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INNER UNIFORM DOMAINS, THE QUASIHYPERBOLIC METRIC AND WEAK BLOCH FUNCTIONS

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Abstract. We characterize the class of inner uniform domains in terms of the quasihyperbolic metric and the quasihyperbolic geodesic. We also characterize uniform domains and inner uniform domains in terms of weak Bloch functions.

1. Introduction

Suppose that D is a subdomain of euclidean n-space $\mathbb{R}^n, n \geq 2$. Let $\overline{\mathbb{B}}(x,r)$ be the closure of $\mathbb{B}(x,r) = \{w : |w-x| < r\}$ for $x \in \mathbb{R}^n$ and $r > 0$. Let $\ell(\gamma)$ denote the euclidean length of an arc γ and $\operatorname{dist}(A,B)$ denote the euclidean distance from A to B for two sets $A, B \subset \mathbb{R}^n$.

A domain D in $\mathbb{R}^n$ is said to be b-*uniform* if there is a constant $b \geq 1$ such that each pair of points x_1, $x_2 \in D$ can be joined by a rectifiable arc γ in D with

$$\ell(\gamma) \leq b|x_1 - x_2|$$

and with

$$\min_{j=1,2} \ell(\gamma(x_j, x)) \leq b \operatorname{dist}(x, \partial D) \tag{1}$$

for each $x \in \gamma$, where $\gamma(x_j, x)$ is the part of γ between x_j and x. We call γ satisfying (1) a *double b-cone arc*.

We say that a domain D in $\mathbb{R}^n$ is b-*inner uniform* if there is a constant $b \geq 1$ such that each pair of points x_1, $x_2 \in D$ can be joined by a double b-cone arc γ in D which satisfies

$$\ell(\gamma) \leq b\lambda_D(x_1, x_2), \tag{2}$$

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where $\lambda_D(x_1, x_2) = \inf \ell(\alpha)$ and infimum is taken over all rectifiable arcs α which join x_1 and x_2 in D. We say that γ satisfies the *Gehring-Hayman inequality* if it satisfies (2). Obviously $|x_1 - x_2| \le \lambda_D(x_1, x_2)$.

A domain D in $\mathbb{R}^n$ is said to be *b-John* if there is a constant $b \ge 1$ such that each pair of points $x_1, x_2 \in D$ can be joined by a double b-cone arc γ in D.

An inner uniform domain is a domain intermediate between a uniform domain and a John domain. Balogh and Volberg introduced an inner uniform domain in connection with conformal dynamics [1], [2]. See also [3], [14].

For each pair of $x_1, x_2 \in D \subset \mathbb{R}^n$, we define the *quasihyperbolic metric* k_D in D by

$$k_D(x_1, x_2) = \inf_{\gamma} \int_{\gamma} \frac{ds}{\operatorname{dist}(x, \partial D)},$$

where the infimum is taken over all rectifiable arcs γ joining x_1 to x_2 in D. A *quasihyperbolic geodesic* is an arc γ along which the above infimum is obtained.

If D is c-uniform, then a quasihyperbolic geodesic $\gamma \subset D$ joining two points in D is a double b-cone arc with $b = b(c)$ [7]. For John domains, it is true when $n = 2$ and D is simply connected, but in general it is not true when $n > 2$ or D is multiply connected [5].

In a simply connected domain $D \subset \mathbb{R}^2$, quasihyperbolic geodesics satisfy the Gehring-Hayman inequality with an absolute constant b [6]. In a domain $D \subset \mathbb{R}^n$, quasihyperbolic geodesics satisfy the inequality with $b = b(a, K, n)$ if D is the image of an a-uniform domain under a K-quasiconformal mapping [9].

In Section 2, we show that an inner uniform domain $D \subset \mathbb{R}^n$ is a domain in which a quasihyperbolic geodesic is a double cone arc and satisfies the Gehring-Hayman inequality (see Theorem 2.1).

We have some important bounds for quasihyperbolic metric and the bounds involve the translation invariant metric j_D, introduced by Gehring and Osgood [7], given by

$$j_D(x_1, x_2) = \frac{1}{2} \log(1 + r_D(x_1, x_2))$$

for $x_1, x_2 \in D \subset \mathbb{R}^n$, where

$$r_D(x_1, x_2) = \frac{|x_1 - x_2|}{\min_{j=1,2} \operatorname{dist}(x_j, \partial D)}.$$

For any proper subdomain D of $\mathbb{R}^n$ we have

$$j_D(x_1, x_2) \le k_D(x_1, x_2) \tag{3}$$

for $x_1, x_2 \in D$ [8]. In [7], Gehring and Osgood observed that this bound may be reversed exactly if the domain is uniform as follows (see also [4, Theorem 5.3.5], [14]).

Theorem 1.1. *A domain D in $\mathbb{R}^n$ is b-uniform if and only if there is a constant a such that*

$$k_D(x_1, x_2) \le a j_D(x_1, x_2)$$

for all $x_1, x_2 \in D$, where a and b depend only on each other.

We now define a metric j'_D by

$$j'_D(x_1, x_2) = \frac{1}{2}\log(1 + r'_D(x_1, x_2))$$

for x_1, $x_2 \in D \subset \mathbb{R}^n$, where

$$r'_D(x_1, x_2) = \frac{\lambda_D(x_1, x_2)}{\min_{j=1,2} \operatorname{dist}(x_j, \partial D)}.$$

In Section 2, we also give a characterization of inner uniform domains in terms of k_D and j'_D which is an analogue of Theorem 1.1 (see Theorem 2.1).

