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THE MINIMAL FREE RESOLUTION OF A STAR-CONFIGURATION IN $\mathbb{P}^n$ AND THE WEAK LEFSCHETZ PROPERTY

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Abstract. We find the Hilbert function and the minimal free resolution of a star-configuration in $\mathbb{P}^n$. The conditions are provided under which the Hilbert function of a star-configuration in $\mathbb{P}^2$ is generic or non-generic. We also prove that if $\mathbb{X}$ and $\mathbb{Y}$ are linear star-configurations in $\mathbb{P}^2$ of types t and s, respectively, with $s \geq t \geq 3$, then the Artinian k-algebra $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ has the weak Lefschetz property.

1. Introduction

Throughout the paper, $R = k[x_0, x_1, \ldots, x_n]$ will be an $(n+1)$-variable polynomial ring over an algebraically closed field k of characteristic 0, and the symbol $\mathbb{P}^n$ will denote the projective n-space over a field k. Let I be a homogeneous ideal of R. Then the numerical function

$$\mathbf{H}_{R/I}(t) := \dim_k R_t - \dim_k I_t$$

is called the *Hilbert function* of the ring R/I. If $I := I_{\mathbb{X}}$ is the ideal of a subscheme $\mathbb{X}$ in $\mathbb{P}^n$, then we denote

$$\mathbf{H}_{R/I_{\mathbb{X}}}(t) := \mathbf{H}_{\mathbb{X}}(t) \quad \text{for } t \geq 0$$

and call it the *Hilbert function* of $\mathbb{X}$. Many interesting problems in the study of Hilbert functions and minimal free resolutions of standard graded algebras have been studied (see [12, 13, 14, 15, 16, 19]).

A graded Artinian k-algebra $A = \bigoplus_{i=0}^{s} A_i$ $(A_s \neq 0)$ has the *weak Lefschetz property* if the homomorphism $(\times L) : A_i \to A_{i+1}$ induced by multiplication by a general linear form L has maximal rank for all i. In this case, we call

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L a *Lefschetz element*. This fundamental property has been studied by many authors (see [3, 6, 12, 17, 21, 22, 24, 25]).

In [2], the following interesting result has been proved.

Proposition 1.1 (Proposition 3.4, [2]). *Let $F_1, F_2, \ldots, F_r$ be general forms in $R = k[x_0, x_1, \ldots, x_n]$ with $r \geq 3$. Then*

$$\bigcap_{1\leq i<j\leq r}(F_i, F_j) = \sum_{i=1}^{r}(F_1 \cdots \hat{F_i} \cdots F_r),$$

where $\hat{*}$ means that we omit $*$.

The variety $\mathbb{X}$ in $\mathbb{P}^n$ of the ideal

$$\bigcap_{1\leq i<j\leq r}(F_i, F_j) = \sum_{i=1}^{r}(F_1 \cdots \hat{F_i} \cdots F_r)$$

in Proposition 1.1 is called a *star-configuration* in $\mathbb{P}^n$ of type r. Furthermore, if the F_i are all general linear forms in R, the star-configuration $\mathbb{X}$ is called a *linear star-configuration* in $\mathbb{P}^n$.

The Terracini Lemma in [27] says that the Hilbert function of the union of star-configurations in $\mathbb{P}^n$ gives the dimensions of *the secant varieties of the varieties of reducible forms* (see also [2, 4, 5, 9, 10, 23]). In [18], Geramita, Migliore, and Sabourin showed that a linear star-configuration in $\mathbb{P}^2$ has a generic Hilbert function. In this paper we study Hilbert functions and minimal free resolutions of star-configurations in $\mathbb{P}^n$, and give answers to the following two interesting questions.

Question 1.2. Let $F_1, \ldots, F_r$ be general forms in $R = k[x_0, x_1, \ldots, x_n]$ of degrees $1 \leq d_1 \leq \cdots \leq d_r$, respectively.

(a) What is the Hilbert function of the ideal of a star-configuration defined by $F_1, \ldots, F_r$?
(b) What is the minimal free resolution of the ideal of a star-configuration defined by $F_1, \ldots, F_r$?

Question 1.3. Let $\mathbb{X}$ and $\mathbb{Y}$ be star-configurations in $\mathbb{P}^2$ defined by general forms of degree $d \geq 1$. Does the Artinian ring $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ have the weak Lefschetz property?

