

## THE LOG-CONVEXITY OF ANOTHER CLASS OF ONE-PARAMETER MEANS AND ITS APPLICATIONS

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ABSTRACT. In this paper, the log-convexity of another class one-parameter mean is investigated. As applications, some new upper and lower bounds of logarithmic mean, new estimations for identric mean and new inequalities for power-exponential mean and exponential-geometric mean are first given.

### 1. Introduction and main results

Let  $p, q \in \mathbb{R}$  and  $a, b \in \mathbb{R}_+$  – the positive semi-axis. For  $a \neq b$  the Stolarsky mean is defined as

$$(1.1) \quad S_{p,q}(a,b) = \begin{cases} \left( \frac{q a^p - b^p}{p a^q - b^q} \right)^{1/(p-q)}, & pq(p-q) \neq 0, \\ \left( \frac{1}{p} \frac{a^p - b^p}{\ln a - \ln b} \right)^{1/p}, & p \neq 0, q = 0, \\ \left( \frac{1}{q} \frac{a^q - b^q}{\ln a - \ln b} \right)^{1/q}, & q \neq 0, p = 0, \\ \exp \left( \frac{a^p \ln a - b^p \ln b}{a^p - b^p} - \frac{1}{p} \right), & p = q \neq 0, \\ \sqrt{ab}, & p = q = 0, \end{cases}$$

and  $S_{p,q}(a,a) = a$  (see [33]). Another two-parameter family of means was introduced by C. Gini in [13]. That is defined as

$$(1.2) \quad G_{p,q}(a,b) = \begin{cases} \left( \frac{a^p + b^p}{a^q + b^q} \right)^{1/(p-q)}, & p \neq q, \\ \exp \left( \frac{a^p \ln a + b^p \ln b}{a^p + b^p} \right), & p = q. \end{cases}$$

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Recently, a more general two-parameter family has been established by the author in [40], which is stated as follows.

**Definition 1.1.** Assume  $f: \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+ \cup \{0\}$  is  $n$ -order homogeneous, continuous and has the first partial derivatives and  $(a, b) \in \mathbb{R}_+ \times \mathbb{R}_+$ ,  $(p, q) \in \mathbb{R} \times \mathbb{R}$ .

If  $f(x, y) > 0$  for  $(x, y) \in \mathbb{R}_+ \times \mathbb{R}_+$  with  $x \neq y$  and  $f(x, x) = 0$  for all  $x \in \mathbb{R}_+$ , then we define that

$$(1.3) \quad \mathcal{H}_f(p, q; a, b) = \left( \frac{f(a^p, b^p)}{f(a^q, b^q)} \right)^{1(p-q)} \quad (p \neq q, pq \neq 0),$$

$$(1.4) \quad \mathcal{H}_f(p, p; a, b) = \lim_{q \rightarrow p} \mathcal{H}_f(p, q; a, b) = G_{f,p}(a, b) \quad (p = q \neq 0),$$

where

$$(1.5) \quad G_{f,p}(a, b) = G_f^{\frac{1}{p}}(a^p, b^p), \quad G_f(x, y) = \exp \left( \frac{x f_x(x, y) \ln x + y f_y(x, y) \ln y}{f(x, y)} \right).$$

Here  $f_x(x, y)$  and  $f_y(x, y)$  denote the first-order partial derivative with respect to the first and the second component of  $f(x, y)$ , respectively.

If  $f(x, y) > 0$  for all  $(x, y) \in \mathbb{R}_+ \times \mathbb{R}_+$ , then we define further

$$(1.6) \quad \mathcal{H}_f(p, 0; a, b) = \left( \frac{f(a^p, b^p)}{f(1, 1)} \right)^{1/p} \quad (p \neq 0, q = 0),$$

$$(1.7) \quad \mathcal{H}_f(0, q; a, b) = \left( \frac{f(a^q, b^q)}{f(1, 1)} \right)^{1/q} \quad (p = 0, q \neq 0),$$

$$(1.8) \quad \mathcal{H}_f(0, 0; a, b) = \lim_{p \rightarrow 0} \mathcal{H}_f(p, 0; a, b) = a^{\frac{f_x(1,1)}{f(1,1)}} b^{\frac{f_y(1,1)}{f(1,1)}} \quad (p = q = 0).$$

Since  $f(x, y)$  is a homogeneous function,  $\mathcal{H}_f(p, q; a, b)$  is also one and called a homogeneous function with parameters  $p$  and  $q$ , and simply denoted by  $\mathcal{H}_f(p, q)$  sometimes.

The monotonicity, convexity and compatibility of Stolarsky and Gini mean have been completely solved (see [17, 18, 27, 25]). It was not until quite recently that unified treatments of their monotonicity and log-convexity were given (see [40, 41]).

Let  $q = p + 1$  in the Stolarsky mean and Gini mean. Then  $S_{p,p+1}(a, b)$  and  $G_{p,p+1}(a, b)$  become the so-called one-parameter mean and Lehmer mean, respectively. Concerning the two one-parameter family of means there are many useful and interesting results (see [4, 5, 6, 36, 37, 11]). As a general form, the one-parameter homogeneous functions were also investigated by the author (see [38]).