A function f analytic in $D \subset \mathbb{R}^2$ is said to be a *Bloch* function, or $f \in \mathcal{B}(D)$, if

$$||f||_{\mathcal{B}(D)} = \sup_{z \in D} |f'(z)| \operatorname{dist}(z, \partial D) < \infty.$$

A real valued harmonic function u in $D \subset \mathbb{R}^n$ is said to be a *Bloch* function, or $u \in \mathcal{B}_h(D)$, if

$$||u||_{\mathcal{B}_h(D)} = \sup_{x \in D} |\nabla u(x)| \operatorname{dist}(x, \partial D) < \infty.$$

If $f \in \mathcal{B}(D)$, then

$$|f'(z)| \le ||f||_{\mathcal{B}(D)} \frac{1}{\operatorname{dist}(z, \partial D)},$$

and thus

$$|f(x_1) - f(x_2)| \le ||f||_{\mathcal{B}(D)} \int_\gamma \frac{ds}{\operatorname{dist}(x, \partial D)},$$

where γ is a rectifiable arc joining x_1 to x_2 in D. Hence we generalize Bloch function in terms of quasihyperbolic metric as follows, see [4].

A function $f : D \to \mathbb{R}^p$ in $D \subset \mathbb{R}^n$ is said to be a *weak Bloch* function, or $f \in \mathcal{B}_w(D)$, if there is a constant $m > 0$ such that

$$|f(x_1) - f(x_2)| \le m k_D(x_1, x_2),\ \forall x_1, x_2 \in D.$$

Let

$$||f||_{\mathcal{B}_w(D)} = \inf\{m > 0 \,|\, |f(x_1) - f(x_2)| \le m k_D(x_1, x_2),\ \forall x_1, x_2 \in D\}.$$

Remark 1.2. For $D \subset \mathbb{R}^2$, $\mathcal{B}(D)$ is the intersection of $\mathcal{B}_w(D)$ with the class of analytic functions in D.

The following two theorems in [4] and [13] show that a simply connected uniform (or John) domain $D \subset \mathbb{R}^2$ is characterized by moduli of continuity of Bloch functions with respect to j_D (or j'_D).

Theorem 1.3. *Let $D \subset \mathbb{R}^2$ be a simply connected proper domain. Then the followings are equivalent.*

(i) *D is c-uniform.*

(ii) *There is a constant c such that for $f \in \mathcal{B}(D)$*

$$|f(z_1) - f(z_2)| \le c||f||_{\mathcal{B}(D)} j_D(z_1, z_2),\ \forall z_1, z_2 \in D.$$

(iii) *There is a constant c such that for* $u \in \mathcal{B}_h(D)$

$$|u(z_1) - u(z_2)| \le c||u||_{\mathcal{B}_h(D)} j_D(z_1, z_2),\ \forall z_1, z_2 \in D.$$

The constants c are not necessarily the same, but they depend only on each other.

Theorem 1.4. *Let* $D \subset \mathbb{R}^2$ *be a simply connected proper domain. Then the followings are equivalent.*

(i) D *is c-John.*
(ii) D *is c-inner uniform.*
(iii) *There is a constant c such that for* $f \in \mathcal{B}(D)$

$$|f(z_1) - f(z_2)| \le c||f||_{\mathcal{B}(D)} j'_D(z_1, z_2),\ \forall z_1, z_2 \in D.$$

(iv) *There is a constant c such that for* $u \in \mathcal{B}_h(D)$

$$|u(z_1) - u(z_2)| \le c||u||_{\mathcal{B}_h(D)} j'_D(z_1, z_2),\ \forall z_1, z_2 \in D.$$

The constants c are not necessarily the same, but they depend only on each other.

The equivalence of (i) and (ii) in Theorem 1.4 is from [5] and [6] and see also [14, 2.18 examples]. In Section 3, we characterize weak Bloch functions in terms of moduli of continuity with respect to j_D (see Theorem 3.1). Then we give higher dimensional versions of Theorem 1.3 and Theorem 1.4 by using Theorem 2.1 and Theorem 3.1.

2. Inner uniform domains and the quasihyperbolic metric

Theorem 2.1. *Let* D *be a proper subdomain of* $\mathbb{R}^n$*. Then the followings are equivalent.*

(i) D *is b-inner uniform.*
(ii) *There is a constant b such that*

$$k_D(x_1, x_2) \le b j'_D(x_1, x_2), \forall x_1,\ x_2 \in D. \tag{4}$$

(iii) *Every quasihyperbolic geodesic in D is a double b-cone arc and satisfies the Gehring-Hayman inequality.*

The constants b are not necessarily the same, but they depend only on each other.

Remark 2.2. In [14, Theorem 3.5] it shows that the class of inner uniform domains is actually equal to the class of uniformly John domains. Therefore (i) and (ii) of Theorem 2.1 are equivalent to Theorem 2.1 in [10]. The proofs are similar and here we use the inner length metric instead of the inner diameter metric. For a simply connected domain $D \subset \mathbb{R}^2$, the equivalence of (i) and (ii) was proved in [11, Theorem 4.1]. For a domain $D \subset \mathbb{R}^n$, $n > 2$, the proof of Theorem 3.6 in [12] shows that (ii) implies that every quasihyperbolic geodesic in D is a double b-cone arc and thus (ii) implies that D is John. But in [12] she did not show the converse, and here (iii) in Theorem 2.1 gives two conditions for the converse. For the converse we need to show that (ii) of Theorem 2.1 implies that every quasihyperbolic geodesic in D satisfies the Gehring-Hayman inequality. To deal with it, we need the inner length metric instead of the inner diameter metric.

