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SEMIPRIME SUBMODULES OF GRADED MULTIPLICATION MODULES

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Abstract. Let G be a group. Let R be a G-graded commutative ring with identity and M be a G-graded multiplication module over R. A proper graded submodule Q of M is *semiprime* if whenever $I^n K \subseteq Q$, where $I \subseteq h(R)$, n is a positive integer, and $K \subseteq h(M)$, then $IK \subseteq Q$. We characterize semiprime submodules of M. For example, we show that a proper graded submodule Q of M is semiprime if and only if $grad(Q) \cap h(M) = Q \cap h(M)$. Furthermore if M is finitely generated, then we prove that every proper graded submodule of M is contained in a graded semiprime submodule of M. A proper graded submodule Q of M is said to be *almost semiprime* if

$$(grad(Q) \cap h(M))\backslash(grad(0_M) \cap h(M))$$
$$= (Q \cap h(M))\backslash(grad(0_M) \cap Q \cap h(M)).$$

Let K, Q be graded submodules of M. If K and Q are almost semiprime in M such that $Q + K \neq M$ and $Q \cap K \subseteq M_g$ for all $g \in G$, then we prove that $Q + K$ is almost semiprime in M.

1. Introduction

Let G be a group. Then we define a G-graded ring R and a G-graded module over R in the same way as in [2], [3], and [5]. The notations which the authors use are slightly different but basically the same.

Throughout this paper G is a group, R is a G-graded commutative ring with identity and M is a G-graded module over R. From now on, by *graded* we mean G-graded, unless otherwise indicated.

Lemma 1.1. *Let R be a graded ring.*

(i) *If $\mathfrak{a}$ and $\mathfrak{b}$ are graded ideals of R, then $\mathfrak{a} + \mathfrak{b}$, $\mathfrak{a} \cap \mathfrak{b}$, and $\mathfrak{a}\mathfrak{b}$ are graded ideals of R.*

(ii) *If a is an element of $h(R)$, then the cyclic ideal aR of R is graded.*

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Let $M = \oplus_{g\in G} M_g$ be a graded R-module. Let N be a submodule of M. The factor R-module M/N becomes a G-graded module over R with g-component $(M/N)_g = (M_g + N)/N$ for $g \in G$. A submodule N of M is called to be *graded* if $N = \oplus_{g\in G} N_g$ where $N_g = N \cap M_g$ for $g \in G$. Clearly, 0 is a graded submodule of M.

If N and K are submodules of an R-module M, the set of all elements $r \in R$ satisfying $rK \subseteq N$ becomes an ideal of R and is denoted by $(N :_R K)$ as usual.

Lemma 1.2. *Let R be a graded ring and M be a graded R-module.*

(i) *If N and K are graded submodules of M, then $N + K$ and $N \cap K$ are graded submodules of M.*

(ii) *If a is an element of $h(R)$ and x is an element of $h(M)$, then aM and Rx are graded submodules of M.*

(iii) *If N is a graded submodule of M and K is a graded submodule of M, then $(N :_R K)$ is a graded ideal of R.*

Proof. Clearly, (i) holds. See [3, Lemma 2.2] for (ii). For the proof of (iii), see [2, Lemma 2.1] and [5, Lemma 1(ii)]. We give a proof of (iii) for our record.

To show that $(N :_R K)$ is a graded ideal of R, let $I = (N :_R K)$. We show $I = \oplus_{g\in G} I_g$. For all $g \in G$, $I_g = I \cap R_g \subseteq I$. Hence $\oplus_{g\in G} I_g \subseteq I$. Conversely, let x be any element of I. Since R is graded, there exist $g_1, g_2, \ldots, g_n \in G$ such that $x = \sum_{j=1}^{n} x_{g_j}$. To show that $I \subseteq \oplus_{g\in G} I_g$, it suffices to show that $x_{g_j} \in I$ since then $x_{g_j} \in R_{g_j} \cap I = I_{g_j}$. In turn, it suffices to show that $x_{g_j} K \subseteq N$.

Since K is graded, $xK \subseteq N$, and N is graded, we have

$$\begin{aligned} x_{g_j} K &= x_{g_j}(\oplus_{h\in G} K_h) = \oplus_{h\in G} x_{g_j} K_h \\ &\subseteq \oplus_{h\in G} (xK)_{g_j h} \subseteq \oplus_{h\in G} N_{g_j h} \subseteq N, \end{aligned}$$

as required. □

Corollary 1.3. *Let R be a graded ring. If $\mathfrak{a}$ and $\mathfrak{b}$ are graded ideals of R, then $(\mathfrak{a} :_R \mathfrak{b})$ is a graded ideal of R.*

Let R be a graded ring and M be a graded R-module. We recall that a proper graded submodule P of M is *prime* if whenever $rm \in P$, where $r \in h(R)$ and $m \in h(M)$, then either $r \in (P :_R M)$ or $m \in P$.