The *one-parameter mean* is relative to the *two-parameter mean*. In general,  $S_{p,q}(a, b)$  and  $G_{p,q}(a, b)$  may be called a one-parameter family of means, respectively, provided that there exist certain given functional relations between

its parameters  $p$  and  $q$ . For example, let  $q = 2c - p$  with  $c \neq 0$  in  $S_{p,q}(a, b)$  and  $G_{p,q}(a, b)$ . Then  $S_{p,2c-p}(a, b)$  and  $G_{p,2c-p}(a, b)$  may be also called one-parameter family of means, respectively. For avoiding confusion,  $S_{p,p+1}(a, b)$  and  $G_{p,p+1}(a, b)$  are called the first one-parameter family of means, respectively; while  $S_{p,2c-p}(a, b)$  and  $G_{p,2c-p}(a, b)$  are called the second ones, respectively. Generally,  $\mathcal{H}_f(p, p+1; a, b)$  is called the first one-parameter family and  $\mathcal{H}_f(p, 2c-p; a, b)$  is called the second one. However, as far as monotonicity and convexity are concerned in parameter  $p$ ,  $\mathcal{H}_f(p, 2c-p; a, b)$  is actually equivalent to  $\mathcal{H}_f(p, 1-p; a, b)$  in view of  $\mathcal{H}_f^{2c}(p, 2c-p; a, b) = \mathcal{H}_f(p/2c, 1-p/2c; a^{2c}, b^{2c})$ . For this reason, in what follows we only consider the case of  $2c = 1$  and call  $\mathcal{H}_f(p, 1-p; a, b)$  the second one-parameter family.

Substituting logarithmic mean

$$(1.9) \quad L(x, y) = \frac{x-y}{\ln x - \ln y} \quad (x, y > 0, x \neq y), \quad L(x, x) = x$$

and arithmetic mean

$$(1.10) \quad A(x, y) = \frac{x+y}{2} \quad (x, y > 0)$$

for  $f$  in  $\mathcal{H}_f(p, 1-p; a, b)$  yields  $S_{p,1-p}(a, b)$  and  $G_{p,1-p}(a, b)$ . For the sake of unification, we adopt our notations to denote them by  $\mathcal{H}_L(p, 1-p; a, b)$  and  $\mathcal{H}_A(p, 1-p; a, b)$ , respectively. Their concrete expressions are as follows:

$$(1.11) \quad \mathcal{H}_L(p, 1-p; a, b) = \begin{cases} \left( \frac{(1-p)(a^p - b^p)}{p(a^{1-p} - b^{1-p})} \right)^{1/(2p-1)}, & p \neq 1/2, \\ I^2(\sqrt{a}, \sqrt{b}), & p = 1/2, \end{cases}$$

where

$$(1.12) \quad I(x, y) = e^{-1}(x^x/y^y)^{1/(x-y)} \quad (x, y > 0, x \neq y), \quad I(x, x) = x$$

is well-known identric mean (exponential mean).

$$(1.13) \quad \mathcal{H}_A(p, 1-p; a, b) = \begin{cases} \left( \frac{a^p + b^p}{a^{1-p} + b^{1-p}} \right)^{1/(2p-1)}, & p \neq 1/2, \\ Z^2(\sqrt{a}, \sqrt{b}), & p = 1/2, \end{cases}$$

where

$$(1.14) \quad Z(x, y) = x^{x/(x+y)} y^{y/(x+y)} \quad (x, y > 0)$$

is called power-exponential mean.

In addition, we also consider other two the second one-parameter families. Substituting identric mean  $I$  defined by (1.12) for  $f$  in  $\mathcal{H}_f(p, 1-p; a, b)$  yields

$$(1.15) \quad \mathcal{H}_I(p, 1-p; a, b) = \begin{cases} \left( \frac{I(a^p, b^p)}{I(a^{1-p}, b^{1-p})} \right)^{1/(2p-1)}, & p \neq 1/2, \\ Y^2(\sqrt{a}, \sqrt{b}), & p = 1/2, \end{cases}$$

where

$$(1.16) \quad Y(x, y) = I(x, y) \exp(1 - G^2(x, y)/L^2(x, y)) \quad (x, y > 0, x \neq y), \quad Y(x, x) = x$$

is called exponential-geometric mean.

Substituting  $D(x, y) = |x - y|$  ( $x, y > 0, x \neq y$ ) for  $f$  in  $\mathcal{H}_f(p, 1 - p; a, b)$  yields

$$(1.17) \quad \mathcal{H}_D(p, 1 - p; a, b) = \begin{cases} \left| \frac{a^p - b^p}{a^{1-p} - b^{1-p}} \right|^{1/(2p-1)}, & p \neq 0, 1, 1/2, \\ e^2 I^2(\sqrt{a}, \sqrt{b}), & p = 1/2. \end{cases}$$

It should be noted that  $\mathcal{H}_D(p, 1 - p; a, b)$  is not a mean of positive numbers  $a$  and  $b$ , which has merely a form of means.

The monotonicity of  $S_{p,1-p}(a, b)$  or  $\mathcal{H}_L(p, 1 - p; a, b)$  was surveyed by E. Leach and M. Sholander at the earliest [18], who gave such a result: For  $c > 0$  and  $-\infty < p \leq c$ ,  $S_{p,2c-p}(a, b)$  increases from  $G$  to  $I_c$ , where  $I_c$  stands for the identric mean of order  $c$ . Recently, the author proved more general result involving the monotonicity of  $\mathcal{H}_f(p, 1 - p; a, b)$ . It reads as follows.

