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BETA-EXPANSIONS WITH PISOT BASES OVER $\mathbb{F}_q((x^{-1}))$

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Abstract. It is well known that if the β-expansion of any nonnegative integer is finite, then β is a Pisot or Salem number. We prove here that in $\mathbb{F}_q((x^{-1}))$, the β-expansion of the polynomial part of β is finite if and only if β is a Pisot series. Consequently we give an other proof of Scheicher theorem about finiteness property in $\mathbb{F}_q((x^{-1}))$. Finally we show that if the base β is a Pisot series, then there is a bound of the length of the fractional part of β-expansion of any polynomial P in $\mathbb{F}_q[x]$.

1. Introduction

The β-expansions of real numbers were introduced by A. Rényi [7]. Since then, their arithmetic, diophantine and ergodic properties have been extensively studied by several authors.

Let $\beta > 1$ be a real number. The β-expansion of a real number $x \in [0, 1)$ is defined as the sequence $d_\beta(x) = (x_i)_{i\geq 1}$ with values in $\{0, 1, \ldots, [\beta]\}$ produced by the β-transformation $T_\beta : \ x \to \ \beta x \pmod 1$ as follows:

$$\forall\ i \geq 1,\ x_i = [\beta T_\beta^{i-1}(x)], \quad \text{and thus} \quad x = \sum_{i\geq 1} \frac{x_i}{\beta^i}.$$

Now let $x \in \mathbb{R}_+$ with $x \geq 1$, then there is a unique $k \in \mathbb{N}$ such that $|\beta|^k \leq x < |\beta|^{k+1}$. Hence $|\frac{x}{\beta^{k+1}}| < 1$ and we can represent x by shifting $d_\beta(\frac{x}{\beta^{k+1}})$ by $k+1$ digits to the left. Therefore, if $d_\beta(x) = 0.d_1d_2d_3\ldots$, then $d_\beta(\beta x) = d_1.d_2d_3\ldots$. An expansion is finite if $(x_i)_{i\geq 1}$ is eventually 0. A β-expansion is periodic if there exists $p \geq 1$ and $m \geq 1$ such that $x_k = x_{k+p}$ holds for all $k \geq m$; if $x_k = x_{k+p}$ holds for all $k \geq 1$, then it is purely periodic. We denote by $\mathrm{Fin}(\beta)$ the nonnegative numbers with finite β-expansions.

Let $\mathbb{Z}[\beta]$ be the smallest ring containing $\mathbb{Z}$ and β. Denote by $\mathbb{Z}[\beta]_{\geq 0}$ the non negative elements of $\mathbb{Z}[\beta]$. We say that the number β has the finiteness property (F) if

$$\mathrm{Fin}(\beta) = \mathbb{Z}[\beta^{-1}]_{\geq 0}$$

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holds. This property was introduced by Frougny and Solomyak [3]. They showed that it implies β is a Pisot number, i.e., a real algebraic integer greater than 1 with all conjugates strictly inside the unit circle, and they found the following class of Pisot numbers satisfying this property.

Theorem 1.1 ([3]). *If β is the dominant root of polynomial $x^d - b_1x^{d-1} - b_2x^{d-2} - \cdots - b_d \in \mathbb{Z}[x]$ with $b_1 \geq b_2 \geq \cdots \geq b_d$, then β is a Pisot number and has the property* (F).

Another class of Pisot numbers with property (F) was found by Hollander.

Theorem 1.2 ([5]). *If β is the dominant root of polynomial*

$$x^d - b_1x^{d-1} - b_2x^{d-2} - \cdots - b_d \text{ with } b_1 > \sum_{i=2}^{d} b_i \text{ and } b_i \geq 0 \ (1 \leq i \leq d),$$

then β is a Pisot number satisfying the finiteness property.

Note that there are Pisot numbers without property (F), in particular all numbers with infinite expansion of one.

It is proved in [3] that if $\mathbb{N} \subset \mathrm{Fin}(\beta)$, then β is a Pisot or a Salem number and it is not the case if we have only $d_\beta(1)$ finite. Also S. Akiyama has proved in [1] that if β has the finiteness property, then there exists a positive constant c such that any rational in $[0, c[$ has purely periodic β-expansion.