Definition 1.4. Let R be a graded ring and M be a graded R-module. A proper graded submodule Q of M is *semiprime* if whenever $I^n K \subseteq Q$, where $I \subseteq h(R)$, n is a positive integer, and $K \subseteq h(M)$, then $IK \subseteq Q$.

Remark 1.5. It is easy to check that a proper graded ideal I of a graded ring R is semiprime if and only if whenever $x^t y \in I$, where $x, y \in h(R)$ and t is a positive integer, then $xy \in I$.

Proposition 1.6. *Let R be a graded ring and M be a graded R-module. Then every graded prime submodule of M is semiprime. Moreover, every graded prime ideal of R is semiprime.*

Proof. Assume that $I^nK \subseteq N$, where n is a positive integer, $I \subseteq h(R)$ and $K \subseteq h(M)$. Now, since N is a graded prime, we have either $I \subseteq (N : M) \subseteq (N : K)$ or $I^{n-1}K \subseteq N$. In the first case $IK \subseteq N$ and we are done. If $I^{n-1}K \subseteq N$, then $I \subseteq (N : M)$ or $I^{n-2}K \subseteq N$. In this way we have $IK \subseteq N$. Hence N is a graded semiprime submodule of M. $\square$

For basic properties of a multiplication module one may refer to [1], [4] and [6].

A graded R-module M is said to be a *graded multiplication module* if for every graded submodule N of M, there exists a graded ideal $\mathfrak{a}$ of R such that $N = \mathfrak{a}M$. Let M be a graded R-module. Assume that M is a graded multiplication module. If N and K are graded submodules of M, then there exist graded ideals $\mathfrak{a}$ and $\mathfrak{b}$ of R such that $N = \mathfrak{a}M$ and $K = \mathfrak{b}M$. Then the *product* of N and K is defined to be $(\mathfrak{a}\mathfrak{b})M$ and is denoted by $N \cdot K$. It is well-known in [1, Theorem 3.4] and [5, Theorem 4] that the product is well-defined. In fact, $\mathfrak{a}\mathfrak{b}$ is a graded ideal of R by Lemma 1.1 and $N \cdot K$ is independent of the choices of $\mathfrak{a}$ and $\mathfrak{b}$. Also, for every positive integer k, N^k is defined to be

$$\overbrace{N \cdot N \cdot \cdots \cdot N}^{k \text{ times}}.$$

Let R be a graded ring and M be a graded multiplication module over R. The *graded radical* of a graded submodule N of M is the set of all elements m of M such that $(Rm)^k \subseteq N$ for some positive integer k and is denoted by $grad(N)$.

Remark 1.7. There were several authors who would like to define the *product* $x \cdot y$ of two elements x and y of M to be $Rx \cdot Ry$ and then they used the notation "$x^n \subseteq N$ for some positive integer n" in their papers, such as in [1, Theorem 3.13] and in [5, Corollary 4 to Theorem 12]. If $n = 1$, then $x \subseteq N$. This does not make sense, because $x \in M$. Hence it is natural not to define the product of two elements of M. However, we define the product of two submodules of M as in the second paragraph just posterior to the proof of Proposition 1.6.

Let R be a graded ring and M be a graded multiplication module over R. A graded submodule N of M is called *nilpotent* if $N^t = 0$ for some positive integer t. If a graded submodule N of M is nilpotent, then $grad(0) = grad(N)$.

A nonempty subset S of M is said to be *multiplicatively closed* if $(Rx)^n \cap S \neq \emptyset$ for each positive integer n and each $x \in S$.

The present paper will proceed as follows. Let R be a graded ring and M be a graded multiplication module over R.

In Section 2, we characterize graded semiprime submodules of M as follows.

(1) (Theorem 2.1 and its corollary) The following ten statements are equivalent for a proper graded submodule P of M.

(i) P is semiprime.

(ii) If $(Rx)^n \subseteq P$, where $x \in h(M)$ and n is a positive integer, then $x \in P$.

(iii) If $K^n \subseteq P$, where K is a graded submodule of M and n is a positive integer, then $K \subseteq P$.
(iv) If L is a graded submodule of M such that $P \subset L \subseteq M$, then $(P :_R L)$ is a graded semiprime ideal of R.
(v) $(P :_R M)$ is a graded semiprime ideal of R.
(vi) $grad(P) = P$.
(vii) If $Rx \cdot Ry \subseteq P$, where $x, y \in h(M)$, then $Rx \cap Ry \subseteq P$.
(viii) The factor R-module M/P has no nonzero nilpotent submodule.
(ix) There exits a graded semiprime ideal $\mathfrak{p}$ of R with $(0 :_R M) \subseteq \mathfrak{p}$ such that $P = \mathfrak{p}M$.
(x) $M \setminus P$ is multiplicatively closed.