**Theorem 1.1** ([41, Corollary 1]). *Let  $f : \mathbb{R}_+ \times \mathbb{R}_+ \setminus \{(x, x) : x \in \mathbb{R}_+\} \rightarrow \mathbb{R}_+$  be a symmetric, homogenous and three-time differentiable function. If  $\mathcal{J} = (x - y)(x\mathcal{I})_x < (>)0$  where  $\mathcal{I} = (\ln f)_{xy}$ , then  $\mathcal{H}_f(p, 1 - p)$  is strictly decreasing (increasing) in  $p \in (0, 1/2)$  and increasing (decreasing) in  $p \in (1/2, 1)$ .*

*Furthermore, if  $f(x, y)$  is defined on  $\mathbb{R}_+ \times \mathbb{R}_+$  and symmetric with respect to  $x$  and  $y$ , then  $\mathcal{H}_f(p, 1 - p)$  is strictly decreasing (increasing) in  $p \in (-\infty, 1/2)$  and increasing (decreasing) in  $p \in (1/2, +\infty)$ .*

The main objective of this paper is to give a new proof of Theorem 1.1 and investigate the log-convexity of the second one-parameter family. Our main results are as follows:

**Theorem 1.2.** *Let  $f : \mathbb{R}_+ \times \mathbb{R}_+ \setminus \{(x, x) : x \in \mathbb{R}_+\} \rightarrow \mathbb{R}_+$  be a symmetric, homogenous and three-time differentiable function. Then  $\mathcal{H}_f(p, 1 - p)$  is log-convex (log-concave) in  $p$  on  $(0, 1)$  if  $\mathcal{J} = (x - y)(x\mathcal{I})_x < (>)0$ , where  $\mathcal{I} = (\ln f)_{xy}$ .*

**Corollary 1.1.** *That  $\mathcal{H}_L(p, 1 - p; a, b)$ ,  $\mathcal{H}_A(p, 1 - p; a, b)$  and  $\mathcal{H}_I(p, 1 - p; a, b)$  all are strictly increasing in  $p \in (-\infty, 1/2)$  and decreasing in  $p \in (1/2, \infty)$ , and log-concave in  $p$  on  $[0, 1]$ .*

**Corollary 1.2.** *That  $\mathcal{H}_D(p, 1 - p; a, b)$  is strictly decreasing in  $p \in (0, 1/2)$  and increasing in  $p \in (1/2, 1)$ , and log-convex in  $p$  on  $(0, 1)$ .*

## 2. Properties and lemmas

Before formulating our main results, let us recall the properties of two-parameter homogeneous functions.

**Property 2.1.**  $\mathcal{H}_f(p, q)$  is symmetric with respect to  $p, q$ , i.e.,

$$(2.1) \quad \mathcal{H}_f(p, q) = \mathcal{H}_f(q, p).$$

**Property 2.2.** If  $f(x, y)$  is symmetric with respect to  $x$  and  $y$ , then

$$(2.2) \quad \mathcal{H}_f(-p, -q; a, b) = \frac{G^{2n}}{\mathcal{H}_f(p, q; a, b)},$$

$$(2.3) \quad \mathcal{H}_f(p, -p; a, b) = G^n,$$

where  $G = \sqrt{ab}$ .

**Property 2.3** ([41, (1.13)]). If  $G_{f,t}$  is continuous on  $[q, p]$  or  $[p, q]$ , then

$$(2.4) \quad \ln \mathcal{H}_f(p, q) = \frac{1}{p-q} \int_q^p \ln G_{f,t} dt,$$

where  $G_{f,t}$  is defined by (1.5).

It is worth mentioning that the following function

$$(2.5) \quad T(t) := \ln f(a^t, b^t), t \neq 0$$

is well-behaved, whose properties as useful lemmas are read as follows.

**Lemma 2.1** ([41, (1.14), (1.15), (2.10), (2.11)]). *Suppose that  $f : \mathbb{R}_+ \times \mathbb{R}_+ \setminus \{(x, x) : x \in \mathbb{R}_+\} \rightarrow \mathbb{R}_+$  is a symmetric,  $n$ -order homogenous and three-time differentiable function. Then*

$$(2.6) \quad T(t) - T(-t) = 2nt \ln G,$$

$$(2.7) \quad T'(t) + T'(-t) = 2n \ln G,$$

$$(2.8) \quad T''(-t) = T''(t),$$

$$(2.9) \quad T'''(-t) = -T'''(t),$$

where  $G = \sqrt{ab}$ .

*Remark 2.1.* If  $f(1, 1) := \lim_{x \rightarrow 1} f(x, 1) > 0$ , then  $T(t)$  can be extended continuously by defining  $T(0) := \lim_{t \rightarrow 0} T(t) = \ln f(1, 1)$ . With the result that  $T(t)$  is also three-time derivable at  $t = 0$  and  $T'(0) := n \ln G$ . Thus (2.7) can be written as

$$(2.10) \quad T'(t) + T'(-t) = 2T'(0).$$

**Lemma 2.2** ([41, Lemma 3 and Lemma 4]). *Suppose that  $f : \mathbb{R}_+ \times \mathbb{R}_+ \setminus \{(x, x) : x \in \mathbb{R}_+\} \rightarrow \mathbb{R}_+$  is a homogenous and three-time differentiable function. Then*

$$(2.11) \quad T'(t) = \frac{a^t f_x(a^t, b^t) \ln a + b^t f_b(a^t, b^t) \ln b}{f(a^t, b^t)},$$

$$(2.12) \quad T''(t) = -xy \mathcal{I} \ln^2(b/a), \quad \mathcal{I} = (\ln f)_{xy},$$

$$(2.13) \quad T'''(t) = -Ct^{-3} \mathcal{J}, \quad \mathcal{J} = (x-y)(x\mathcal{I})_x, C = \frac{xy \ln^3(x/y)}{x-y} > 0,$$

where  $x = a^t, y = b^t$ .