2. β-expansions in $\mathbb{F}_q((x^{-1}))$

Let $\mathbb{F}_q$ be a finite field of q elements, $\mathbb{F}_q[x]$ the ring of polynomials with coefficients in $\mathbb{F}_q$, $\mathbb{F}_q(x)$ the field of rational functions and $\mathbb{F}_q((x^{-1}))$ the field of formal power series of the form:

$$f = \sum_{k=-\infty}^{l} f_k x^k, \quad f_k \in \mathbb{F}_q,$$

where

$$l = \deg f := \begin{cases} \max\{k : f_k \neq 0\} & \text{if} \quad f \neq 0, \\ -\infty & \text{if} \quad f = 0. \end{cases}$$

Define the absolute value

$$|f| = \begin{cases} q^{\deg f} & \text{for } f \neq 0; \\ 0 & \text{for } f = 0. \end{cases}$$

Since $|\cdot|$ is not Archimedean, then it fulfills the strict triangle inequality:

$$|f + g| \leq \max(|f|, |g|) \quad \text{and}$$

$$|f + g| = \max(|f|, |g|) \quad \text{if} \quad |f| \neq |g|.$$

For $f \in \mathbb{F}_q((x^{-1}))$, define the integer (polynomial) part $[f] = \sum_{k=0}^{l} f_k x^k$ where the empty sum, as usual, is defined to be zero. Therefore $[f] \in \mathbb{F}_q[x]$ and $f - [f] \in D(0, 1)$ (the open unit disc) for all $f \in \mathbb{F}_q((x^{-1}))$.

Proposition 2.1 ([6]). *Let K be a complete field with respect to a non Archimedian absolute value $|\cdot|$ and $(K \subset L)$ be an algebraic extension of degree m. Then $|\cdot|$ has a unique extension to L defined by $|a| = \sqrt[m]{|N_{L/K}(a)|}$ and L is complete with respect to this extension.*

We apply this proposition to algebraic elements of $\mathbb{F}_q((x^{-1}))$. Since $\mathbb{F}_q[x] \subset \mathbb{F}_q((x^{-1}))$, then every algebraic element in $\mathbb{F}_q[x]$ can be valuated. However, since $\mathbb{F}_q((x^{-1}))$ is not algebraically closed, such an element need not be necessarily a formal power series.

An element $\beta \in \mathbb{F}_q((x^{-1}))$ is called a Pisot (resp Salem) element if it is an algebraic integer over $\mathbb{F}_q[x]$ such that $|\beta| > 1$ and $|\beta_j| < 1$ for all conjugates β_j (resp $|\beta_j| \leq 1$ and there exists at least one conjugate β_k such that $|\beta_k| = 1$). Bateman and Duquette [2] characterized the Pisot and Salem elements in $\mathbb{F}_q((x^{-1}))$:

Theorem 2.1. *Let $\beta \in \mathbb{F}_q((x^{-1}))$ be an algebraic integer over $\mathbb{F}_q[x]$ and*

$$P(y) = y^n - A_1 y^{n-1} - \cdots - A_n, \quad A_i \in \mathbb{F}_q[x],$$

be its minimal polynomial. Then

(i) *β is a Pisot element if and only if $|A_1| > \max_{2\leq j\leq n} |A_i|$.*
(ii) *β is a Salem element if and only if $|A_1| = \max_{2\leq j\leq n} |A_i|$.*

Let $\beta, f \in \mathbb{F}_q((x^{-1}))$ where $|\beta| > 1$ and $f \in M_0$. A representation in base β (or β-representation) of f is a sequence $(d_i)_{i\geq 1}$, $d_i \in \mathbb{F}_q[x]$, such that

$$f = \sum_{i\geq 1} \frac{d_i}{\beta^i}.$$

A particular β-representation of f is called the β-expansion of f and noted $d_\beta(f)$. It is obtained by using the β-transformation T_β in M_0 which is given by the mapping:

$$\begin{aligned} T_\beta : D(0,1) &\longrightarrow D(0,1) \\ f &\longmapsto \beta f - [\beta f]. \end{aligned}$$

Thus, $d_\beta(f) = (d_i)_{i\geq 1}$ if and only if $d_i = [\beta T_\beta^{i-1}(f)]$. Note that $d_\beta(f)$ is finite if and only if there is a $k \geq 0$ such that $T_\beta^k(f) = 0$, $d_\beta(f)$ is ultimately periodic if and only if there is some smallest $p \geq 0$ (the pre-period length) and $s \geq 1$ (the period length) for which $T_\beta^{p+s}(f) = T_\beta^p(f)$.