The proof of Theorem 2.1 is similar to those of Theorem 1 and Theorem 2 in [7] and also to the proof of Theorem 2.1 in [10]. But for the completeness we give the whole proof.

Proof of Theorem 2.1. First we show that (i) implies (ii). Suppose that D is b-inner uniform. Then there is a constant $b \geq 1$ such that each pair of points x_1, $x_2 \in D$ can be joined by an arc γ in D which satisfies (1) and (2). Choose $x_0 \in \gamma$ so that $\ell(\gamma(x_0, x_1)) = \ell(\gamma(x_0, x_2))$. Then by the triangle inequality it is sufficient to show that

$$k_D(x_j, x_0) \leq b' \log\left(\frac{\lambda_D(x_1, x_2)}{\mathrm{dist}(x_j, \partial D)} + 1\right) \tag{5}$$

for $j = 1, 2$, where $b' = 2b(2b+1)$. By symmetry we may assume that $j = 1$.

Suppose first that

$$\ell(\gamma(x_1, x_0)) \leq \frac{b}{b+1}\mathrm{dist}(x_1, \partial D). \tag{6}$$

Then $x_0 \in \overline{\mathbb{B}}\left(x_1, \frac{b}{b+1}\mathrm{dist}(x_1, \partial D)\right)$. If $x \in [x_1, x_0]$, then

$$\mathrm{dist}(x, \partial D) \geq \mathrm{dist}(x_1, \partial D) - |x_1 - x| \geq \frac{1}{b+1}\mathrm{dist}(x_1, \partial D)$$

and hence

$$\begin{aligned} |x_1 - x| + \mathrm{dist}(x_1, \partial D) &\leq \frac{b}{b+1}\mathrm{dist}(x_1, \partial D) + \mathrm{dist}(x_1, \partial D) \\ &\leq (2b+1)\mathrm{dist}(x, \partial D). \end{aligned} \tag{7}$$

Thus by (2), (7) and [11, Lemma 4.3]

$$\begin{aligned} k_D(x_1, x_0) &\leq \int_{[x_1, x_0]} \frac{ds}{\mathrm{dist}(x, \partial D)} \\ &\leq \int_0^{|x_1 - x_0|} \frac{2b+1}{s + \mathrm{dist}(x_1, \partial D)} ds \\ &\leq (2b+1)\log\left(\frac{\ell(\gamma)}{\mathrm{dist}(x_1, \partial D)} + 1\right) \\ &\leq (2b+1)b\log\left(\frac{\lambda_D(x_1, x_2)}{\mathrm{dist}(x_1, \partial D)} + 1\right). \end{aligned}$$

This implies (5).

Next suppose that (6) does not hold and choose $y_1 \in \gamma(x_1, x_0)$ so that

$$\ell(\gamma(x_1, y_1)) = \frac{b}{b+1}\mathrm{dist}(x_1, \partial D).$$

If $x \in \gamma(y_1, x_0)$, then by (1)

$$\operatorname{dist}(x, \partial D) \geq \frac{1}{b}\ell(\gamma(x_1, x))$$

and hence again by (2) and [11, Lemma 4.3]

$$\begin{aligned}
k_D(y_1, x_0) &\leq \int_{\gamma(y_1,x_0)} \frac{ds}{\operatorname{dist}(x, \partial D)} \\
&\leq b \int_{\gamma(y_1,x_0)} \frac{ds}{\ell(\gamma(x_1, y_1)) + \ell(\gamma(y_1, x))} \\
&= b \int_0^{\ell(\gamma(y_1,x_0))} \frac{ds}{\frac{b}{b+1}\operatorname{dist}(x_1, \partial D) + s} \\
&\leq b \log\left(\frac{b+1}{b}\frac{\ell(\gamma(x_1, x_0))}{\operatorname{dist}(x_1, \partial D)} + 1\right) \\
&\leq (b+1)\log\left(\frac{\ell(\gamma)}{\operatorname{dist}(x_1, \partial D)} + 1\right) \\
&\leq (b+1)b\log\left(\frac{\lambda_D(x_1, x_2)}{\operatorname{dist}(x_1, \partial D)} + 1\right).
\end{aligned}$$

We also have

$$k_D(x_1, y_1) \leq (2b+1)b\log\left(\frac{\lambda_D(x_1, x_2)}{\operatorname{dist}(x_1, \partial D)} + 1\right)$$

by what was proved above. Then (5) follows from the triangle inequality. Thus (i) implies (ii).

Next we show that (ii) implies (iii). Suppose that (ii) holds. Fix $x_1, x_2 \in D$ and let γ be the quasihyperbolic geodesic joining x_1, x_2 in D. We may assume that $\operatorname{dist}(x_1, \partial D) \geq \operatorname{dist}(x_2, \partial D)$. We want to show that (1) and (2) with $b' = \max\{e^2, 2a(2+e^a)e^a\}$, $a = 4b^2$. Set

$$r = \min\{\sup_{x\in\gamma} \operatorname{dist}(x, \partial D), 2\lambda_D(x_1, x_2)\}.$$

We shall consider the cases where

$$r < \operatorname{dist}(x_1, \partial D)$$

and where

$$r \geq \operatorname{dist}(x_1, \partial D) \tag{8}$$

separately.