*Remark 2.2.* Compared (1.5) with (2.10), it is easy to see that

$$T'(t) = \ln G_{f,t}(a, b).$$

Thus (2.4) can be expressed in integral form as

$$(2.14) \quad \ln \mathcal{H}_f(p, q) = \begin{cases} \frac{1}{p-q} \int_q^p T'(t) dt & \text{if } p \neq q \\ T'(q) & \text{if } p = q \end{cases} = \int_0^1 T'(tp + (1-t)q) dt.$$

### 3. Proofs of main results

*New proof of Theorem 1.1.* 1) By (2.4) and (2.14),  $\ln \mathcal{H}_f(p, 1-p)$  can be expressed in integral form as

$$(3.1) \quad \ln \mathcal{H}_f(p, 1-p) = \int_0^1 T'(t_1(t)) dt$$

for  $p \in (0, 1)$ , where  $t_1(t) = tp + (1-t)(1-p)$ . A direct partial derivative calculation leads to

$$(3.2) \quad \frac{\partial \ln \mathcal{H}_f(p, 1-p)}{\partial p} = \int_0^1 (2t-1) T''(t_1(t)) dt,$$

which can be splitted into two parts:

$$\int_0^{1/2} (2t-1) T''(t_1(t)) dt + \int_{1/2}^1 (2t-1) T''(t_1(t)) dt.$$

Substituting  $t = 1-v$  in the first integral above yields

$$\int_0^{1/2} (2t-1) T''(t_1(t)) dt = - \int_{1/2}^1 (2v-1) T''(t_2(v)) dv,$$

where  $t_2(t) = (1-t)p + t(1-p)$ . Hence

$$(3.3) \quad \begin{aligned} \frac{\partial \ln \mathcal{H}_f(p, 1-p)}{\partial p} &= - \int_{1/2}^1 (2v-1) T''(t_2(v)) dv + \int_{1/2}^1 (2t-1) T''(t_1(t)) dt \\ &= \int_{1/2}^1 (2t-1) (T''(t_1(t)) - T''(t_2(t))) dt \\ &= \int_{1/2}^1 (2t-1) \left( \int_{t_2(t)}^{t_1(t)} T'''(s) ds \right) dt. \end{aligned}$$

Obviously,  $t_1(t), t_2(t) > 0$  due to  $0 < p < 1$  and  $1/2 \leq t \leq 1$ , hence  $T'''(s) = -Cs^{-3} \mathcal{J} > (<) 0$  if  $\mathcal{J} = (x-y)(x\mathcal{L})_x < (>) 0$ , where  $C = \frac{xy \ln^3(x/y)}{x-y} > 0$ ,  $x = a^s$ ,  $y = b^s$ ,  $s$  lies between  $t_1(t)$  and  $t_2(t)$ . It follows that  $\frac{\partial \ln \mathcal{H}_f(p, 1-p)}{\partial p}$  is positive (negative) if  $t_1(t) > t_2(t)$  and negative (positive) if  $t_1(t) < t_2(t)$ . While

$$t_1(t) - t_2(t) = (2t-1)(2p-1),$$

hence

$$\frac{\partial \ln \mathcal{H}_f(p, 1-p)}{\partial p} \begin{cases} < (>)0 \text{ for } p \in (0, 1/2) \\ > (<)0 \text{ for } p \in (1/2, 1) \end{cases} \text{ if } \mathcal{J} < (>)0.$$

2) If  $f(x, y)$  is defined on  $\mathbb{R}_+ \times \mathbb{R}_+$  and symmetric with respect to  $x$  and  $y$  further, then  $T(t)$  is defined on  $\mathbb{R}$  which is three-time derivable and  $T''(t)$  is even by (2.8). Thus (3.1) holds for  $p \in \mathbb{R}$ , and then from (3.3) we have

$$\begin{aligned} \frac{\partial \ln \mathcal{H}_f(p, 1-p)}{\partial p} &= \int_{1/2}^1 (2t-1) (T''(t_1(t)) - T''(t_2(t))) dt \\ &= \int_{1/2}^1 (2t-1) (T''(|t_1(t)|) - T''(|t_2(t)|)) dt \\ &= \int_{1/2}^1 (2t-1) \left( \int_{|t_2(t)|}^{|t_1(t)|} T'''(s) ds \right) dt, \end{aligned}$$

where  $t_1(t) = tp + (1-t)(1-p)$ ,  $t_2(t) = (1-t)p + t(1-p)$ .

Clearly,  $T'''(s) = -Cs^{-3}\mathcal{J} > (<) 0$  if  $\mathcal{J} = (x-y)(x\mathcal{I})_x < (>) 0$ , where  $C = \frac{xy \ln^3(x/y)}{x-y} > 0$ ,  $x = a^s$ ,  $y = b^s$ ,  $s$  lies between  $|t_1(t)|$  and  $|t_2(t)|$ . It follows that  $\frac{\partial \ln \mathcal{H}_f(p, 1-p)}{\partial p}$  is positive (negative) if  $|t_1(t)| > |t_2(t)|$  and negative (positive) if  $|t_1(t)| < |t_2(t)|$ . However,

$$|t_1(t)|^2 - |t_2(t)|^2 = (2t-1)(2p-1),$$

hence

$$\frac{\partial \ln \mathcal{H}_f(p, 1-p)}{\partial p} \begin{cases} < (>)0 \text{ for } p < 1/2, \\ > (<)0 \text{ for } p > 1/2 \end{cases} \text{ if } \mathcal{J} < (>)0.$$

This completes the proof.  $\square$

*Proof of Theorem 1.2.* A partial derivative calculation for (3.2) leads to

$$\frac{\partial^2 \ln \mathcal{H}_f(p, 1-p)}{\partial p^2} = \int_0^1 (2t-1)^2 T'''(t_1) dt.$$

Obviously,  $t_1(t) = tp + (1-t)(1-p) > 0$  due to  $0 < p < 1$  and  $0 \leq t \leq 1$ , hence  $T'''(t_1) = -Ct_1^{-3}\mathcal{J} > (<)0$  if  $\mathcal{J} = (x-y)(x\mathcal{I})_x < (>)0$ , where  $C = \frac{xy \ln^3(x/y)}{x-y} > 0$ ,  $x = a^{t_1}$ ,  $y = b^{t_1}$ . It follows that

$$\frac{\partial^2 \ln \mathcal{H}_f(p, 1-p)}{\partial p^2} > (<)0 \text{ for } p \in (0, 1) \text{ if } \mathcal{J} = (x-y)(x\mathcal{I})_x < (>)0.$$

