APPLICATIONS OF HILBERT SPACE DISSIPATIVE NORM

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Abstract. The concept of Hilbert space dissipative norm was introduced in [8] to obtain necessary and sufficient conditions for exponential stability of contraction semigroups. In the present paper we show that the same concept can also be used to derive further properties of contraction semigroups, as well as to characterize strongly stable semigroups that are not exponentially stable.

1. Introduction

Strong stability of continuous and discrete operator semigroups on Banach and Hilbert spaces have been extensively studied in current literature. We refer to [2] for a recent and comprehensive discussion. The present paper is a sequel to our effort to go after strong stability of continuous and discrete Hilbert space contraction semigroups [1, 7, 8, 9, 10].

Necessary and sufficient conditions for e-stability (i.e., exponential stability) and s-stability (i.e., strong stability) of Hilbert space contraction semigroups were recently obtained in [8] in terms of an inequality involving the Hilbert space norm and the dissipative norm. This note is a sequel to [8]. Here we show several applications of dissipative norm to Hilbert space contraction semigroups.

In the following $[T(t)] = \{T(t) \, ; t \geq 0\}$ will always denote a C_0-semigroup (i.e., a strongly continuous semigroup) of contraction operators over a complex Hilbert space $\mathcal{H}$, with inner product $\langle \cdot \, ; \cdot \rangle$ and norm $\| \cdot \|$. It is said to be *exponentially stable* (or *e-stable*) if there exist real constants $\alpha > 0$ and $M \geq 1$ such that

$$\|T(t)\| \leq Me^{-\alpha t} \quad \text{for every } t \geq 0$$

(equivalently, if $\|T(t)x\| \leq Me^{-\alpha t}\|x\|$ for every $x \in \mathcal{H}$ and every $t \geq 0$). It is said to be *plain-e-stable* if it is e-stable with $M = 1$. Recall that $[T(t)]$ is *uniformly stable* (i.e., $\lim_{t \to \infty} \|T(t)\| = 0$) if and only if it is exponentially stable

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[4]. A semigroup $[T(t)]$ is *strongly stable* (or *s-stable*) if

$$\lim_{t \to \infty} \|T(t)x\| = 0 \quad \text{for every} \ x \in \mathcal{H}.$$

The generator $A \colon \mathcal{D} \to \mathcal{H}$ of $[T(t)]$ is a closed linear transformation on a dense linear manifold $\mathcal{D} = \mathcal{D}(A)$ of $\mathcal{H}$. Moreover, A is maximal dissipative; that is,

$$\operatorname{Re}\langle Ax \, ; x \rangle \leq 0 \quad \text{for every} \ x \in \mathcal{D},$$

and maximal because it does not admit any dissipative extension in $\mathcal{H}$ [5, 6]. The generator A is called strictly dissipative if

$$\operatorname{Re}\langle Ax \, ; x \rangle < 0 \quad \text{for every} \ 0 \neq x \in \mathcal{D}.$$

It is readily verified that if the generator A is strictly dissipative, then the functional $(-2 \operatorname{Re}\langle A\cdot \, ; \cdot\rangle)^{\frac{1}{2}} \colon \mathcal{D} \to \mathbb{R}$ defines a norm on the linear manifold $\mathcal{D}$. Such a norm on $\mathcal{D}$ is called a dissipative norm [8] and is denoted by $\|\cdot\|_d$; that is,

$$\|x\|_d = (-2 \operatorname{Re}\langle Ax \, ; x \rangle)^{\frac{1}{2}} \quad \text{for every} \ x \in \mathcal{D}.$$

Expressions for $\|\cdot\|_d$ in terms of the generator A as well as of the cogenerator S that will be needed throughout the paper are considered in Section 2, and then we obtain some inequalities involving both norms $\|\cdot\|$ and $\|\cdot\|_d$ that will be required for deriving further properties of the generator and cogenerator of $[T(t)]$. The main results appear in Section 3. They deal with strongly and exponentially stable semigroups. Exponential and strong stability are compared, leading to a characterization of strongly stable semigroups that are not exponentially stable.