Moreover, if M is regular, then we show that every proper graded submodule of M is semiprime.

We give an example showing that the condition "M being a multiplication module" cannot be omitted.

Using the result above, we show that the three statements are true.

(2) (Theorem 2.6) If K is a graded submodule of M and S is a multiplicatively closed subset of M such that $K \cap S = \emptyset$, then there is a graded semiprime submodule P of M which is maximal with respect to the properties that $K \subseteq P$ and $P \cap S = \emptyset$.

(3) (Proposition 2.8) If N is a graded semiprime submodule of M, then it contains a minimal graded semiprime submodule.

(4) (Theorem 2.9) If N is a proper graded submodule of M and M is finitely generated, then there exists a graded semiprime submodule of M that contains N.

In Section 3, we define an almost semiprime submodule of M.

(5) (Theorem 3.5) Let Q, K be graded submodules of M. If Q and K are almost semiprime in M such that $Q + K \neq M$ and $Q \cap K \subseteq M_g$ for all $g \in G$, then we prove that $Q + K$ is almost semiprime in M.

2. Semiprime submodules

In this section, we deal with graded multiplication modules over graded rings. We define a semiprime submodule of a graded multiplication module over a graded ring to characterize it. And then we discuss several properties of semiprime submodules.

Let M be a multiplication module over a ring R. Let K be a submodule of M. Then there exists an ideal I of R such that $K = IM$. Consider the following descending chain of ideals of R:

$$I \supseteq I^2 \supseteq \cdots .$$

Then we can get a descending chain of submodules of M

$$K \supseteq K^2 \supseteq \cdots .$$

From this, we can see the following: if $K \subseteq N$, where N is a submodule of M, then $K^n \subseteq N$ for every positive integer n. In view of this it is natural to ask a question: when $K^n \subseteq N$, where n is a positive integer, under what conditions can we get $K \subseteq N$? The following result deals with this question.

Theorem 2.1. *Let M be a graded multiplication module over R and P be a proper graded R-submodule of M. Then the following statements are equivalent.*

(i) *P is semiprime.*
(ii) *If $(Rx)^n \subseteq P$, where $x \in h(M)$ and n is a positive integer, then $x \in P$.*
(iii) *If $K^n \subseteq P$, where K is a graded submodule of M and n is a positive integer, then $K \subseteq P$.*
(iv) *If L is a graded submodule of M such that $P \subset L \subseteq M$, then $(P :_R L)$ is a graded semiprime ideal of R.*
(v) *$(P :_R M)$ is a graded semiprime ideal of R.*
(vi) *$grad(P) = P$.*
(vii) *If $Rx \cdot Ry \subseteq P$, where $x, y \in h(M)$, then $Rx \cap Ry \subseteq P$.*
(viii) *The factor R-module M/P has no nonzero nilpotent submodule.*
(ix) *There exits a graded semiprime ideal $\mathfrak{p}$ of R with $(0 :_R M) \subseteq \mathfrak{p}$ such that $P = \mathfrak{p}M$.*

Proof. (i) $\Rightarrow$ (ii) Let P be a graded semiprime submodule of M. Assume that $(Rx)^n \subseteq P$, where $x \in h(M)$ and n is a positive integer. Since M is a multiplication module, there exists a graded ideal $\mathfrak{a}$ of R such that $Rx = \mathfrak{a}M$. Then

$$\mathfrak{a}^n M = (\mathfrak{a}M)^n = (Rx)^n \subseteq P.$$

Since P is a graded semiprime submodule of M, we have $Rx = \mathfrak{a}M \subseteq P$. Therefore $x \in P$.

(ii) $\Rightarrow$ (iii) Assume that $K^n \subseteq P$, where K is a graded submodule of M and n is a positive integer. To show that $K \subseteq P$, it suffices to show that every element x of $h(K)$ belongs to P. Let x be an arbitrary element of $h(K)$. Then $x \in h(M)$ and $(Rx)^n \subseteq K^n \subseteq P$. By (ii), $x \in P$.

(iii) $\Rightarrow$ (iv) Assume that (iii) is true. Assume that L is a graded submodule of M such that $P \subset L \subseteq M$. Then $(P :_R L)$ is proper. By Lemma 1.2, $(P :_R L)$ is graded.