Suppose first that $r < \operatorname{dist}(x_1, \partial D)$. Then $r = 2\lambda_D(x_1, x_2)$ and

$$|x_1 - x_2| < \frac{1}{2}\operatorname{dist}(x_1, \partial D) \leq \operatorname{dist}(x, \partial D)$$

for all x on the segment $\beta = [x_1, x_2] \subset D$. Thus $\lambda_D(x_1, x_2) = |x_1 - x_2|$ and hence

$$k_D(x_1, x_2) \le \int_\beta \frac{ds}{\mathrm{dist}(x, \partial D)} \le \frac{2|x_1 - x_2|}{\mathrm{dist}(x_1, \partial D)} \le 1.$$

Since $k_D(x, x_1) \le k_D(x_1, x_2)$ for $x \in \gamma$, from [8, Lemma 2.1]

$$e^{-1}\mathrm{dist}(x_1, \partial D) \le \mathrm{dist}(x, \partial D) \le e\mathrm{dist}(x_1, \partial D)$$

for each $x \in \gamma$. Thus

$$\begin{aligned}\ell(\gamma) &\le e\mathrm{dist}(x_1, \partial D) \int_\gamma \frac{1}{\mathrm{dist}(x, \partial D)} ds \\ &= e\mathrm{dist}(x_1, \partial D) k_D(x_1, x_2) \le 2e\lambda_D(x_1, x_2)\end{aligned}$$

and that for each $x \in \gamma$

$$\begin{aligned}\ell(\gamma(x_1, x)) &\le \ell(\gamma) \le e\mathrm{dist}(x_1, \partial D) k_D(x_1, x_2) \\ &\le e\mathrm{dist}(x_1, \partial D) \le e^2 \mathrm{dist}(x, \partial D)\end{aligned}$$

and hence γ holds (1) and (2).

Suppose next that (8) holds. By compactness there is $x_0 \in \gamma$ with

$$r \le \sup_{x \in \gamma} \mathrm{dist}(x, \partial D) = \mathrm{dist}(x_0, \partial D).$$

For $j = 1, 2$ let m_j be the largest integer for which

$$2^{m_j} \mathrm{dist}(x_j, \partial D) \le r,$$

and let y_j be the first point of $\gamma(x_j, x_0)$ with

$$\mathrm{dist}(y_j, \partial D) = 2^{m_j} \mathrm{dist}(x_j, \partial D)$$

as we traverse γ from x_j towards x_0. Then

$$\mathrm{dist}(y_j, \partial D) \le r < 2\mathrm{dist}(y_j, \partial D). \tag{9}$$

We first show that

$$\ell(\gamma(x_j, y_j)) \le a\mathrm{dist}(y_j, \partial D), \ell(\gamma(x_j, x)) \le ae^{\frac{a}{2}} \mathrm{dist}(x, \partial D), \forall x \in \gamma(x_j, y_j) \tag{10}$$

for $j = 1, 2$ and $a = 4b^2$. We need only consider the case where $j = 1$ and $m_1 \ge 1$. Choose points $z_1, \ldots, z_{m_1+1} \in \gamma(x_1, y_1)$ so that $z_1 = x_1$ and z_k is the first point of $\gamma(x_1, y_1)$ with

$$\mathrm{dist}(z_k, \partial D) = 2^{k-1} \mathrm{dist}(x_1, \partial D) \tag{11}$$

as we traverse γ from x_1 towards y_1. Then $z_{m_1+1} = y_1$. Fix k, $1 \le k \le m_1$, and let

$$\gamma_k = \gamma(z_k, z_{k+1}) \quad \text{and} \quad t = \frac{\ell(\gamma_k)}{\mathrm{dist}(z_k, \partial D)}.$$

If $x \in \gamma_k$, then

$$\mathrm{dist}(x, \partial D) \le \mathrm{dist}(z_{k+1}, \partial D) = 2\mathrm{dist}(z_k, \partial D)$$

and

$$t \le 2\int_{\gamma_k} \frac{ds}{\operatorname{dist}(x, \partial D)} = 2k_D(z_k, z_{k+1}).$$

Next since the function $f(x) = \sqrt{x} - \log(x+1)$ is increasing for $x > 0$ with $f(0) = 0$,

$$j'_D(z_k, z_{k+1}) \le \log\left(\frac{\lambda_D(z_k, z_{k+1})}{\operatorname{dist}(z_k, \partial D)} + 1\right) \le \log(t+1) \le \sqrt{t}.$$

Hence (4) implies that

$$t \le 2k_D(z_k, z_{k+1}) \le 2bj'_D(z_k, z_{k+1}) \le 2b\sqrt{t},$$

whence $t \le 4b^2 = a$ and

$$k_D(z_k, z_{k+1}) \le b\sqrt{t} \le \frac{a}{2}. \tag{12}$$

Next if $x \in \gamma_k$, then from [8, Lemma 2.1] and (12)

$$0 < \log \frac{\operatorname{dist}(z_{k+1}, \partial D)}{\operatorname{dist}(x, \partial D)} \le k_D(x, z_{k+1}) \le k_D(z_k, z_{k+1}) < \frac{a}{2}.$$

