z))) \\
&= ((x - z) - y) \vee (y - (y - (x - z))) \\
&= (x - z) - (((x - z) - (y - (y - (x - z)))) - ((x - z) - y))
\end{aligned}$$

$$= (x - z) - (((x - z) - ((x - z) - y)) - (y - (y - (x - z))))$$
$$= (x - z) - ((y - (y - (x - z))) - (y - (y - (x - z))))$$
$$= (x - z) - 0 \; = \; x - z$$

so from (c2) that

$$\mu(x - z) = \mu(((x - y) - z) \vee y) \geq \min\{\mu((x - y) - z), \mu(y)\}$$

for all $x, y \in X$. □

We finally answer to the first question.

Theorem 3.7. *Every fuzzy ideal μ of X satisfies the inequality* (c3).

Proof. Suppose that μ is a fuzzy ideal of X. Then the nonempty level set $U(\mu; t)$ of μ is an ideal of X for all $t \in (0, 1]$. Let $\theta_{U(\mu;t)}$ be a relation on X defined by

$$(\forall x, y \in X) \ \big((x, y) \in \theta_{U(\mu;t)} \ \Leftrightarrow \ x - y \in U(\mu; t), \ y - x \in U(\mu; t)\big).$$

Then $\theta_{U(\mu;t)}$ is a congruence relation on X. For any $a, b \in U(\mu; t)$ and $x \in X$, we have $(x, x) \in \theta_{U(\mu;t)}$, $(a, 0) \in \theta_{U(\mu;t)}$ and $(b, 0) \in \theta_{U(\mu;t)}$. Hence

$$(x - ((x - a) - b), 0) = (x - ((x - a) - b), x - ((x - 0) - 0)) \in \theta_{U(\mu;t)},$$

and so $x - ((x - a) - b) \in U(\mu; t)$. It follows that

$$\mu(x - ((x - a) - b)) \geq \min\{\mu(a), \mu(b)\}$$

for all $a, b, x \in X$ because if there exist $a_0, b_0 \in X$ such that

$$\mu(x - ((x - a_0) - b_0)) < \min\{\mu(a_0), \mu(b_0)\},$$

then $\mu(x - ((x - a_0) - b_0)) < t_0 \leq \min\{\mu(a_0), \mu(b_0)\}$ for some $t_0 \in (0, 1]$. Thus $a_0 \in U(\mu; t_0)$ and $b_0 \in U(\mu; t_0)$, but $x - ((x - a_0) - b_0) \notin U(\mu; t_0)$. This is a contradiction. □

4. Characterizations of fuzzy ideals

Using [7] and the previous section, we have the following characterizations of fuzzy ideals.

Theorem 4.1. *For a fuzzy set μ in X, the following assertions are equivalent*:

(1) *μ is a fuzzy ideal of X.*
(2) *μ satisfies the condition* (3.2).
(3) *μ satisfies the condition* (c3).
(4) *μ satisfies the conditions* (c4) *and* (c5).

Theorem 4.2. *A fuzzy set μ in X is a fuzzy ideal of X if and only if it satisfies the condition* (c4) *and*

$$(\forall x, y \in X) \ (\mu(x) \geq \min\{\mu(x - y), \mu(y)\}). \tag{4.1}$$

Proof. Assume that μ is a fuzzy ideal of X. Then the condition (c4) is valid by [7, Proposition 3.4]. The condition (4.1) is by taking $z = 0$ in (c5) and using (a2).

Conversely, let μ satisfy (c4) and (4.1). Note that

$$(x - ((x - a) - b)) - b = (x - b) - ((x - a) - b) \leq x - (x - a) \leq a,$$

that is, $((x - ((x - a) - b)) - b) - a = 0$ for all $x, a, b \in X$. It follows from (c4) and (4.1) that

$$\begin{aligned}
&\mu(x - ((x - a) - b)) \\
\geq\ &\min\{\mu((x - ((x - a) - b)) - b), \mu(b)\} \\
\geq\ &\min\{\min\{\mu(((x - ((x - a) - b)) - b) - a), \mu(a)\}, \mu(b)\} \\
=\ &\min\{\min\{\mu(0), \mu(a)\}, \mu(b)\} \\
=\ &\min\{\mu(a), \mu(b)\}
\end{aligned}$$

for all $x, a, b \in X$. Therefore μ is a fuzzy ideal of X by Proposition 3.2. □

Theorem 4.3. *For fixed elements $a, b \in X$, let μ_a^b be a fuzzy set in X defined by*

$$\mu_a^b(x) = \begin{cases} t_1 & \text{if } (x - a) - b = 0, \\ t_2 & \text{otherwise} \end{cases}$$

for all $x \in X$ and $t_1, t_2 \in (0, 1]$ with $t_1 > t_2$. Then μ_a^b is a fuzzy ideal of X.

Proof. Since $(0 - a) - b = 0$, we have $\mu_a^b(0) = t_1 \geq \mu_a^b(x)$ for all $x \in X$. Let $x, y \in X$. If $(x - a) - b = 0$, then $\mu_a^b(x) = t_1 \geq \min\{\mu_a^b(x - y), \mu_a^b(y)\}$. Suppose that $(x - a) - b \neq 0$. If $(y - a) - b = 0$ and $((x - y) - a) - b = 0$, then

$$\begin{aligned}
(x - a) - b &= ((x - a) - b) - 0 \\
&= ((x - a) - b) - ((y - a) - b) \\
&= ((x - a) - (y - a)) - b \\
&= ((x - y) - a) - b = 0,
\end{aligned}$$

a contradiction. Hence $(y - a) - b \neq 0$ or $((x - y) - a) - b \neq 0$, and thus $\mu_a^b(y) = t_2$ or $\mu_a^b(x - y) = t_2$. It follows that

$$\mu_a^b(x) = t_2 = \min\{\mu_a^b(x - y), \mu_a^b(y)\}.$$

Hence, by Theorem 4.2, μ_a^b is a fuzzy ideal of X. □

Theorem 4.4. *A fuzzy set μ is a fuzzy ideal of X if and only if it satisfies:*

(4.2) $(\forall a, b, x \in X)\ (x - a \leq b \implies \mu(x) \geq \min\{\mu(a), \mu(b)\}).$

Proof. Assume that μ is a fuzzy ideal of X. Let $a, b, x \in X$ be such that $x - a \leq b$. Then $(x - a) - b = 0$, and so

$$\begin{aligned}\mu(x) &\geq \min\{\mu(x-a), \mu(a)\} \\ &\geq \min\{\min\{\mu((x-a)-b), \mu(b)\}, \mu(a)\} \\ &= \min\{\min\{\mu(0), \mu(b)\}, \mu(a)\} \\ &= \min\{\mu(a), \mu(b)\}\end{aligned}$$

by (4.1) and (c4).

Conversely, let μ be satisfy the condition (4.2). Since $0 - x \leq x$ for all $x \in X$, it follows from (4.2) that

$$\mu(0) \geq \min\{\mu(x), \mu(x)\} = \mu(x)$$

for all $x \in X$. Note that $x - (x - y) \leq y$ for all $x, y \in X$. Using (4.2), we have $\mu(x) \geq \min\{\mu(x-y), \mu(y)\}$ for all $x, y \in X$. Hence μ is a fuzzy ideal of X by Theorem 4.2. □

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