Now let $f \in \mathbb{F}_q((x^{-1}))$ be an element with $|f| \geq 1$. Then there is a unique $k \in \mathbb{N}$ such that $|\beta|^k \leq |f| < |\beta|^{k+1}$. Hence $|\frac{f}{\beta^{k+1}}| < 1$ and we can represent f by shifting $d_\beta(\frac{f}{\beta^{k+1}})$ by $k+1$ digits to the left. Therefore, if $d_\beta(f) = 0.d_1 d_2 d_3 \ldots$, then $d_\beta(\beta f) = d_1.d_2 d_3 \ldots$.

We say that $d_\beta(f)$ is finite when $d_i = 0$ for all sufficiently large i. This is the case when there is an integer $i \geq 0$ such that $T_\beta^i(f) = 0$. If $d_\beta(f) =$

$d_l d_{l-1} \dots d_0, d_{-1} \dots d_m$, let $\deg_\beta(f) = k$ and $\text{ord}_\beta(f) = m$, where m and l are in $\mathbb{Z}$.

In the sequel, we will use the following notation:

$$\text{Fin}(\beta) = \{f \in \mathbb{F}_q((x^{-1})) : d_\beta(f) \text{ is finite}\}.$$

Remark 2.2. In contrast to the real case, there is no carry occurring, when we add two digits. Therefore, if z, $w \in \mathbb{F}_q((x^{-1}))$, we have $d_\beta(z+w) = d_\beta(z) + d_\beta(w)$ digitwise. We have also $d_\beta(cf) = cd_\beta(f)$ for every $c \in \mathbb{F}_q$.

Lemma 2.3 ([4]). *Let $P(y) = A_n y^n + A_{n-1}y^{n-1} + \dots + A_0$, where $A_i \in \mathbb{F}_q[x]$ for $i = 1, \dots, n$. Then P admits a unique root in $\mathbb{F}_q((x^{-1}))$ with absolute value > 1 if and only if $|A_{n-1}| > |A_i|$ for $i \neq n-1$.*

Theorem 2.2 ([4]). *An infinite sequence $(d_j)_{j \geq 1}$ is the β-expansion of $f \in M_0$ if and only if it is a β-representation of f and $|d_j| < |\beta|$ for $j \geq 1$.*

In the field of formal series, it was proved independently by Hbaib - Mkaouar and Scheicher the following theorems:

Theorem 2.3 ([8]). *β is a Pisot element if and only if $Fin(\beta) = \mathbb{F}_q[x, \beta^{-1}]$.*

Theorem 2.4 ([4]). *β is a Pisot element if and only if $d_\beta(1)$ is finite.*

3. Results

In this section, we concentrate on the case that β is a Pisot series of algebraical degree d. First, we begin with this theorem which gives a characterization of Pisot series:

Theorem 3.1. *Let $\beta \in \mathbb{F}_q((x^{-1}))$ such that $|\beta| > 1$. Then β is a Pisot series if and only if the β-expansion of (x^m) is finite, where $m = \deg(\beta)$.*

Proof. Let $P(y) = y^d - A_{d-1}y^{d-1} - A_{d-2}y^{d-2} - \dots - A_0$ be the minimal polynomial of β. Since β is a Pisot series, then $|A_{d-1}| = |\beta|$ and $|A_i| < |\beta|$ for all $i < d-1$. However, $\deg(\beta) = m$, then A_{d-1} is the unique polynomial A_i of degree m and let c be his dominant coefficient, so

$$cx^m\beta^{d-1} = -\beta^d - (A_{d-1} - cx^m)\beta^{d-1} - A_{d-2}\beta^{d-2} - \dots - A_0.$$

Therefore

$$cx^m = \beta - (A_{d-1} - cx^m) - \frac{A_{d-2}}{\beta} - \dots - \frac{A_0}{\beta^{d-1}}.$$

According to Theorem 2.2, the last equality is the β-expansion of cx^m, which implies that $d_\beta(x^m)$ is finite and $\text{ord}_\beta(x^m) = 1 - d$.