The proof is accomplished.  $\square$

*Proof of Corollary 1.1.* By Theorem 1.2, the monotonicity and log-convexity of  $\mathcal{H}_f(p, 1-p)$  depend on the sign of  $\mathcal{J} = (x-y)(x\mathcal{I})_x$ . From Section 4 of [41, 4. Some conclusions concerning  $L, A, E$  and  $D$ ] we see that  $\mathcal{J} > 0$  for  $f(x, y) = L(x, y), A(x, y), I(x, y)$ . Using Theorems 1.1 and 1.2, we see that  $\mathcal{H}_f(p, 1-p)$  is strictly increasing in  $p \in (-\infty, 1/2)$  and decreasing in  $p \in (1/2, \infty)$ , and log-concave in  $p$  on  $(0, 1)$ .

However, owing to

$$f(1, 1) := \lim_{x \rightarrow 1} f(x, 1) = 1 \text{ for } f = L, A, I,$$

then

$$\mathcal{H}_f(0, 1) := \lim_{p \rightarrow 0} \mathcal{H}_f(p, 1-p) = f(a, b) \text{ for } f = L, A, I;$$

similarly,  $\mathcal{H}_f(1, 0)$  is also equal to  $f(a, b)$ . Hence the log-concave interval of  $\mathcal{H}_f(p, 1-p)$  can be extended to  $[0, 1]$ .

This corollary follows.  $\square$

*Proof of Corollary 1.2.* By Theorem 1.2, the monotonicity and log-convexity of  $\mathcal{H}_f(p, 1-p)$  depend on the sign of  $\mathcal{J} = (x-y)(x\mathcal{I})_x$ . From Section 4 of [41] we see that  $\mathcal{J} < 0$  for  $f(x, y) = D(x, y)$ . Using Theorems 1.1 and 1.2, we see that  $\mathcal{H}_D(p, 1-p; a, b)$  is strictly decreasing in  $p \in (0, 1/2)$  and increasing in  $p \in (1/2, 1)$ , and log-convex in  $p$  on  $(0, 1)$ .

While

$$f(1, 1) := \lim_{x \rightarrow 1} f(x, 1) = 0 \text{ for } f = D,$$

hence  $\mathcal{H}_f(0, 1)$  and  $\mathcal{H}_f(1, 0)$  both do not exist. Consequently, the log-convex interval of  $\mathcal{H}_D(p, 1-p)$  cannot be extended to  $[0, 1]$ .

This corollary is proved.  $\square$

#### 4. Some new inequalities for means

It is well-known that monotonicity can lead to many inequalities but the convexity sometimes yields more refined ones. In this section we will apply main results to present some new and inequalities for known means.

To begin with, let us note that the known fact:  $f(x)$  is concave (convex) on  $\Omega$  if and only if for  $x, y \in \Omega$  with  $x \neq y$  the function  $\frac{f(x)-f(y)}{x-y}$  is decreasing (increasing) in either  $x$  or  $y$ . It follows from Theorem 1.2 that the function

$$\frac{\ln \mathcal{H}_f(p, 1-p) - \ln \mathcal{H}_f(q, 1-q)}{p-q} = \ln \left( \frac{\mathcal{H}_f(p, 1-p)}{\mathcal{H}_f(q, 1-q)} \right)^{1/(p-q)} \quad (p \neq q)$$

is increasing (decreasing) in either  $p$  or  $q$  on  $(0, 1)$  if  $\mathcal{J} = (x-y)(x\mathcal{I})_x < (>)0$ . In other words, the function  $R_{2f}(p, q)$  defined by

$$(4.1) \quad R_{2f}(p, q) := \begin{cases} \left( \frac{\mathcal{H}_f(p, 1-p)}{\mathcal{H}_f(q, 1-q)} \right)^{1/(p-q)}, & p \neq q, p, q \in (0, 1); \\ \left( \frac{G_{f,p} G_{f,1-p}}{\mathcal{H}_f^2(p, 1-p)} \right)^{1/(2p-1)}, & p = q \neq 1/2, p, q \in (0, 1); \\ 1, & p = q = 1/2. \end{cases}$$

is increasing (decreasing) in either  $p$  or  $q$  on  $(0, 1)$  if  $\mathcal{J} = (x-y)(x\mathcal{I})_x < (>)0$ , where  $G_{f,p}$  is defined by (1.5). It should be noted that

$$R_f(p, p) := \lim_{q \rightarrow p} \left( \frac{\mathcal{H}_f(p, 1-p)}{\mathcal{H}_f(q, 1-q)} \right)^{1/(p-q)},$$



$$R_f(1/2, 1/2) := \lim_{p \rightarrow 1/2} R_f(p, p),$$

whose computation processes are omitted here.

Let  $a$  and  $b$  be positive numbers with  $a \neq b$ . Define

$$M_p := \begin{cases} M^{1/p}(a^p, b^p) & \text{if } p \neq 0 \\ G(a, b) & \text{if } p = 0 \end{cases}, \quad M = A, H, L, I, Z \text{ and } Y,$$

where  $A, L, I, Z$  and  $Y$  stand for the arithmetic mean, logarithmic mean, identric mean (exponential mean), power-exponential mean and exponential-geometric mean, which are defined by (1.10), (1.9), (1.12), (1.14) and (1.16), respectively; while Heronian mean is defined by  $H_e = (a + \sqrt{ab} + b)/3$ .