We conclude that

$$\ell(\gamma_k) \le a\operatorname{dist}(z_k, \partial D), \operatorname{dist}(z_{k+1}, \partial D) \le e^{\frac{a}{2}} \operatorname{dist}(x, \partial D), \quad \forall x \in \gamma_k, \tag{13}$$

for $k = 1, \dots, m_1$. Hence by (11) and (13)

$$\begin{aligned} \ell(\gamma(x_1, y_1)) &= \sum_{k=1}^{m_1} \ell(\gamma_k) \le a \sum_{k=1}^{m_1} \operatorname{dist}(z_k, \partial D) \\ &= a(2^{m_1} - 1)\operatorname{dist}(x_1, \partial D) < a\operatorname{dist}(y_1, \partial D). \end{aligned}$$

This proves the first inequality in (10). For the second, if $x \in \gamma(x_1, y_1)$, then $x \in \gamma_k$ for some k, $1 \le k \le m_1$, and

$$\begin{aligned} \ell(\gamma(x_1, x)) &\le \sum_{i=1}^{k} \ell(\gamma_i) \le a \sum_{i=1}^{k} \operatorname{dist}(z_i, \partial D) \\ &< a\operatorname{dist}(z_{k+1}, \partial D) \le ae^{\frac{a}{2}} \operatorname{dist}(x, \partial D) \end{aligned}$$

again by (11) and (13). This completes the proof of (10).

We show next that if $\operatorname{dist}(y_1, \partial D) \le \operatorname{dist}(y_2, \partial D)$, then

$$\begin{aligned} &\ell(\gamma(y_1, y_2)) \le ae^a \operatorname{dist}(y_1, \partial D), \\ &\operatorname{dist}(y_2, \partial D) \le e^a \operatorname{dist}(x, \partial D), \quad \forall x \in \gamma(y_1, y_2). \end{aligned} \tag{14}$$

We may assume that $y_1 \ne y_2$ since otherwise there is nothing to prove. By the hypothesis (8), we have the following two possible subcases:

$$r = \sup_{x \in \gamma} \operatorname{dist}(x, \partial D), \tag{15}$$

$$r = 2\lambda_D(x_1, x_2). \tag{16}$$

If (15) holds, set

$$t = \frac{\ell(\gamma(y_1, y_2))}{\mathrm{dist}(y_1, \partial D)}.$$

If $x \in \gamma(y_1, y_2)$, then by (9)

$$\mathrm{dist}(x, \partial D) \le r \le 2\mathrm{dist}(y_1, \partial D),$$

and we can repeat the proof of the first part of (13), with y_1 in place of z_k and y_2 in place of z_{k+1} to obtain (14).

Next if (16) holds, then by (9) and (10)

$$\begin{aligned}\lambda_D(y_1, y_2) &\le \ell(\gamma(x_1, y_1)) + \ell(\gamma(x_2, y_2)) + \lambda_D(x_1, x_2)\\ &\le a\mathrm{dist}(y_1, \partial D) + a\mathrm{dist}(y_2, \partial D) + \frac{r}{2}\\ &\le 4a\mathrm{dist}(y_1, \partial D).\end{aligned}$$

Therefore

$$\begin{aligned}k_D(y_1, y_2) \le bj'_D(y_1, y_2) &\le b \log\left(\frac{\lambda_D(y_1, y_2)}{\mathrm{dist}(y_1, \partial D)} + 1\right)\\ &= b\log(4a + 1) \le b\sqrt{4a} = a.\end{aligned}$$

If $x \in \gamma(y_1, y_2)$, then by [8, Lemma 2.1]

$$e^{-a}\mathrm{dist}(y_2, \partial D) \le \mathrm{dist}(x, \partial D) \le e^a\mathrm{dist}(y_1, \partial D)$$

and thus

$$\ell(\gamma(y_1, y_2)) \le e^a\mathrm{dist}(y_1, \partial D)k_D(y_1, y_2) \le ae^a\mathrm{dist}(y_1, \partial D)$$

and again we obtain (14).

We now complete the proof that (ii) implies (iii) as follows. By relabelling we may assume that $\mathrm{dist}(y_1, \partial D) \le \mathrm{dist}(y_2, \partial D)$. Then

$$\begin{aligned}\ell(\gamma) &= \ell(\gamma(x_1, y_1)) + \ell(\gamma(x_2, y_2)) + \ell(\gamma(y_1, y_2))\\ &\le a(2 + e^a)\mathrm{dist}(y_2, \partial D) \le a(2 + e^a)r \le 2a(2 + e^a)\lambda_D(x_1, x_2)\end{aligned}$$

by (9), (10) and (14). This establishes (2). Next if $x \in \gamma$, then either $x \in \gamma(x_j, y_j)$ and

$$\min_{j=1,2} \ell(\gamma(x_j, x)) \le \ell(\gamma(x_j, x)) \le ae^{\frac{a}{2}}\mathrm{dist}(x, \partial D)$$

by (10), or $x \in \gamma(y_1, y_2)$ and

$$\min_{j=1,2} \ell(\gamma(x_j, x)) \le \frac{1}{2}\ell(\gamma) \le \frac{1}{2}a(2 + e^a)\mathrm{dist}(y_2, \partial D) \le \frac{1}{2}a(2 + e^a)e^a\mathrm{dist}(x, \partial D)$$

by (14). In each case we obtain (1).

It is obvious that (iii) implies (i) and the proof is complete. $\square$

3. Weak Bloch extension property on the domains

We characterize weak Bloch functions in terms of moduli of continuity with respect to j_D.