In Section 2, we introduce preliminary results and definitions and then find the Hilbert function and the minimal free resolution of a star-configuration in $\mathbb{P}^n$ (Theorem 2.1 and Corollary 2.5), which is the complete answer to Question 1.2. In Section 3 we show that the star-configuration $\mathbb{X}$ in $\mathbb{P}^2$, defined by general forms $F_1, \ldots, F_r$ of the same degree d with $d = 1, 2$, has a generic Hilbert function (see Proposition 3.1), which slightly generalizes the result of [18]. In other words,

$$\mathbf{H}_{\mathbb{X}}(t) = \min\left\{\binom{t+2}{2}, \deg(\mathbb{X})\right\} \quad \text{for} \quad t \geq 0.$$

However, if the star-configuration $\mathbb{X}$ is defined by general forms $F_1, \ldots, F_r$ of the same degree d with $d \geq 3$, then the Hilbert function of the star-configuration is NEVER generic (see Example 3.2, Proposition 3.3, and Remark 3.4). In Section 4, we show that if $\mathbb{X}$ and $\mathbb{Y}$ are linear star-configurations in $\mathbb{P}^2$ of types t and s with $s \geq t \geq 3$, then the Artinian ring $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ has the weak Lefschetz property (see Theorem 4.2), which is the answer to Question 1.3 for $d = 1$. However, Question 1.3 is still open for $d > 1$.

2. Star-configurations in $\mathbb{P}^n$

Let $\mathbb{X}$ be a star-configuration in $\mathbb{P}^n$ defined by general forms $F_1, \ldots, F_r$ in $R = k[x_0, \ldots, x_n]$ of degrees $1 \leq d_1 \leq \cdots \leq d_r$, respectively. By Proposition 1.1, a star-configuration $\mathbb{X}$ in $\mathbb{P}^n$ is a closed subscheme of codimension 2, and the ideal of a linear star-configuration $\mathbb{X}$ has r generators of degree $r-1$. For any matrix M with entries in an arbitrary ring R we write $I_t(M)$ for the ideal generated by the $t \times t$ minors of M. We begin with the following theorem, which gives an answer to Question 1.2.

Theorem 2.1. *Let $\mathbb{X}$ be a star-configuration in $\mathbb{P}^n$ defined by general forms $F_1, \ldots, F_r$ in $R = k[x_0, \ldots, x_n]$ of degrees $1 \leq d_1 \leq \cdots \leq d_r$, respectively, and let $d = d_1 + d_2 + \cdots + d_r$. Then the minimal free resolution of $R/I_{\mathbb{X}}$ is*

$$0 \rightarrow R^{r-1}(-d) \rightarrow \bigoplus_{i=1}^{r} R(-(d-d_i)) \rightarrow R \rightarrow R/I_{\mathbb{X}} \rightarrow 0.$$

Proof. For the proof, we first introduce some notations.

- Let $\mathbf{e}_i = [0, \ldots, \overset{i\text{-th}}{1}, \ldots, 0]^T$ be an i-th standard vector in R^r for $i = 1, \ldots, r$.
- Define $\sigma_{i,j} = F_i\mathbf{e}_i - F_j\mathbf{e}_j$ for $1 \leq i < j \leq r$.
- Let M be an $r \times (r-1)$ matrix whose column vectors are $\sigma_{1,2}$, $\sigma_{2,3}$, $\ldots$, $\sigma_{r-1,r}$, that is,

$$M := \begin{pmatrix} F_1 & 0 & 0 & \cdots & 0 & 0 \\ -F_2 & F_2 & 0 & \cdots & 0 & 0 \\ 0 & -F_3 & F_3 & \cdots & 0 & 0 \\ 0 & 0 & -F_4 & & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & -F_{r-1} & F_{r-1} \\ 0 & 0 & \cdots & \cdots & 0 & -F_r \end{pmatrix}.$$

- Let $\delta_i = F_1 \cdots \hat{F_i} \cdots F_r$ be a homogeneous polynomial of degree $d - d_i$ for $i = 1, \ldots, r$.
- Define two maps ψ and φ as

$$\psi \ : \ \bigoplus_{i=1}^{r} R(-(d-d_i)) \overset{[\delta_1,\ldots,\delta_r]}{\longrightarrow} R, \quad \text{and}$$
$$\varphi \ : \ R^{r-1}(-d) \overset{M}{\longrightarrow} \bigoplus_{i=1}^{r} R(-(d-d_i)).$$