#### 4.1. Some new upper and lower bounds of logarithmic mean

For the logarithmic mean, since B. Ostle and H. L. Terwilliger [23] proved the following inequalities

$$G < L < A,$$

many researchers like B. C. Carlson, T. P. Lin, B. C. Stolarsky, P. O. Pittinger, E. B. Leach, Zs. Páles, J. Sándor, H. Alzer, E. Neuman, R. Yang, G. Jia and T. Trif and others have presented various upper and lower bounds of logarithmic mean and identric mean (exponential mean) (see [10, 19, 33, 7, 34, 26, 18, 1, 2, 24, 25, 29, 31, 32, 14, 15, 21, 20, 22, 35, 16]), some of which can be formed as a chain of inequalities for means in turn:

$$(4.2) \quad \begin{aligned} G &< A^{1/3}G^{2/3} < \sqrt{IG} < L < He_{1/2} < A_{1/3} \\ &< \frac{A+2G}{3} < He < A_{2/3} < I < A_{\ln 2} < A. \end{aligned}$$

Recently, the author [41] has proved again the following chain of inequalities for means:

$$(4.3) \quad \begin{aligned} G &< \dots < G^{2/3}A^{1/3} < \sqrt{GHe} < G^{2/5}A^{1/5}A_{2/3}^{2/5} \\ &< L < A_{1/5}^{1/3}A_{2/5}^{2/3} < He_{1/2} < A_{1/3} < He_{2/5}^2A_{1/5}^{-1} < I_{1/2}, \end{aligned}$$

and established interesting inequalities involving  $L, He, A, I, Z, Y$ , that is:

$$(4.4) \quad L_2 < He < A_{2/3} < I < Z_{1/3} < Y_{1/2}.$$

We now give some new upper and lower bound of logarithmic mean.

**Theorem 4.1.** *The following inequalities*

$$(4.5) \quad \begin{aligned} A_{1/3} &> A_{1/3}^2 I_{1/2}^{-1} > \sqrt{I_{1/3} I_{2/3}} > He_{1/2}^2 A_{1/3}^{-1} > He_{1/2}^4 A_{1/3}^{-3} > \\ &\sqrt{I_{1/4} I_{3/4}} > L > (\sqrt{IG})^{1/3} He_{1/2}^{2/3} > (\sqrt{IG})^{2/5} A_{1/3}^{3/5} > \sqrt{I_{1/2} \sqrt{IG}} \end{aligned}$$

hold.

*Proof.* By Corollary 1.1, we see that  $\mathcal{H}_L(p, 1-p)$  is log-concave on  $[0, 1]$ . Therefore,  $R_{2L}(p, q)$  is decreasing either  $p$  or  $q$  on  $[0, 1]$ .

Firstly, the first inequality is equivalent to  $I_{1/2} > A_{1/3}$  given in [34].

Secondly, we prove the second and third inequalities. From  $R_{2L}(2/3, 1/2) > R_{2L}(2/3, 2/3) > R_{2L}(2/3, 3/4)$  we have

$$\left(\frac{A_{1/3}}{I_{1/2}}\right)^6 > \left(\frac{I_{1/3}I_{2/3}}{A_{1/3}^2}\right)^3 > \left(\frac{He_{1/2}}{A_{1/3}}\right)^{12}.$$

A simple transformation yields

$$(4.6) \quad A_{1/3}^2 I_{1/2}^{-1} > \sqrt{I_{1/3}I_{2/3}} > He_{1/2}^2 A_{1/3}^{-1}.$$

Thirdly, the fourth inequality is obvious due to  $A_{1/3} > He_{1/2}$  (see [15, 22, 41]).

Fourthly, let us prove the fifth and sixth inequalities

$$(4.7) \quad He_{1/2}^4 A_{1/3}^{-3} > \sqrt{I_{1/4}I_{3/4}} > L.$$

From  $R_{2L}(3/4, 2/3) > R_{2L}(3/4, 3/4) > R_{2L}(3/4, 1)$  we get

$$\left(\frac{He_{1/2}}{A_{1/3}}\right)^{12} > \left(\frac{I_{1/4}I_{3/4}}{He_{1/2}^2}\right)^2 > \left(\frac{L}{He_{1/2}}\right)^4.$$

Taking fourth roots for all items of inequalities above and multiplying by  $He_{1/2}$  result in (4.7).

Finally, we prove the seventh, eighth and ninth inequalities

$$(4.8) \quad L > (\sqrt{IG})^{1/3} He_{1/2}^{2/3} > (\sqrt{IG})^{2/5} A_{1/3}^{3/5} > \sqrt{I_{1/2}\sqrt{IG}}.$$

From  $R_{2L}(1, 1/2), R_{2L}(1, 2/3), R_{2L}(1, 3/4) > R_{2L}(1, 1)$  we have

$$(4.9) \quad \left(\frac{L}{I_{1/2}}\right)^2, \left(\frac{L}{A_{1/3}}\right)^3, \left(\frac{L}{He_{1/2}}\right)^4 > \frac{IG}{L^2},$$

which are equivalent to

$$(4.10) \quad L > \sqrt{I_{1/2}\sqrt{IG}},$$

$$(4.11) \quad L > (\sqrt{IG})^{2/5} A_{1/3}^{3/5},$$

$$(4.12) \quad L > (\sqrt{IG})^{1/3} He_{1/2}^{2/3},$$

respectively.

(4.12) is the first inequality of (4.8), it remains to be proved that the second and third inequalities of (4.8). In fact,

$$\left(\frac{(\sqrt{IG})^{1/3} He_{1/2}^{2/3}}{(\sqrt{IG})^{2/5} A_{1/3}^{3/5}}\right)^{15} = \frac{He_{1/2}^{10} A_{1/3}^{-9}}{\sqrt{IG}} = \frac{He_{1/2}^{12} A_{1/3}^{-9}}{He_{1/2}^2 \sqrt{IG}}$$

$$= \frac{\left(He_{1/2}^4 A_{1/3}^{-3}\right)^3}{\sqrt{IG}He_{1/2}^2} > \frac{L^3}{\sqrt{IG}He_{1/2}^2} > 1,$$

where the first inequality due to  $He_{1/2}^4 A_{1/3}^{-3} > \sqrt{I_{1/4}I_{3/4}} > L$  in (4.7) and the second inequality due to (4.12). It follows that the second inequality of (4.8).