Theorem 3.1. *Let $f : D \to \mathbb{R}^p$ be a function in $D \subset \mathbb{R}^n$. Then the followings are equivalent.*

(i) $f \in \mathcal{B}_w(D)$.

(ii) *There is a constant m such that*

$$|f(x_1) - f(x_2)| \le m j_D(x_1, x_2),$$

for all $x_1, x_2 \in D$ with $|x_1 - x_2| < \operatorname{dist}(x_1, \partial D)$.

Here all constants depend only on each other.

Proof. Suppose that (i) holds and let $x_1, x_2 \in D$ with $|x_1 - x_2| < \operatorname{dist}(x_1, \partial D)$. Let γ be the segment $[x_1, x_2] \subset D$. Then

$$\ell(\gamma) = |x_1 - x_2|$$

and

$$\min_{j=1,2} \ell(\gamma(x_j, x)) \le \operatorname{dist}(x, \partial \mathbb{B}) \le \operatorname{dist}(x, \partial D)$$

for all $x \in \gamma$, where $\mathbb{B} = \mathbb{B}(x_1, \operatorname{dist}(x_1, \partial D)$. Following the same argument used in the proof of [7, Theorem 1] or [11, Theorem 4.1] we get

$$k_D(x_1, x_2) \le \int_\gamma \frac{ds}{\operatorname{dist}(x, \partial D)} \le c_0 \log(1 + r_D(x_1, x_2)) = c_0 j_D(x_1, x_2),$$

where c_0 is an absolute constant. Thus by (i)

$$|f(x_1) - f(x_2)| \le m k_D(x_1, x_2) \le m c_0 j_D(x_1, x_2)$$

for some constant $m > 0$. Now suppose that (ii) holds. Fix $x_1, x_2 \in D$ and let γ be the quasihyperbolic geodesic in D with endpoints x_1 and x_2. Let $\gamma(s)$ be the parameterization of γ with respect to arc length measured from x_1, $\ell = \ell(\gamma)$. Let $y_1 = x_1$. We choose positive numbers r_i and ℓ_i, and points $y_i \in \gamma$ as follows:

$$r_1 = \frac{1}{2}\operatorname{dist}(y_1, \partial D),\ \ell_1 = \max\{s : \gamma(s) \in \overline{\mathbb{B}}(y_1, r_1)\},\ y_2 = \gamma(\ell_1);$$

$$r_2 = \frac{1}{2}\operatorname{dist}(y_2, \partial D),\ \ell_2 = \max\{s : \gamma(s) \in \overline{\mathbb{B}}(y_2, r_2)\},\ y_3 = \gamma(\ell_2);$$

and so on. After a finite number of steps, say N, $\ell_N = \ell$ and the process stops. Let $y_{N+1} = x_2$. So by (ii) and [5, Lemma 2.6] for some constant $m > 0$

$$\begin{aligned} |f(x_1) - f(x_2)| &\le \sum_{i=1}^N |f(y_i) - f(y_{i+1})| \le \sum_{i=1}^N m \log\left(1 + \frac{|y_i - y_{i+1}|}{\operatorname{dist}(y_{i+1}, \partial D)}\right) \\ &\le m \sum_{i=1}^N k_D(\gamma(y_i, y_{i+1})) = m k_D(x_1, x_2). \end{aligned}$$

□

If a function f in $D \subset \mathbb{R}^n$ satisfies

$$|f(x_1) - f(x_2)| \le m j_D(x_1, x_2), \tag{17}$$

for all $x_1, x_2 \in D$, then $f \in \mathcal{B}_w(D)$ by Theorem 3.1 or (3). Conversely, $f \in \mathcal{B}_w(D)$ holds (17) locally by Theorem 3.1. Thus it is natural to ask when $f \in \mathcal{B}_w(D)$ holds (17) globally. We called it weak Bloch extension property.

For $n = 2$, Theorem 1.3 gives an answer. Higher dimensional versions of Theorem 1.3 was partly given in [4] and [13] as follows.

Theorem 3.2. *Suppose that $D \subset \mathbb{R}^n$ is a proper subdomain and $f : D \to \mathbb{R}^p$ is a function. Then D is b-uniform if and only if there is a constant c such that*

$$|f(x_1) - f(x_2)| \le j_D(x_1, x_2)$$

for all $x_1, x_2 \in D$ with $|x_1 - x_2| < \operatorname{dist}(x_1, \partial D)$ implies

$$|f(x_1) - f(x_2)| \le c j_D(x_1, x_2),\ \forall x_1, x_2 \in D.$$

Here b and c depend only on each other [13, Theorem 6.1].

Theorem 3.3. *Suppose that $D \subset \mathbb{R}^n$ is a b-uniform domain. Then for $u : D \to \mathbb{R}$, $u \in \mathcal{B}_h(D)$,*

$$|u(x_1) - u(x_2)| \le 4b^2 ||u||_{\mathcal{B}_h(D)} j_D(x_1, x_2)$$

for all $x_1, x_2 \in D$ [4, Corollary 5.4.17].

We can rewrite Theorem 3.2 by using Theorem 3.1 as follows.

Corollary 3.4. *Let $D \subset \mathbb{R}^n$ be a proper subdomain. Then the followings are equivalent.*

(i) *D is b-uniform.*

(ii) *There is a constant c such that for $f \in \mathcal{B}_w(D)$, $f : D \to \mathbb{R}^p$, with $||f||_{\mathcal{B}_w(D)} \le 1$,*

$$|f(x_1) - f(x_2)| \le c j_D(x_1, x_2),\ \forall x_1, x_2 \in D.$$

Here b and c depend only on each other.