We shall show that the following sequence is the minimal free resolution of $R/I_{\mathbb{X}}$.

$$0 \;\to\; R^{r-1}(-d) \;\overset{\varphi}{\to}\; \bigoplus_{i=1}^{r} R(-(d-d_i)) \;\overset{\psi}{\to}\; R \;\to\; R/I_{\mathbb{X}} \;\to\; 0.$$

First, we prove that $\operatorname{Im}\varphi = \operatorname{Ker}\psi$. It is obvious that $\operatorname{Im}\varphi \subseteq \operatorname{Ker}\psi$. Conversely, suppose that

$$(a_1, \ldots, a_r) \in \operatorname{Ker}\psi, \text{ where } a_i \in R.$$

Since $a_1\delta_1 + \cdots + a_r\delta_r = 0$, we have that, for $i = 1, \ldots, r$,

$$F_i \mid (a_1\delta_1 + \cdots + a_{i-1}\delta_{i-1} + a_{i+1}\delta_{i+1} + \cdots + a_r\delta_r) = -a_i\delta_i, \quad \text{i.e.,} \quad F_i \mid a_i.$$

Let $a_i = b_iF_i$ for such i. Then we have

$$a_1\delta_1 + \cdots + a_r\delta_r = (b_1 + \cdots + b_r)F_1 \cdots F_r = 0, \quad \text{that is,} \quad b_1 + \cdots + b_r = 0,$$

and so

$$\begin{aligned}
(a_1, \ldots, a_r) &= (b_1F_1, b_2F_2, \ldots, b_{r-1}F_{r-1}, b_rF_r)\\
&= (b_1F_1, b_2F_2, \ldots, b_{r-1}F_{r-1}, -(b_1 + \cdots + b_{r-1})F_r)\\
&= b_1\sigma_{1,r} + b_2\sigma_{2,r} + \cdots + b_{r-1}\sigma_{r-1,r}\\
&= b_1(\sigma_{1,2} + \cdots + \sigma_{r-1,r}) + b_2(\sigma_{2,3} + \cdots + \sigma_{r-1,r}) + \cdots + b_{r-1}\sigma_{r-1,r}\\
&= b_1\sigma_{1,2} + (b_1 + b_2)\sigma_{2,3} + \cdots + (b_1 + \cdots + b_{r-1})\sigma_{r-1,r}\\
&= M[b_1, b_1 + b_2, \ldots, b_1 + \cdots + b_{r-1}]^T\\
&\in \operatorname{Im}\varphi,
\end{aligned}$$

as we wished.

Second, we show that the map φ is injective. If $\varphi(a_1, \ldots, a_{r-1}) = (0, \ldots, 0)$ in R^r, then we have

$$\begin{aligned}
&\varphi(a_1, \ldots, a_{r-1})\\
&= M[a_1, \ldots, a_{r-1}]^T\\
&= a_1\sigma_{1,2} + a_2\sigma_{2,3} + \cdots + a_{r-1}\sigma_{r-1,r}\\
&= a_1F_1\mathbf{e}_1 + (a_2 - a_1)F_2\mathbf{e}_2 + \cdots + (a_{r-2} - a_{r-1})F_{r-1}\mathbf{e}_{r-1} - a_rF_r\mathbf{e}_{r-1}\\
&= (0, \ldots, 0).
\end{aligned}$$

This implies that $a_1F_1 = (a_2 - a_1)F_2 = \cdots = (a_{r-2} - a_{r-1})F_{r-1} = -a_rF_r = 0$. Therefore

$$a_1 = \cdots = a_r = 0,$$

which completes the proof. □

Remark 2.2. Let $\mathbb{X}$ be a star-configuration in $\mathbb{P}^n$ defined by general forms $F_1, \ldots, F_r$ in $R = k[x_0, \ldots, x_n]$ of degrees $1 \le d_1 \le \cdots \le d_r$, respectively.