Likewise,

$$\begin{aligned} \left(\frac{(\sqrt{IG})^{2/5} A_{1/3}^{3/5}}{\sqrt{I_{1/2}\sqrt{IG}}}\right)^{10} &= \frac{A_{1/3}^{10} I_{1/2}^{-5}}{A_{1/3}^4 \sqrt{IG}} = \frac{\left(A_{1/3}^2 I_{1/2}^{-1}\right)^5}{A_{1/3}^4 \sqrt{IG}} > \frac{\left(He_{1/2}^2 A_{1/3}^{-1}\right)^5}{A_{1/3}^4 \sqrt{IG}} \\ &= \frac{\left(He_{1/2}^4 A_{1/3}^{-3}\right)^3}{He_{1/2}^2 \sqrt{IG}} > \frac{L^3}{He_{1/2}^2 \sqrt{IG}} > 1, \end{aligned}$$

where the first inequality due to  $A_{1/3}^2 I_{1/2}^{-1} > \sqrt{I_{1/3}I_{2/3}} > He_{1/2}^2 A_{1/3}^{-1}$  in (4.6), the second inequality due to  $He_{1/2}^4 A_{1/3}^{-3} > \sqrt{I_{1/4}I_{3/4}} > L$  in (4.7) and the third inequality due to (4.12). It follows that the third inequality of (4.8).

Combined (4.7) with (4.8), inequalities (4.5) hold.

This proof ends.  $\square$

*Remark 4.1.* Inequalities (4.5) contain the following inequality

$$(4.13) \quad L < He_{1/2}^4 A_{1/3}^{-3}.$$

It is superior to Lin's [19] and Jia's [15] inequalities since  $L < He_{1/2} < A_{1/3} < I_{1/2}$ . While  $L > (\sqrt{IG})^{1/3} He_{1/2}^{2/3}$  is stronger than  $L > \sqrt{IG} > A^{1/3} G^{2/3}$  because that

$$(\sqrt{IG})^{1/3} He_{1/2}^{2/3} > (\sqrt{IG})^{2/5} A_{1/3}^{3/5} > \sqrt{I_{1/2}\sqrt{IG}} > \sqrt{IG} > \sqrt{GHe} > A^{1/3} G^{2/3}.$$

In addition, from the second and third inequality of (4.5) it follows that

$$(4.14) \quad I_{1/2} < He_{1/2}^{-2} A_{1/3}^3.$$

From (4.13) and (4.14) it follows that

$$(4.15) \quad LI_{1/2} < He_{1/2}^2,$$

$$(4.16) \quad LI_{1/2}^2 < A_{1/3}^3.$$

#### 4.2. Some new estimations for identric mean (exponential mean)

To estimate identric mean  $I$  by another mean is very interesting. In 1988, H. Alzer [3] obtained that

$$2e^{-1}A < I < A.$$

E. Neuman and J. Sándor [20] proved that

$$\frac{A+G}{2} < I < 4e^{-1} \frac{A+G}{2}.$$

The author [40] obtained more general results:

$$\begin{aligned} S_{p_1,1}(a,b) &< I(a,b) < e^{-1}p_1^{1/(p_1-1)}S_{p_1,1}(a,b), p_1 \in (0,1), \\ e^{-1}p_2^{1/(p_2-1)}S_{p_2,1}(a,b) &< I(a,b) < S_{p_2,1}(a,b), p_2 \in (1,+\infty), \end{aligned}$$

where  $S_{p,q}(a,b)$  is the Stolarsky mean. In 2007 the author [41] presented more precise estimations for  $I$  :

$$\begin{aligned} 1 &< I/A_{2/5}^{1/3}A_{4/5}^{2/3} < \sqrt[3]{32}/e \approx 1.16794, \\ 1 &< I/He < 3/e \approx 1.10364, \\ 1 &< I/A_{2/3} < \sqrt{8}/e \approx 1.04052, \\ 1 &< I/He_{4/5}^2A_{2/5}^{-1} < \sqrt{486}/8e \approx 1.01376. \end{aligned}$$

As a general form of the four previous estimations, we have:

**Theorem 4.2.** For  $p_1, p_2 \in [1/2, 1)$  with  $p_1 < p_2$ , we have

$$(4.17) \quad 1 < \frac{\mathcal{H}_L(p_1, 1-p_1)}{\mathcal{H}_L(p_2, 1-p_2)} < \exp\left(\frac{1}{L(p_2, 1-p_2)} - \frac{1}{L(p_1, 1-p_1)}\right).$$

In particular, for  $p \in (1/2, 1)$  the following inequalities

$$(4.18) \quad 1 < \frac{I_{1/2}}{\mathcal{H}_L(p, 1-p)} < \exp\left(\frac{1}{L(p, 1-p)} - 2\right)$$

hold.

*Proof.* For  $p_1, p_2 \in [1/2, 1)$  with  $p_1 < p_2$ , by Corollary 1.1 we get

$$\mathcal{H}_L(p_1, 1-p_1) > \mathcal{H}_L(p_2, 1-p_2),$$

which implies the first inequality of (4.17).