Also, assume that $\mathfrak{a}^n\mathfrak{b} \subseteq (P :_R L)$, where n is a positive integer and $\mathfrak{a}$ and $\mathfrak{b}$ are graded ideals of R. Then

$$((\mathfrak{a}\mathfrak{b})L)^n = (\mathfrak{a}\mathfrak{b})^n L = \mathfrak{b}^{n-1}((\mathfrak{a}^n\mathfrak{b})L) \subseteq \mathfrak{b}^{n-1}P \subseteq P.$$

Notice that $(\mathfrak{a}\mathfrak{b})L$ is a graded submodule of M. Then by (iii) we have $(\mathfrak{a}\mathfrak{b})L \subseteq P$. This shows that $\mathfrak{a}\mathfrak{b} \subseteq (P :_R L)$. Hence $(P :_R L)$ is a semiprime ideal.

(iv) $\Rightarrow$ (v) Assume that (iv) is true. Taking L by M, we can see that $(P :_R M)$ is a graded semiprime ideal of R.

(v) $\Rightarrow$ (vi) Assume that (v) is true. Clearly, $P \subseteq grad(P)$. Conversely, assume that $(Rx)^n \subseteq P$ for some positive integer n. Then we need to show that $x \in P$. If $n = 1$, then $x \in P$; we are done. Assume that $n > 1$. Since M is a graded multiplication module, there is a graded ideal $\mathfrak{a}$ of R such that $Rx = \mathfrak{a}M$. Then

$$\mathfrak{a}^n M = (Rx)^n \subseteq P.$$

So, $\mathfrak{a}^{n-1}\mathfrak{a} = \mathfrak{a}^n \subseteq (P :_R M)$. Since $(P :_R M)$ is graded semiprime, we get $\mathfrak{a} \subseteq (P :_R M)$. Hence

$$x \in Rx = \mathfrak{a}M \subseteq (P :_R M)M = P,$$

as required.

(vi) $\Rightarrow$ (vii) Assume that (vi) is true. Assume that $Rx \cdot Ry \subseteq P$, where $x, y \in h(M)$. Let m be an arbitrary element of $Rx \cap Ry$. Then $Rm \subseteq Rx$ and $Rm \subseteq Ry$. Hence

$$(Rm)^2 \subseteq (Rx) \cdot (Ry) \subseteq P.$$

By (vi), $Rm \subseteq P$. Hence $m \in P$. This shows that $Rx \cap Ry \subseteq P$.

(vii) $\Rightarrow$ (viii) Assume that (vii) is true. Let $x + P$ be an arbitrary nilpotent element of M/P. Then there exists a positive integer n such that $((Rx + P)^n/P) = 0$ in M/P. There exists a graded ideal $\mathfrak{a}$ of R such that $Rx = \mathfrak{a}M$. So,

$$((Rx)^n + P)/P = (\mathfrak{a}^n M + P)/P = \mathfrak{a}^n(M/P) = ((Rx + P)^n/P) = 0.$$

This implies that $(Rx)^n \subseteq P$. By (vii),

$$x \in Rx = \overbrace{Rx \cap Rx \cap \cdots \cap Rx}^{n \text{ times}} \subseteq P.$$

Hence $x + P = 0 + P$.

(viii) $\Rightarrow$ (ix) Assume that (viii) is true. Since M is a graded multiplication module, there exists a graded ideal $\mathfrak{p}$ of R such that $P = \mathfrak{p}M$. To show that $\mathfrak{p}$ is semiprime, assume that $\mathfrak{a}^n\mathfrak{b} \subseteq \mathfrak{p}$, where $\mathfrak{a}$ and $\mathfrak{b}$ are graded ideals of R. Then $(\mathfrak{a}\mathfrak{b})^n \subseteq \mathfrak{p}$. So,

$$((\mathfrak{a}\mathfrak{b})M)^n = (\mathfrak{a}\mathfrak{b})^n M \subseteq \mathfrak{p}M = P.$$

This means that

$$(((\mathfrak{a}\mathfrak{b})M + P)/P)^n = (((\mathfrak{a}\mathfrak{b})M)^n + P)/P = \{0 + P\}.$$

By (viii), $((\mathfrak{a}\mathfrak{b})M + P)/P = \{0 + P\}$. This implies that

$$(\mathfrak{a}\mathfrak{b})M \subseteq ((\mathfrak{a}\mathfrak{b})M + P = P = \mathfrak{p}M.$$

Since M is multiplication, it follows that $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{p}$. Therefore $\mathfrak{p}$ is semiprime.

Also, let a be an arbitrary element of $(0 :_R M)$. Then $aM = 0 \subseteq \mathfrak{p}M$. Since M is multiplication, it follows that $a \in \mathfrak{p}$. Hence $(0 :_R M) \subseteq \mathfrak{p}$.