Reciprocally, assume that $d_\beta(x^m) = a_1 a_0 \bullet a_{-1} \cdots a_{-n}$ is finite. We have then:

$$x^m = a_1\beta + a_0 + \frac{a_{-1}}{\beta} + \frac{a_{-2}}{\beta^2} + \dots + \frac{a_{-n}}{\beta^{-n}}.$$

Multiplying by β^n, we have

$$-a_1\beta^{n+1} + \beta^n(x^m - a_0) - a_1\beta^{n-1} - a_2\beta^{n-2} - \dots - a_{n-1}\beta = 0.$$

Since $|a_i| < |\beta| = |x^m|$, then $|x^m - a_0| > |a_i|$ for every $i \leq 1$. Then according to Lemma 2.3 β is a Pisot series. □

Combining Theorem 3.1 with Remark 2.2 we obtain:

Corollary 3.2. *β is a Pisot series if and only if the β-expansion of the polynomial part of β is finite.*

The following result proved by K. Scheicher [8] can be derived from Theorem 3.1.

Theorem 3.3. *Let $\beta \in \mathbb{F}_q((x^{-1}))$, $|\beta| > 1$. Then β is a Pisot series if and only if* $\mathrm{Fin}(\beta) = \mathbb{F}_q[x, \beta^{-1}]$.

Proof. It is trivial that $\mathrm{Fin}(\beta) \subset \mathbb{F}_q[x, \beta^{-1}]$, we need only to prove the opposite inclusion. Suppose that $|\beta| = q^m$, i.e., $\deg(\beta) = m \geq 1$, then $d_\beta(x^k)$ is finite for all $0 \leq k \leq m$. We will prove now by induction that $d_\beta(x^k)$ is finite for all $k \geq m$. According to Theorem 2.4, this is true for $k = m$. Assume now that $d_\beta(x^k)$ is finite, i.e.,

$$x^k = a_s\beta^s + \cdots + a_0 + \frac{a_{-1}}{\beta} + \cdots + \frac{a_{-n}}{\beta^n} \quad \text{where} \quad a_h = \sum_{j=0}^{m-1} c_j^h x^j, \; -n \leq h \leq s.$$

Then,

$$x^{k+1} = xa_s\beta^s + \cdots + xa_0 + \frac{xa_{-1}}{\beta} + \cdots + \frac{xa_{-n}}{\beta^n}.$$

However $\deg(a_i) \leq m - 1$, so, $\deg(xa_i) \leq m$, which implies

$$\begin{aligned} x^{k+1} = {} & (c_{m-1}^s x^m + \cdots + c_0^s x)\beta^s + \cdots + (c_{m-1}^0 x^m + \cdots + c_0^0 x) \\ & + \frac{c_{m-1}^{-1} x^m + \cdots + c_0^{-1} x}{\beta} + \cdots + \frac{c_{m-1}^{-n} x^m + \cdots + c_0^{-n} x}{\beta^n}. \end{aligned}$$

If we replace in the last equality x^m by its finite β-expansion, we get a β-representation of x^{k+1} which is the β-expansion of x^{k+1} according to Theorem 2.1.

Finally, we conclude that $d_\beta(x^k)$ is finite for all $k \geq m$ and then all polynomials admits finite β-expansion (Remark 2.2), and if we divide by β^{-i} for all $i \geq 1$, we get also a finite β-expansion.

Reciprocally, assume that $\mathbb{F}_q[x, \beta^{-1}]$=Fin($\beta$), especially, $d_\beta(x^m)$ is finite. Therefore by Theorem 3.1, β is Pisot. □

We give now a quantitative version of the results above. One may ask if there is a bound on the increase of the length of the beta-expansion of polynomials. The answer is yes if the base is a Pisot element.