2. Preliminaries

Let A be the generator of the semigroup $[T(t)]$. Recall that

$$(2.1) \qquad \|(A \pm I)x\|^2 = \|Ax\|^2 + \|x\|^2 \pm 2 \operatorname{Re}\langle Ax \, ; x \rangle$$

for every $x \in \mathcal{D}$, and that A is dissipative. If, in addition, A is strictly dissipative, then the dissipative norm

$$(2.2) \qquad \|x\|_d^2 = -2 \operatorname{Re}\langle Ax \, ; x \rangle$$

is expressed as

$$(2.3) \qquad \|x\|_d^2 = \tfrac{1}{2}\big(\|(A - I)x\|^2 - \|(A + I)x\|^2\big)$$

for every $x \in \mathcal{D}$. The cogenerator $S = (A + I)(A - I)^{-1}$ of $[T(t)]$ is the Cayley transform of the generator A. Since A is dissipative, it follows by (2.1) that $\|x\| \leq \|(A - I)x\|$ for every $x \in \mathcal{D}$, and so $A - I$ is injective. Actually, in this case, $A - I$ has a bounded inverse on its range. Therefore [5, 6],

$$(2.4) \qquad S = (A + I)(A - I)^{-1} = I + 2(A - I)^{-1} = -I + 2A(A - I)^{-1}.$$

Moreover, $1 \notin \sigma_P(S)$, with $\sigma_P(S)$ standing for the point spectrum of S (i.e., $S - I$ is injective), and the domain of $S - I$ (and so the domain of S) coincides with the range of $A - I$ (i.e., $\mathcal{D}(S - I) = \mathcal{R}(A - I)$) which is all of $\mathcal{H}$ (i.e.,

$(A - I)\mathcal{D} = \mathcal{H}$ because A is densely defined) [5, 6]. Conversely, the generator A can be expressed in terms of the cogenerator S as follows [5, 6].

$$(2.5) \qquad A = (S + I)(S - I)^{-1} = I + 2(S - I)^{-1} = -I + 2S(S - I)^{-1}.$$

Recall that the range of $S - I$ coincides with the domain of A (i.e., $\mathcal{R}(S - I) = \mathcal{D}$).

The following expressions of $\| \cdot \|_d$ in terms of S can be readily verified by using the previous identities from (2.1) to (2.5).

Lemma 1. *Let $[T(t)]$ be a contraction semigroup with a strictly dissipative generator A and cogenerator S. Then, for every $x \in \mathcal{D}$,*

$$(2.6) \qquad \|x\|_d^2 \;=\; 2\big(\|(S - I)^{-1}x\|^2 - \|S(S - I)^{-1}x\|^2\big).$$

Moreover, if $x = (S - I)y \in \mathcal{D}$ for some $y \in \mathcal{H}$,

$$(2.7) \qquad \|x\|_d^2 \;=\; 2\big(\|y\|^2 - \|Sy\|^2\big)$$
$$(2.8) \qquad\qquad =\; 2\|Dy\|^2$$
$$(2.9) \qquad\qquad =\; 2\|D(S - I)^{-1}x\|^2 \;=\; \tfrac{1}{2}\|D(A - I)x\|^2,$$

where

$$(2.10) \qquad\qquad D = (I - S^*S)^{\frac{1}{2}}$$

is the defect operator of the contraction S [11].

Note that, under the assumption of Lemma 1, the cogenerator S is, in fact, a proper contraction, that is,

$$(2.11) \qquad\qquad \|Sy\| < \|y\| \quad \text{for every } 0 \neq y \in \mathcal{H},$$

because the left side of (2.7) is positive (Recall that $\mathcal{D}(S) = \mathcal{D}(S - I) = \mathcal{H}$). Since $\|x\|_d^2 = -2\,\mathrm{Re}\,\langle Ax\,;x\rangle$ for every $x \in \mathcal{D}$ (cf. (2.2)), it is also easy to see from (2.1) and (2.3) that the following inequalities hold.

Lemma 2. *Consider the assumptions of the previous lemma. Then, for $x \in \mathcal{D}$,*

$$(2.12) \qquad \|x\|^2 \;\leq\; \|(A - I)x\|^2,$$
$$(2.13) \qquad \|x\|_d^2 \;\leq\; \tfrac{1}{2}\|(A - I)x\|^2,$$
$$(2.14) \qquad \|x\|^2 \;\leq\; \|x\|_d^2 + \|(A + I)x\|^2 \;\leq\; \|x\|_d^2 + \|(A - I)x\|^2,$$
$$(2.15) \qquad \|x\|_d^2 \;\leq\; \|Ax\|^2 + \|x\|^2,$$
$$(2.16) \qquad \|Ax\|^2 \;\leq\; \|x\|_d^2 + \|(A + I)x\|^2 \;\leq\; \|x\|_d^2 + \|(A - I)x\|^2.$$

In general, neither $\| \cdot \|$ nor $\| \cdot \|_d$ dominates the other. However, if $[T(t)]$ is e-stable or s-stable, then the situation becomes clearer, as we shall see in the sequel.