(ix) $\Rightarrow$ (i) Assume that (ix) is true. To show that P is semiprime, assume that $\mathfrak{a}^n K \subseteq P$, where $\mathfrak{a}$ is a graded ideal of R and K is a graded submodule of M, and n is a positive integer. Since M is a graded multiplication module, there exists a graded ideal $\mathfrak{b}$ of R such that $K = \mathfrak{b}M$. Then

$$(\mathfrak{a}^n\mathfrak{b})M = \mathfrak{a}^n K \subseteq P = \mathfrak{p}M.$$

Since $\mathfrak{p} + (0 :_R M) = \mathfrak{p}$, it follows from [6, Theorem 9, p. 231] that either $\mathfrak{a}^n\mathfrak{b} \subseteq \mathfrak{p}$ or $M = (\mathfrak{p} :_R \mathfrak{a}^n\mathfrak{b})M$. If $\mathfrak{a}^n\mathfrak{b} \subseteq \mathfrak{p}$, then we have $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{p}$ since $\mathfrak{p}$ is semiprime. Hence $\mathfrak{a}K = \mathfrak{a}(\mathfrak{b}M) = (\mathfrak{a}\mathfrak{b})M \subseteq \mathfrak{p}M = P$; we are done. Or, assume that $M = (\mathfrak{p} :_R \mathfrak{a}^n\mathfrak{b})M$. Notice that

$$\mathfrak{a}^n(\mathfrak{p} :_R \mathfrak{a}^n\mathfrak{b})\mathfrak{b} = (\mathfrak{p} :_R \mathfrak{a}^n\mathfrak{b})\mathfrak{a}^n\mathfrak{b} \subseteq \mathfrak{p}.$$

Since $\mathfrak{p}$ is semiprime, we have $(\mathfrak{p} :_R \mathfrak{a}^n\mathfrak{b})\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{p}$. Hence

$$\mathfrak{a}K = \mathfrak{a}(\mathfrak{b}M) = (\mathfrak{a}\mathfrak{b})M = ((\mathfrak{p} :_R \mathfrak{a}^n\mathfrak{b})\mathfrak{a}\mathfrak{b})M \subseteq \mathfrak{p}M = P.$$

Hence P is semiprime. □

Corollary 2.2. *Let R be a graded ring and M be a graded multiplication module over R. Then a proper graded submodule P of M is semiprime if and only if $M \setminus P$ is multiplicatively closed.*

Proof. Let P be a graded semiprime submodule of M and let $x \in M \setminus P$. Since P is graded semiprime, it follows from Theorem 2.1 that $(Rx)^n \nsubseteq P$ for every positive integer n. Hence $(Rx)^n \cap (M \setminus P) \neq \emptyset$. This shows that $M \setminus P$ is multiplicatively closed.

Conversely, assume that $M \setminus P$ is multiplicatively closed. To show that P is semiprime, assume that $(Rx)^n \subseteq P$, where n is a positive integer and $x \in h(M)$. We need to show that $x \in P$. Suppose on the contrary that $x \notin P$. Then $x \in M \setminus P$. By our assumption, $(Rx)^n \cap (M \setminus P) \neq \emptyset$. Take $y \in (Rx)^n \cap (M \setminus P)$. Then $y \in (Rx)^n \subseteq P$. This contradiction shows that $x \in P$, as needed. □

Let M be a graded multiplication module over a graded ring R. Then $N \cdot K \subseteq N \cap K$ for each pair of graded submodules N and K of M. M is said to be *regular* if for each pair of graded submodules N and K of M, $N \cdot K = N \cap K$.

Corollary 2.3. *Let R be a graded ring and M be a regular graded multiplication module over R. Then every proper graded submodule of M is semiprime.*

The condition "M being multiplication" in Theorem 2.1 cannot be omitted. The example of this is given below.

Example 2.4. First, consider the set $\mathbb{Z}$ of all integers. Then $(\mathbb{Z}, +)$ is a group with additive identity 0 and $(\mathbb{Z}, +, \cdot)$ is a commutative ring with identity 1. Take $G = (\mathbb{Z}, +)$ and $R = (\mathbb{Z}, +, \cdot)$. Define

$$R_g = \begin{cases} \mathbb{Z} & \text{if } g = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then each R_g is an additive subgroup of R and R is their internal direct sum. In fact, $1 \in R_0$ and $R_gR_h \subseteq R_{g+h}$. That is, $R = \oplus_{g\in G}R_g$. Hence R is a G-graded ring. In other words, the ring $(\mathbb{Z}, +, \cdot)$ of integers is a $(\mathbb{Z}, +)$-graded ring.

Next, let M be the set $\mathbb{Z}\times\mathbb{Z}$. Then M can be given a $\mathbb{Z}$-module structure. Define

$$M_g = \begin{cases} \mathbb{Z}\times 0 & \text{if } g=0 \\ 0\times\mathbb{Z} & \text{if } g=1 \\ 0\times 0 & \text{otherwise.} \end{cases}$$

Then $M=\oplus_{g\in G}M_g$. Hence M is a G-graded R-module. In other words, the $\mathbb{Z}$-module $(\mathbb{Z}\times\mathbb{Z}, +, \cdot)$ is a $\mathbb{Z}$-graded $\mathbb{Z}$-module.