Theorem 3.4. *Let β be a Pisot series of algebraical degree d and let $k \geq m =$* $\deg(\beta)$. *Then*

$$\mathrm{ord}_\beta(x^k) \geq (k - m + 1)(1 - d).$$

Proof. Since β is Pisot of algebraical degree d, from Lemma 3.2 $\text{ord}_\beta(x^m)$ is finite and equal to $(1-d)$. Let

$$x^m = a_1\beta + a_0 + \frac{a_{-1}}{\beta} + \frac{a_{-2}}{\beta^2} + \cdots + \frac{m_{1-d}}{\beta^{d-1}}$$

be the beta-expansion of x^m. Now let $f \in \mathbb{F}_q[x, \beta^{-1}]$ such that $d_\beta(f) = b_s \cdots b_0 \bullet b_{-1} \cdots b_{-n}$. We have then:

$$f = b_s\beta^s + \cdots + b_0 + \frac{b_{-1}}{\beta} + \frac{b_{-2}}{\beta^2} + \cdots + \frac{b_{-n}}{\beta^n}.$$

Multiplying by x, we get

$$xf = b_s x\beta^s + \cdots + b_0 x + \frac{b_{-1}x}{\beta} + \frac{b_{-2}x}{\beta^2} + \cdots + \frac{b_{-n}x}{\beta^n}.$$

However $\text{ord}_\beta(b_i x) \geq 1-d$ because $\deg(b_i) < m$ for all $s \leq i \leq -n$, so $\text{ord}_\beta(xf) \geq \text{ord}_\beta(f) + 1 - d$. If we replace f by x^m, we will have: $\text{ord}_\beta(x^{m+1}) \geq 2(1-d)$ and by a simple induction we get $\text{ord}_\beta(x^k) \geq (k-m+1)(1-d)$ for all $k \geq m$. □

Theorem 3.5. *Let β be a Pisot unit series of algebraical degree d and let $k \geq m = \deg(\beta)$. Then*

$$(k-m+1)(1-d) \leq \text{ord}_\beta(x^k) \leq (\frac{k}{m} - 1)(1-d).$$

Proof. It suffices to show the first inequality. Let $\beta_2, \ldots, \beta_d$ the conjugates of β in the algebraic closure of $\mathbb{F}_d((x^{-1}))$. Since β is unit we have $|\beta\beta_2 \ldots \beta_d| = 1$. It implies that $|\beta_2 \ldots \beta_d| = \frac{1}{|\beta|}$, so there exists at least one conjugate β_j such that

$$|\beta_j| > \frac{1}{|\beta|^{\frac{1}{d-1}}}. \tag{1}$$

Let

$$x^k = a_{-s}\beta^s + \cdots + a_0 + \frac{a_1}{\beta} + \frac{a_2}{\beta^2} + \cdots + \frac{a_n}{\beta^n}$$

be the expansion of x^k. We have then,

$$x^k = a_{-s}\beta_j^s + \cdots + a_0 + \frac{a_1}{\beta_j} + \frac{a_2}{\beta_j^2} + \cdots + \frac{a_n}{\beta_j^n}. \tag{2}$$

So from (1) and (2), we get

$$|x^k| < \frac{|\beta|}{|\beta_j|^n} < |\beta|^{1+\frac{n}{d-1}},$$

hence $k < m(1 + \frac{n}{d-1})$ which implies that

$$(\frac{k}{m} - 1)(d-1) < n = -\text{ord}_\beta(x^k).$$

□

Theorem 3.6. *Let β be a quadratic Pisot unit with $\deg(\beta) = 1$. Then for all $k \geq 1$, $\text{ord}_\beta(x^k) = -k$.*

Proof. According to Theorem 3.5 and for $m = 1$, we have

$$k(1-d) \leq \mathrm{ord}_\beta(x^k) < (k-1)(1-d).$$

Since β is quadratic, then $d = 2$ and $k - 1 < -\mathrm{ord}_\beta(x^k) \leq k$. Therefore $\mathrm{ord}_\beta(x^k) = -k$. □

Corollary 3.7. *Let β be a quadratic Pisot unit with $\deg(\beta) = 1$. Then for all $P \in \mathbb{F}_q[x]$,*

$$\mathrm{ord}_\beta(P) = -\deg(P).$$

Corollary 3.8. *Let β be a Pisot series of algebraical degree d with $\deg(\beta) = m$. Then for all polynomials P of degree $\geq m$,*

$$\mathrm{ord}_\beta(P) \geq (\deg P - m + 1)(1-d).$$

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