3. Exponential and strong stabilities

We begin this section on the characterization of strong and exponential stabilities by recalling the following results from [8].

Lemma 3. *If* $[T(t)]$ *is a contraction semigroup with a strictly dissipative generator* A, *then, for every* $x \in \mathcal{D}$,

(i) $$\int_0^\infty \|T(t)x\|_d^2 \, dt \le \|x\|^2,$$

(ii) $[T(t)]$ *is plain-e-stable if and only if there exists a constant* $\alpha > 0$ *such that*

$$\alpha\|x\| \le \|x\|_d,$$

(iii) $[T(t)]$ *is s-stable if and only if*

$$\int_0^\infty \|T(t)x\|_d^2 \, dt = \|x\|^2.$$

Proof. See Lemma 1, Theorem 2, and Remark 1 in [8]; see also [1] for part (iii). □

Our first theorem presents further properties of the cogenerator S as a consequence of the equivalent condition for e-stability in Lemma 3(ii).

Theorem 1. *Let* $[T(t)]$ *be a contraction semigroup with a strictly dissipative generator* A *and cogenerator* S. *Then the following propositions hold.*
    (i) *The defect operator* D *of* S *is a contraction.*
    (ii) *Suppose* $[T(t)]$ *is plain-e-stable. Then there exists an* $\alpha > 0$ *such that*

(ii.a) $$\alpha\|(S - I)y\|^2 \le \|y\|^2 - \|Sy\|^2,$$

and so

(ii.b) $$\alpha\big(\|y\| - \|Sy\|\big)^2 \le \alpha\|(S - I)y\|^2 \le \|y\|^2 - \|Sy\|^2$$

for every $y \in \mathcal{H}$. *For such an* α *it follows that:* $S - I$ *is a contraction if* $\alpha \ge 1$, S *is boundedly invertible if* $\alpha > 1$, *and* A *is boundedly invertible if* $\alpha > \frac{1}{2}$.

Proof. First recall that, under the above assumptions, S is a proper contraction.
    (i) From (2.4), (2.8), (2.9) and (2.13), for every $x = (S - I)y \in \mathcal{D}$ and $y \in \mathcal{H}$,

$$2\|Dy\|^2 = \|x\|_d^2 \le \tfrac{1}{2}\|(A - I)x\|^2 = \tfrac{1}{2}\|(A - I)(S - I)y\|^2$$
$$= \tfrac{1}{2}\|(A - I)\, 2\,(A - I)^{-1}y\|^2,$$

and therefore, for every $y \in \mathcal{H}$,

$$\|Dy\| \le \|y\|.$$

    (ii) Under the additional e-stability assumption, Lemma 3(ii) ensures the existence of an $\alpha > 0$ such that, for every $x \in \mathcal{D}$,

$$2\alpha\|x\|^2 \le \|x\|_d^2.$$

Consider this $\alpha > 0$. Thus (ii.a) follows from (2.7) by setting $x = (S - I)y \in \mathcal{D}$. Moreover, it is plain (by the triangle inequality) that, for every $y \in \mathcal{H}$,

$$\big| \|Sy\| - \|y\| \big| \leq \|(S - I)y\|.$$

This and (ii.a) lead to (ii.b). Furthermore, we also have from (ii.a) that

$$\alpha \|(S - I)y\|^2 \leq \|y\|^2$$

for every $y \in \mathcal{H}$. Hence $S - I$ is a contraction if $\alpha \geq 1$. Moreover, from (ii.b),

$$\alpha \big(\|y\| - \|Sy\| \big)^2 \leq \|y\|^2 + \|Sy\|^2,$$

or

$$\tfrac{\alpha-1}{\alpha+1} \, \|y\|^2 \leq \|Sy\|^2$$

for every $y \in \mathcal{H}$. Therefore, if $\alpha > 1$, then S is boundedly invertible (i.e., S has a bounded inverse). Finally, from (2.15),

$$2\alpha \|x\|^2 \leq \|x\|_d^2 \leq \|Ax\|^2 + \|x\|^2,$$

so that

$$(2\alpha - 1)\|x\|^2 \leq \|Ax\|^2$$

for every $x \in \mathcal{D}$. Hence A is boundedly invertible if that α is such that $\alpha > \frac{1}{2}$. $\square$