Now, consider a submodule $N=9\mathbb{Z}\times 0$ of M. Then it is a graded submodule. $(N :_R M)=0$ and so it is a graded semiprime ideal of R. But the graded submodule N is not graded semiprime in M, since $3^2(2,0)\in N$ but $3(2,0)\notin N$.

By Theorem 2.1, we can see that the $\mathbb{Z}$-module $(\mathbb{Z}\times\mathbb{Z}, +, \cdot)$ is not a multiplication module.

Lemma 2.5. *Let R be a graded ring and M be a graded R-module. If P is a graded submodule of M and $x\in h(M)$, then both Rx and $P+Rx$ are graded submodules of M.*

Proof. This follows from Lemma 1.2. □

Theorem 2.6. *Let R be a graded ring and M be a graded multiplication module over R. Let K be a graded submodule of M and S be a multiplicatively closed subset of M such that $K\cap S=\emptyset$. Then there is a graded semiprime submodule P of M which is maximal with respect to the properties that $K\subseteq P$ and $P\cap S=\emptyset$.*

Proof. Let Ω be the set of all graded submodules L of M such that $K\subseteq L$ and $L\cap S=\emptyset$. $K\in\Omega$, so in particular $\Omega\neq\emptyset$. By the Zorn lemma Ω has a maximal element, say P. It is enough to show that P is semiprime. To show that P is semiprime, assume that $(Rx)^n\subseteq P$, where n is a positive integer and $x\in h(M)$. Then we need to show that $x\in P$. Suppose on the contrary that $x\notin P$. Then $P\subset P+Rx$. By Lemma 2.5, $P+Rx$ is graded. By the maximality of P, $P+Rx\notin\Omega$. Hence $(P+Rx)\cap S\neq\emptyset$. Take $y\in(P+Rx)\cap S$. Then $y\in P+Rx$ and $y\in S$. Since M is a multiplication module and $(Rx)^n\subseteq P$, we can show that

$$(P+Rx)^n\subseteq P+(Rx)^n=P.$$

Also, since S is multiplicatively closed and $y\in S$, we have $(Ry)^n\cap S\neq\emptyset$. Hence

$$\emptyset\neq(Ry)^n\cap S\subseteq(P+Rx)^n\cap S\subseteq P\cap S,$$

contradicting the disjointness of P and S. This shows that $x\in P$. Therefore P is a graded semiprime submodule. □

Lemma 2.7. *Let R be a graded ring and M be a graded multiplication module over R. Let Ω be a nonempty family of graded submodules of M.*

(i) *If each member of Ω is semiprime in M, then so is $\cap_{Q\in\Omega}Q$.*
(ii) *If each member of Ω is semiprime in M, Ω is totally ordered by inclusion, and $\cup_{Q\in\Omega}Q\neq M$, then $\cup_{Q\in\Omega}Q$ is a proper graded semiprime submodule of M.*

Proof. (i) Assume that each member of Ω is semiprime in M. Then by Theorem 2.1,

$$\begin{aligned} grad(\cap_{Q\in\Omega}Q)\cap h(M) &\subseteq (\cap_{Q\in\Omega}grad(Q))\cap h(M) \\ &= \cap_{Q\in\Omega}(grad(Q)\cap h(M)) \\ &= \cap_{Q\in\Omega}(Q\cap h(M)) \\ &= (\cap_{Q\in\Omega}Q)\cap h(M). \end{aligned}$$

It is clear that the converse inclusion holds. Hence by Theorem 2.1 again, $\cap_{Q\in\Omega}Q$ is semiprime.

(ii) Assume that Ω is totally ordered by inclusion and $\cup_{Q\in\Omega}Q \neq M$. Then it is clear that $\cup_{Q\in\Omega}Q$ is a proper graded submodule of M. Now assume that each member of Ω is semiprime in M. Then by Theorem 2.1,

$$\begin{aligned} grad(\cup_{Q\in\Omega}Q)\cap h(M) &\subseteq (\cup_{Q\in\Omega}grad(Q))\cap h(M) \\ &= \cup_{Q\in\Omega}(grad(Q)\cap h(M)) \\ &= \cup_{Q\in\Omega}(Q\cap h(M)) \\ &= (\cup_{Q\in\Omega}Q)\cap h(M). \end{aligned}$$

It is clear that the converse inclusion holds. Hence by Theorem 2.1 again, $\cup_{Q\in\Omega}Q$ is semiprime. □

A graded semiprime submodule P of a graded R-module M is said to be *minimal* if whenever $N \subseteq P$ and N is graded semiprime, then $N = P$.