By Corollary 1.2, we have

$$(4.19) \quad \mathcal{H}_D(p_1, 1-p_1) < \mathcal{H}_D(p_2, 1-p_2).$$

Since

$$\mathcal{H}_D(p, 1-p) = (p/(1-p))^{1/(2p-1)} \mathcal{H}_L(p, 1-p) = e^{1/L(p, 1-p)} \mathcal{H}_L(p, 1-p),$$

then (4.19) can be written as

$$(4.20) \quad e^{1/L(p_1, 1-p_1)} \mathcal{H}_L(p_1, 1-p_1) < e^{1/L(p_2, 1-p_2)} \mathcal{H}_L(p_2, 1-p_2),$$

which implies the second inequality of (4.17).

Substituting  $p_1 = 1/2$ ,  $p_2 = p \in (1/2, 1)$  in (4.17) leads to (4.18).

The proof ends.  $\square$

Next, let us apply the log-convexity of  $\mathcal{H}_L(p, 1-p)$  and  $\mathcal{H}_D(p, 1-p)$  to establish a new estimation for identric mean  $I$ .

**Theorem 4.3.** The following inequalities

$$(4.21) \quad 16\sqrt{2}/9e < I/(A_{2/3}^3 H e^{-2}) < 1$$

are true.

*Proof.* By Corollary 1.2,  $\mathcal{H}_D(p, 1 - p)$  is log-concave on  $(0, 1)$ . Therefore,  $R_{2D}(p, q)$  is increasing either  $p$  or  $q$  on  $(0, 1)$ .

From  $R_{2D}(3/4, 1/2) < R_{2D}(3/4, 2/3)$ , i.e.,

$$\left( \frac{(a^{1/2} + \sqrt[4]{ab} + b^{1/2})^2}{e^2 I^2(a^{1/2}, b^{1/2})} \right)^4 < \left( \frac{(a^{1/2} + \sqrt[4]{ab} + b^{1/2})^2}{(a^{1/3} + b^{1/3})^3} \right)^{12},$$

it follows that

$$I_{1/2} > e^{-2} \left( 8^3 A_{1/3}^3 \right) \left( 9^{-2} H e_{1/2}^{-2} \right).$$

Taking (4.14) into account yields

$$e^{-2} 8^3 9^{-2} < I_{1/2} / (A_{1/3}^3 H e_{1/2}^{-2}) < 1.$$

Applied with  $a^{1/2} \rightarrow a, b^{1/2} \rightarrow b$  and by a simple transformation, we obtain that

$$0.924906836 \approx \sqrt{e^{-2} 8^3 9^{-2}} < I / (A_{2/3}^3 H e^{-2}) < 1.$$

This completes proof.  $\square$

### 4.3. Some new inequalities involving power-exponential and exponential-geometric mean

It is worth noting that there are the corresponding relations of  $G_{f,p}, G_{f,0}$  and function values of  $\mathcal{H}_f(p, 1 - p)$  when  $p = 1/2, 2/3, 3/4, 1$  for  $f(x, y) = L(x, y), A(x, y), I(x, y), D(x, y)$  (see Table 1). It should be pointed out that we have used the identity for means of  $I(a^{2p}, b^{2p})/I(a^p, b^p) = Z(a^p, b^p)$  given in [30, 39].

TABLE 1. Corresponding relations table of  $L, A, E$  and  $D$

$f$	$G_{f,p}$	$G_{f,0}$	$\mathcal{H}_f(1/2, 1/2)$	$\mathcal{H}_f(2/3, 1/3)$	$\mathcal{H}_f(3/4, 1/4)$	$\mathcal{H}_{2f}(1, 0)$
$L$	$I_p$	$G$	$I_{1/2}$	$A_{1/3}$	$H e_{1/2}$	$L$
$A$	$Z_p$	$G$	$Z_{1/2}$	$A_{2/3}^2 A_{1/3}^{-1}$	$A_{3/4}^{3/2} A_{1/4}^{-1/2}$	$A$
$I$	$Y_p$	$G$	$Y_{1/2}$	$Z_{1/3}$	$I_{3/4}^{3/2} I_{1/4}^{-1/2}$	$I$
$D$	$e^{1/p} I_p$	doesn't exist	$e^2 I_{1/2}$	$8 A_{1/3}$	$9 H e_{1/2}$	doesn't exist

Therefore, following these corresponding relations given in Table 1 and applying Corollary 1.1 and 1.2, we can get inequalities corresponding to (4.5), (4.13) and (4.14). In this subsection, we only give some succinct inequalities for power-exponential means and exponential-geometric means but leave their proofs to readers.

Substituting  $Z_{1/4}, Z_{3/4}, A$  for  $I_{1/4}, I_{3/4}, L$  in the sixth inequality of (4.5) and  $Z_{1/2}, A_{2/3}^2 A_{1/3}^{-1}, A_{3/4}^{3/2} A_{1/4}^{-1/2}$  for  $I_{1/2}, A_{1/3}, H e_{1/2}$  in (4.14), respectively, we have:

**Theorem 4.4.** *The following inequalities*

$$(4.22) \quad A < \sqrt{Z_{1/4} Z_{3/4}},$$

$$(4.23) \quad Z_{1/2} < A_{2/3}^6 A_{1/3}^{-3} A_{3/4}^{-3} A_{1/4}$$

hold.

Similarly, substituting  $Y_{1/4}$ ,  $Y_{3/4}$ ,  $I$  for  $I_{1/4}$ ,  $I_{3/4}$ ,  $L$  in the sixth inequality of (4.5) and  $Z_{1/2}$ ,  $Z_{1/3}$ ,  $I_{3/4}^{3/2} I_{1/4}^{-1/2}$  for  $I_{1/2}$ ,  $A_{1/3}$ ,  $He_{1/2}$  in (4.14), respectively, we have

**Theorem 4.5.** *The following inequalities*

$$(4.24) \quad I < \sqrt{Y_{1/4} Y_{3/4}},$$

$$(4.25) \quad Y_{1/2} < Z_{1/3}^3 I_{3/4}^{-3} I_{1/4}$$

hold.

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