If a contraction semigroup $[T(t)]$ is only s-stable but not e-stable, then its generator A is injective but it may not be boundedly invertible. In fact, it is injective since, for any $0 \neq x \in \mathcal{D}$, if $Ax = 0$, then $T(t)x = x$ for every $t \geq 0$, thus implying that $T(t)x \to x$ as $t \to \infty$, which is a contradiction. Hence, if $[T(t)]$ is s-stable, then $\mathcal{N}(A) = \{0\}$ (i.e., A is injective), where $\mathcal{N}(A)$ denotes the kernel of A.

The above facts and Theorem 1(ii) may hint on the difference which separates s-stability via e-stability (i.e., e-stability, which clearly implies s-stability) for contraction semigroups and stand-alone s-stability (i.e., s-stability for non-e-stable contraction semigroups).

Corollary 1. *Let A be a strictly dissipative generator of a contraction semigroup $[T(t)]$. If A is boundedly invertible, and if there exists a constant $\gamma \geq 1$ such that*

$$\|(A + I)x\| \leq \gamma \|x\|_d$$

for every $x \in \mathcal{D}$, then

$$\|(A + I)x\| \leq \tfrac{\sqrt{2}\,\gamma}{2} \|(A - I)x\|$$

for every $x \in \mathcal{D}$, and $[T(t)]$ is plain-e-stable.

Proof. If the generator A has a bounded inverse, then there exists a constant $\beta > 0$ such that

$$\beta \|x\|^2 \leq \|Ax\|^2$$

for every $x \in \mathcal{D}$. Hence, by (2.1) with $\|x\|_d^2 = -2\operatorname{Re}\langle Ax\,;x\rangle$,

$$\beta \|x\|^2 \leq \|Ax\|^2 = \|(A+I)x\|^2 - \|x\|^2 + \|x\|_d^2,$$

and therefore

(3.1) $$(\beta + 1)\|x\|^2 \leq \|x\|_d^2 + \|(A+I)x\|^2$$

for every $x \in \mathcal{D}$. Now suppose that there exists a constant $\gamma > 0$ such that

$$\|(A+I)x\| \leq \gamma \|x\|_d$$

for every $x \in \mathcal{D}$. This and (2.13) imply that

$$\|(A+I)x\|^2 \leq \gamma^2 \|x\|_d^2 \leq \tfrac{\gamma^2}{2} \|(A-I)x\|^2,$$

and so

$$\|(A+I)x\| \leq \tfrac{\sqrt{2}\gamma}{2}\|(A-I)x\|$$

for every $x \in \mathcal{D}$; and also that, from (3.1),

$$(\beta + 1)\|x\|^2 \leq (\gamma^2 + 1)\|x\|_d^2$$

for every $x \in \mathcal{D}$, which ensures that $[T(t)]$ is plain-e-stable by Lemma 3(ii). $\qquad\square$

The next result exhibits further necessary and sufficient conditions for e-stability that, as far as we can tell, have not been discussed before.

Theorem 2. *Let $[T(t)]$ be a contraction semigroup with a strictly dissipative generator A. The following assertions are pairwise equivalent.*

(i) $[T(t)]$ *is e-stable.*

(ii) $$\int_0^\infty \|T(t)(A-I)x\|^2 dt < \infty \quad \text{ for every } x \in \mathcal{D}.$$

(iii) $$\int_0^\infty \|T(t)(A+I)x\|^2 dt < \infty \quad \text{ for every } x \in \mathcal{D}.$$