Proposition 2.8. *Let R be a graded ring and M be a graded multiplication module over R. If N is a graded semiprime submodule of M, then it contains a minimal graded semiprime submodule.*

Proof. Consider the set Σ of all graded semiprime submodules P of M such that $N \supseteq P$. Since $N \in \Sigma$ we see that Σ is not empty. Also $\supseteq$ is a partial order on Σ. Let Ω be a non-empty subset of Σ which is totally ordered by $\supseteq$. Therefore by Lemma 2.7(i), $\cap_{P\in\Omega}P$ is a graded semiprime submodule of M. Now the result holds by applying the Zorn lemma. □

Theorem 2.9. *Let R be a graded ring and M be a graded multiplication module over R. If N is a proper graded submodule of M and if M is finitely generated, then there exists a graded semiprime submodule of M that contains N.*

Proof. Assume that N is a proper graded submodule of M and M is finitely generated. Let Σ be the collection of all proper graded submodules of M that contains N. Then $N \in \Sigma$. In particular, $\Sigma \neq \emptyset$. Order Σ by inclusion. Then Σ is partially ordered. Let Ω be any chain of Σ. Take $Q^* = \cup_{Q\in\Omega}Q$. Then by Lemma 2.7(ii), $Q^* \in \Sigma$. Ω has an upper bound in Σ. By the Zorn lemma, Σ has a maximal member, say P. It remains to prove that P is semiprime.

Suppose that $grad(P)\cap h(M) \neq P\cap h(M)$. Then we can take an element $x \in (grad(P)\cap h(M))\backslash(P\cap h(M))$. Then $x \notin P$, so $P \subset P + Rx$. By

Lemma 2.7(ii) and by the maximality of P, we must have $P + Rx = M$. Since $x \in grad(P)$, there exists a positive integer n such that $x^n \in P$. Hence

$$M = M^n = (P + Rx)^n \subseteq P + (Rx)^n \subseteq P,$$

so $M = P$. This contradiction shows that $grad(P) \cap h(M) = P \cap h(M)$. Therefore it follows from Theorem 2.1 that P is semiprime. □

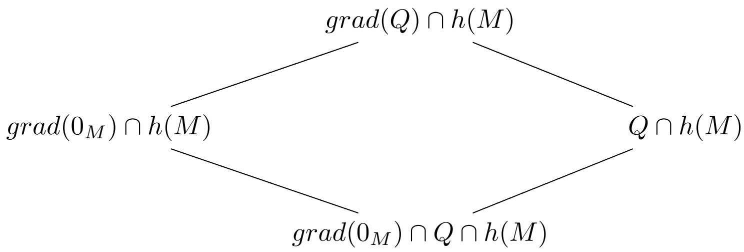
3. Almost semiprime submodules

In this section we define an almost semiprime submodule of a graded multiplication module over a graded ring and discuss the sum of two almost semiprime submodules.

Let R be a graded ring and M be a graded multiplication module over R. Let Q be a proper graded submodule of M. Then $Q \cap h(M) \subseteq grad(Q) \cap h(M)$. The following two statements are true:

$$grad(0_M) \cap h(M) \subseteq grad(Q) \cap h(M),$$
$$grad(0_M) \cap Q \cap h(M) \subseteq Q \cap h(M).$$

More precisely, we can draw their lattice diagram as follows:



Then it is easy to see that

$$\begin{aligned}&(Q \cap h(M))\backslash(grad(0_M) \cap Q \cap h(M))\\ &\subseteq (grad(Q) \cap h(M))\backslash(grad(0_M) \cap h(M)).\end{aligned}$$

Remark 3.1. This statement is the same as the following one but the following one is much easier for us to make sure if it is true.

$$(Q\backslash(Q \cap grad(0_M)) \cap h(M) \subseteq (grad(Q)\backslash grad(0_M)) \cap h(M).$$

Definition 3.2. Let R be a graded ring and M be a graded multiplication module over R. A proper graded submodule Q of M is said to be *almost semiprime* if

$$\begin{aligned}&(grad(Q) \cap h(M))\backslash(grad(0_M) \cap h(M))\\ &= (Q \cap h(M))\backslash(grad(0_M) \cap Q \cap h(M)).\end{aligned} \tag{3.1}$$

Let $g \in G$. Likewise, a proper graded submodule Q_g of the R_e-module M_g is said to be *almost g-semiprime* if

$$(grad(Q_g) \cap M_g)\backslash(grad(0_{M_g}) \cap M_g) = Q_g\backslash(grad(0_{M_g}) \cap Q_g). \tag{3.2}$$

It is immediate that the zero submodule of a graded multiplication module is graded and almost semiprime.