Now in more general situations than Theorem 3.3 and Corollary 3.4 we consider higher dimensional versions of Theorem 1.3. We show that uniform domains have weak Bloch extension Property.

Theorem 3.5. *Let $D \subset \mathbb{R}^n$ be a proper subdomain. Then the followings are equivalent.*

(i) *D is c-uniform.*

(ii) *There is a constant c such that for $f \in \mathcal{B}_w(D)$, $f : D \to \mathbb{R}^p$,*

$$|f(x_1) - f(x_2)| \le c ||f||_{\mathcal{B}_w(D)} j_D(x_1, x_2),\ \forall x_1, x_2 \in D.$$

(iii) *There is a constant c such that for $u \in \mathcal{B}_w(D)$, $u : D \to \mathbb{R}$,*

$$|u(x_1) - u(x_2)| \le c ||u||_{\mathcal{B}_w(D)} j_D(x_1, x_2),\ \forall x_1, x_2 \in D.$$

The constants c are not necessarily the same, but they depend only on each other.

Lemma 3.6. *Let $D \subset \mathbb{R}^n$ be a proper subdomain. Fix $x_0 \in D$ and define a function $u : D \to \mathbb{R}$ by $u(x) = k_D(x, x_0)$. Then $u \in \mathcal{B}_w(D)$ and $||u||_{\mathcal{B}_w(D)} \leq 1$.*

Proof. Let $x_1, x_2 \in D$ and let γ be any curve joining x_1 and x_2 in D. Fix a curve β joining x_0 and x_1 in D. Then

$$u(x_2) \leq \int_{\beta+\gamma} \frac{ds}{\operatorname{dist}(x, \partial D)}.$$

Hence

$$u(x_2) \leq \inf_{\beta} \int_{\beta} \frac{ds}{\operatorname{dist}(x, \partial D)} + \int_{\gamma} \frac{ds}{\operatorname{dist}(x, \partial D)}$$

and thus

$$u(x_2) - u(x_1) \leq \int_{\gamma} \frac{ds}{\operatorname{dist}(x, \partial D)}.$$

Reversing the roles of x_1 and x_2 yields

$$|u(x_1) - u(x_2)| \leq \int_{\gamma} \frac{ds}{\operatorname{dist}(x, \partial D)}.$$

Therefore

$$|u(x_1) - u(x_2)| \leq k_D(x_1, x_2).$$

Thus $u \in \mathcal{B}_w(D)$ and $||u||_{\mathcal{B}_w(D)} \leq 1$. □

Lemma 3.6 gives an example of weak Bloch functions. We can also prove the lemma by using Theorem 3.1 in this paper and the proof of the sufficient condition in [10, Theorem 3.5], but in that way we do not have $||u||_{\mathcal{B}_w(D)} \leq 1$.

Proof of Theorem 3.5. First we show that (i) implies (ii). Suppose that D is c-uniform. Let $f_1(x) = \frac{1}{||f||_{\mathcal{B}_w(D)}} f(x)$ for $f \in \mathcal{B}_w(D)$ and let $x_1, x_2 \in D$. Then

$$|f_1(x_1) - f_1(x_2)| = \frac{1}{||f||_{\mathcal{B}_w(D)}} |f(x_1) - f(x_2)| \leq k_D(x_1, x_2).$$

Thus $f_1 \in \mathcal{B}_w(D)$ and $||f_1||_{\mathcal{B}_w(D)} \leq 1$. Hence

$$|f_1(x_1) - f_1(x_2)| \leq c_1 j_D(x_1, x_2)$$

for some constant $c_1 = c_1(c)$ by Corollary 3.4. Then

$$|f(x_1) - f(x_2)| \leq c_1 ||f||_{\mathcal{B}_w(D)} j_D(x_1, x_2).$$

Next obviously (ii) implies (iii), and we need to show that (iii) implies (i). Suppose that (iii) holds. Fix $x_0 \in D$ and define a function $u : D \to \mathbb{R}$ by $u(x) = k_D(x, x_0)$. Then by Lemma 3.6 and (iii)

$$k_D(x, x_0) = |u(x) - u(x_0)| \leq c j_D(x, x_0), \ \forall x \in D,$$

where c is independent of x_0. Thus

$$k_D(x_1, x_2) \leq c j_D(x_1, x_2), \ \forall x_1, x_2 \in D$$

and by Theorem 1.1 D is c_1-uniform, $c_1 = c_1(c)$. □

Next higher dimensional versions of Theorem 1.4 was partly given in [12, Theorem 7.5] as follows.

Theorem 3.7. *Suppose that $D \subset \mathbb{R}^n$ is a proper subdomain and $f : D \to \mathbb{R}^p$ is a function. Then D is b-John if there is a constant c such that for each ball $\mathbb{B} \subset D$*

$$|f(x_1) - f(x_2)| \leq j_D(x_1, x_2), \; \forall x_1, x_2 \in \mathbb{B}.$$

implies

$$|f(x_1) - f(x_2)| \leq c j'_D(x_1, x_2), \; \forall x_1, x_2 \in D.$$

Here b depend only on c.

We can rewrite Theorem 3.7 by using Theorem 3.1 as follows.