Proof. If A is strictly dissipative, then we have from (2.14) that

$$\|T(t)x\|^2 \leq \|T(t)x\|_d^2 + \|T(t)(A-I)x\|^2$$

for every $x \in \mathcal{D}$ and every $t \geq 0$, since $T(t)x \in \mathcal{D}$ for every $x \in \mathcal{D}$ and $T(t)$ and A commute on $\mathcal{D}$. Integrating the above expression on $[0,t]$ we get

$$\int_0^t \|T(\tau)x\|^2 d\tau \leq \int_0^t \|T(\tau)x\|_d^2\, d\tau + \int_0^t \|T(\tau)(A-I)x\|^2 d\tau$$

for every $x \in \mathcal{D}$ and every $t \geq 0$. Thus it follows by Lemma 3(i) and assertion (ii) that

$$\int_0^\infty \|T(t)x\|^2 dt < \infty \quad \text{for every} \ x \in \mathcal{D},$$

which, when extended by continuity (of the integral functional) over all $\mathcal{H}$ leads to Datko's equivalent condition for e-stability [3]. Conversely, if $[T(t)]$ is e-stable, then Datko's equivalent condition for e-stability [3], namely,

$$(3.2) \qquad \int_0^\infty \|T(t)z\|^2 dt < \infty \quad \text{for every} \ z \in \mathcal{H},$$

holds. In particular, for $z = (A - I)x$ in $\mathcal{R}(A - I) = \mathcal{H}$ this implies that assertion (ii) holds true for every $x \in \mathcal{D}$. A similar argument (now with $z = (A + I)x$ in $\mathcal{R}(A + I) \subseteq \mathcal{H}$) applies to assertion (iii). $\qquad\square$

Note that the conditions of Theorem 2 are milder than that of Datko's in (3.2).

We now turn to s-stability of contraction semigroups.

Lemma 4. *Let $[T(t)]$ be a contraction semigroup with a strictly dissipative generator A. If, for every $x \in \mathcal{D}$,*

$$\lim_{t \to \infty} \|T(t)(A - I)x\| = 0,$$

then $[T(t)]$ is s-stable and also $\lim_{t \to \infty} \|T(t)x\|_d = 0$ for every $x \in \mathcal{D}$.

Proof. Suppose A is strictly dissipative and consider Lemma 2. Recall that $[T(t)]$ and A commute. From (2.13) we get, for every $x \in \mathcal{D}$ and every $t \geq 0$,

$$\|T(t)x\|_d^2 \leq \tfrac{1}{2} \|T(t)(A - I)x\|^2.$$

Moreover, from (2.14), for every $x \in \mathcal{D}$ and every $t \geq 0$,

$$\|T(t)x\|^2 \leq \|T(t)x\|_d^2 + \|T(t)(A - I)x\|^2.$$

Therefore, if

$$\lim_{t \to \infty} \|T(t)(A - I)x\| = 0$$

for every $x \in \mathcal{D}$, then

$$\lim_{t \to \infty} \|T(t)x\| = \lim_{t \to \infty} \|T(t)x\|_d = 0$$

for every $x \in \mathcal{D}$. Furthermore, $\lim_{t \to \infty} \|T(t)x\| = 0$ extends naturally by continuity (of the norm) from the dense $\mathcal{D}$ to the whole space $\mathcal{H}$, since $\mathcal{D}$ is $T(t)$-invariant. $\qquad\square$

Note that the norms $\|\cdot\|$ and $\|\cdot\|_d$ are not necessarily commensurable on $\mathcal{D}$ (i.e., none dominates the other) because A, although closed and densely defined, is generally unbounded (e.g., see the inequalities in Lemma 2) even under the assumption of s-stability. However, these norms are commensurable on $\mathcal{D}$ under the assumption of plain-e-stability by Lemma 3(ii).

Contraction semigroups that are strong but not exponentially stable are characterized in the following corollary, where it is given a full account of stand-alone s-stable contraction semigroups.

Corollary 2. *Let $[T(t)]$ be a contraction semigroup with a strictly dissipative generator A. $[T(t)]$ is s-stable but not plain-e-stable if and only if*

$$\|x\|^2 = \int_0^\infty \|T(t)x\|_d^2 \, dt$$

for every $x \in \mathcal{D}$, and for every $\alpha > 0$ there exists an $x_\alpha \in \mathcal{D}$ such that

$$\alpha\|x_\alpha\|_d^2 < \|x_\alpha\|^2 = \int_0^\infty \|T(t)x_\alpha\|_d^2 \, dt.$$

Proof. Recall that e-stability clearly implies s-stability. The result is a straightforward consequence of Lemma 3(ii), (iii). In fact, Lemma 3(ii) ensures that $[T(t)]$ is not plain-e-stable if and only if for every $\beta > 0$ there exists an $x_\beta \in \mathcal{D}$ such that

$$\|x_\beta\|_d < \beta\|x_\beta\|.$$

$\square$

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