Let R be a graded ring and M be a graded multiplication module over R. Let Q be a proper graded submodule of M. Assume that Q is semiprime. Then it follows from Theorem 2.1 that $grad(Q) \cap h(M) = Q \cap h(M)$, so that $grad(0_M) \cap h(M) = grad(0_M) \cap Q \cap h(M)$. Hence Q is almost semiprime. This shows that every semiprime submodule of M is almost semiprime. Conversely, if Q is almost semiprime and $grad(0_M) \cap h(M) = grad(0_M) \cap Q \cap h(M)$, then Q is semiprime.

Proposition 3.3. *Let R be a graded ring, M be a graded multiplication module over R and Q be a proper graded submodule of M. If Q is almost semiprime, then for every $g \in G$, Q_g is almost g-semiprime in M_g.*

Proof. Assume that Q is almost semiprime. Then the equality (3.1) holds. Let $g \in G$. Note that $Q = \oplus_{g\in G} Q_g$. Then taking the intersection of the equation (3.1) with M_g, we can get (3.2). Hence Q_g is almost semiprime. □

Lemma 3.4. *Let R be a graded ring, M a graded multiplication module over R and K, Q graded submodules of M such that $K \subseteq Q$. Then the following statements are true.*

(i) *If Q is almost semiprime such that $K \subseteq M_g$ for all $g \in G$, then Q/K is almost semiprime in M/K.*

(ii) *If K and Q/K are almost semiprime in M and M/K, respectively, then Q is almost semiprime in M.*

Proof. If $K \subseteq Q$, then we have already known that M/K and Q/K are G-graded.

(i) Assume that Q is almost semiprime such that $K \subseteq M_g$ for all $g \in G$. Then $K \subseteq \cup_{g\in G} M_g = h(M)$ and

$$h(M/K) = \cup_{g\in G}((M_g + K)/K) = \cup_{g\in G}(M_g/K) = h(M)/K.$$

Now since the equality (3.1) holds, direct computation gives

$$\begin{aligned} &(grad(Q/K) \cap h(M/K)) \backslash (grad(0_{M/K}) \cap h(M/K)) \\ &= (Q/K \cap h(M/K)) \backslash (grad(0_{M/K}) \cap Q/K \cap h(M/K)). \end{aligned} \tag{3.3}$$

Hence Q/K is almost semiprime.

(ii) In order to show that Q is almost semiprime, we show that (3.1) holds. Let x belong up in the equality (3.1). Then $(Rx)^s \subseteq Q$ for some positive integer s. This implies that $(R(x + K))^s = ((Rx)^s + K)/K$ is in Q/K. Hence $x + K \in grad(Q/K)$. Now, there are two cases to consider.

Case 1. Assume that $x + K$ is in $grad(0_{M/K})$. Then there exists a positive integer t such that $(R(x + K))^t = 0$ in M/K. So, $(Rx)^t \subseteq K$. This implies that $x \in grad(K)$. Since K is almost semiprime, we have

$$\begin{aligned} x &\in (grad(K) \cap h(M)) \backslash (grad(0_M) \cap h(M)) \\ &= (K \cap h(M)) \backslash (grad(0_M) \cap K \cap h(M)). \end{aligned}$$

Hence since $K \subseteq Q$, x belongs down in the equality (3.1).

Case 2. Assume that $x + K$ is not in $grad(0_{M/K})$. Then $x + K$ belongs up in the equality (3.3). Since Q/K is almost semiprime, the equality (3.3) holds. Hence $x + K$ belongs down in the equality (3.3). This implies that $x + K \in Q/K$. Then there exists an element $y \in Q$ such that $x + K = y + K$. This implies that $x - y \in K$, so that $x = (x - y) + y \in K + Q = Q$ since $K \subseteq Q$. Hence x belongs down in the equality (3.1). This shows that the equality (3.1) holds. Therefore Q is almost semiprime. □

Theorem 3.5. *Let R be a graded ring, M be a graded multiplication module over R and K, Q be graded submodules of M. If K and Q are almost semiprime in M such that $Q + K \neq M$ and $Q \cap K \subseteq M_g$ for all $g \in G$, then $Q + K$ is almost semiprime in M.*

Proof. Assume that Q and K are almost semiprime in M such that $Q+K \neq M$ and $Q \cap K \subseteq M_g$ for all $g \in G$. Then Lemma 3.4(i), $Q/(Q \cap K)$ is also almost semiprime in $M/(Q \cap K)$. Notice that $Q/(Q \cap K) \cong (Q + K)/K$ by the second isomorphism theorem for modules. Then $(Q + K)/K$ is almost semiprime in M/K. Hence by Lemma 3.4(ii), $Q + K$ is almost semiprime. □

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