Corollary 3.8. *Let $D \subset \mathbb{R}^n$ be a proper subdomain. Then D is b-John if there is a constant c such that for $f \in \mathcal{B}_w(D)$, $f : D \to \mathbb{R}^p$, with $||f||_{\mathcal{B}_w(D)} \leq 1$*

$$|f(x_1) - f(x_2)| \leq c j'_D(x_1, x_2), \; \forall x_1, x_2 \in D.$$

Here c depend only on b.

Now in more general situations than Corollary 3.8 we consider higher dimensional versions of Theorem 1.4. We show that inner uniform domains have weak Bloch extension Property with respect to j'_D.

Theorem 3.9. *Let $D \subset \mathbb{R}^n$ be a proper subdomain. Then the followings are equivalent.*

(i) *D is c-inner uniform.*

(ii) *There is a constant c such that for $f \in \mathcal{B}_w(D)$, $f : D \to \mathbb{R}^p$,*

$$|f(x_1) - f(x_2)| \leq c||f||_{\mathcal{B}_w(D)} j'_D(x_1, x_2), \; \forall x_1, x_2 \in D.$$

(iii) *There is a constant c such that for $u \in \mathcal{B}_w(D)$, $u : D \to \mathbb{R}$,*

$$|u(x_1) - u(x_2)| \leq c||u||_{\mathcal{B}_w(D)} j'_D(x_1, x_2), \; \forall x_1, x_2 \in D.$$

The constants c are not necessarily the same, but they depend only on each other.

Proof. First we show that (i) implies (ii). Suppose that $D \subset \mathbb{R}^n$ is c-inner uniform. Let $f \in \mathcal{B}_w(D)$, $f : D \to \mathbb{R}^p$. Then by Theorem 2.1 there is a constant $c_1 = c_1(c)$ such that

$$|f(x_1) - f(x_2)| \leq ||f||_{\mathcal{B}_w(D)} k_D(x_1, x_2) \leq ||f||_{\mathcal{B}_w(D)} c_1 j'_D(x_1, x_2).$$

Next obviously (ii) implies (iii), and we need to show that (iii) implies (i). Suppose that (iii) holds. Fix $x_0 \in D$ and define a function $u : D \to \mathbb{R}$ by $u(x) = k_D(x, x_0)$. By Lemma 3.6 and (iii)

$$k_D(x, x_0) = |u(x) - u(x_0)| \leq c||u||_{\mathcal{B}_w(D)} j'_D(x, x_0) \leq c j'_D(x, x_0), \; \forall x \in D,$$

where c is independent of x_0. Therefore

$$k_D(x_1, x_2) \le c j'_D(x_1, x_2),\ \forall x_1, x_2 \in D$$

and hence by Theorem 2.1 D is c_1-inner uniform, $c_1 = c_1(c)$. □

Remark 3.10. Since the class of inner uniform domains is actually equal to the class of uniformly John domains [14, Theorem 3.5], an inner uniform domain satisfies the necessary condition of Theorem 3.5 in [10]. Then by Theorem 3.1 in this paper we can also get (ii) of Theorem 3.9.

References

[1] Z. Balogh and A. Volberg, *Boundary Harnack principle for separated semihyperbolic repellers, harmonic measure applications*, Rev. Mat. Iberoamericana **12** (1996), no. 2, 299–336.

[2] ———, *Geometric localization, uniformly John property and separated semihyperbolic dynamics*, Ark. Mat. **34** (1996), no. 1, 21–49.

[3] M. Bonk, J. Heinonen, and P. Koskela, *Uniformizing Gromov hyperbolic spaces*, Asteroque **270** (2001), viii+99 pp.

[4] O. J. Broch, *Geometry of John disks*, Norwegian University of Science and Technology Doctoral Thesis, 2005.

[5] F. W. Gehring, K. Hag, and O. Martio, *Quasihyperbolic geodesics in John domains*, Math. Scand. **65** (1989), no. 1, 75–92.

[6] F. W. Gehring and W. F. Hayman, *An inequality in the theory of conformal mapping*, J. Math. Pures Appl. (9) **41** (1962), 353–361.

[7] F. W. Gehring and B. G. Osgood, *Uniform domains and the quasihyperbolic metric*, J. Analyse Math. **36** (1979), 50–74.

[8] F. W. Gehring and B. P. Palka, *Quasiconformal homogeneous domains*, J. Analyse Math. **30** (1976), 172–199.

[9] J. Heinonen and S. Rohde, *The Gehring and Hayman inequality for quasihyperbolic geodesics*, Math. Proc. Cambridge Philos. Soc. **114** (1993), no. 3, 393–405.

[10] K. Kim, *The quasihyperbolic metric and analogues of the Hardy-Littlewood property for* $\alpha = 0$ *in uniformly John domains*, Bull. Korean Math. Soc. **43** (2006), no. 2, 395–410.

[11] K. Kim and N. Langmeyer, *Harmonic Measure and Hyperbolic distance in John disks*, Math. Scand. **83** (1998), no. 2, 283–299.

[12] N. Langmeyer, *The quasihyperbolic metric, growth and John domains*, University of Michigan Ph.D. Thesis, 1996.

[13] ———, *The quasihyperbolic metric, growth and John domains*, Ann. Acad. Sci. Fenn. Math. **23** (1998), no. 1, 205–224.

[14] J. Väisälä, *Relatively and inner uniform domains*, Conform. Geom. Dyn. **2** (1998), 56–88.

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