From Theorem 2.1 and Hilbert-Burch theorem (see [11]), we have that $I_{\mathbb{X}}$ is generated by maximal minors of the matrix

$$M := \begin{pmatrix} F_1 & 0 & 0 & \cdots & 0 & 0 \\ -F_2 & F_2 & 0 & \cdots & 0 & 0 \\ 0 & -F_3 & F_3 & \cdots & 0 & 0 \\ 0 & 0 & -F_4 & & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & -F_{r-1} & F_{r-1} \\ 0 & 0 & \cdots & \cdots & 0 & -F_r \end{pmatrix},$$

and $I_{\mathbb{X}}$ has depth exactly 2. Moreover, by Auslander-Buchsbaum formula (see [11] again), we see that

$$\operatorname{depth}(R/I_{\mathbb{X}}) = \operatorname{depth}(R) - \operatorname{pd}(R/I_{\mathbb{X}}) = (n+1) - 2 = n - 1.$$

Since $\mathbb{X}$ is a closed subscheme in $\mathbb{P}^n$ of codimension 2, we get that

$$\dim(R/I_{\mathbb{X}}) = \dim \mathbb{X} + 1 = n - 1 = \operatorname{depth}(R/I_{\mathbb{X}}).$$

This implies that $R/I_{\mathbb{X}}$ is a Cohen-Macaulay ring, i.e., $\mathbb{X}$ is an arithmetically Cohen-Macaulay subscheme in $\mathbb{P}^n$.

The following two corollaries are the special cases of Theorem 2.1 when $d_1 = \cdots = d_r = d$ and $d_1 = \cdots = d_r = 1$, respectively.

Corollary 2.3. *With notations as in Theorem 2.1 for $d_1 = \cdots = d_r = d$, the minimal free resolution of $R/I_{\mathbb{X}}$ is*

$$0 \to R^{r-1}(-dr) \to R^r(-d(r-1)) \to R \to R/I_{\mathbb{X}} \to 0.$$

Corollary 2.4. *Let $\mathbb{X}$ be a linear star-configuration in $\mathbb{P}^n$ of type r with $r \geq 3$. Then the minimal free resolution of $R/I_{\mathbb{X}}$ is*

$$0 \to R^{r-1}(-r) \to R^r(-(r-1)) \to R \to R/I_{\mathbb{X}} \to 0.$$

From Theorem 2.1, we can immediately find the Hilbert function of a star-configuration in $\mathbb{P}^n$ defined by general forms $F_1, \ldots, F_r$ in $R = k[x_0, \ldots, x_n]$, and thus we have the following corollary.

Corollary 2.5. *Let $\mathbb{X}$ be a star-configuration in $\mathbb{P}^n$ defined by general forms $F_1, \ldots, F_r$ in $R = k[x_0, \ldots, x_n]$ of degrees $1 \leq d_1 \leq \cdots \leq d_r$, respectively.*

Then the Hilbert function of $R/I_{\mathbb{X}}$ is

$$\mathbf{H}_{\mathbb{X}}(i) = \begin{cases} \binom{n+i}{n}, & 1 \le i < d-d_r, \\ \binom{n+i}{n} - \binom{n+i-(d-d_r)}{n}, & d-d_r \le i < d-d_{r-1}, \\ \binom{n+i}{n} - \binom{n+i-(d-d_r)}{n} - \binom{n+i-(d-d_{r-1})}{n}, & d-d_{r-1} \le i < d-d_{r-2}, \\ \vdots & \vdots \\ \binom{n+i}{n} - \sum_{j=2}^{r} \binom{n+i-(d-d_j)}{n}, & d-d_2 \le i < d-d_1, \\ \binom{n+i}{n} - \sum_{j=1}^{r} \binom{n+i-(d-d_j)}{n}, & d-d_1 \le i < d, \\ \binom{n+i}{n} - \sum_{j=1}^{r} \binom{n+i-(d-d_j)}{n} + (r-1)\binom{n+i-d}{n}, & i \ge d. \end{cases}$$

Proof. From the minimal free resolution of $R/I_{\mathbb{X}}$

$$0 \to R^{r-1}(-d) \to \bigoplus_{i=1}^{r} R(-(d-d_i)) \to R \to R/I_{\mathbb{X}} \to 0,$$

the Hilbert function of $\mathbb{X}$ is

$$\begin{aligned} \mathbf{H}_{\mathbb{X}}(i) &= \dim_k R_i - \left[\sum_{j=1}^{r} \dim_k R_{i-(d-d_j)}\right] + (r-1)\dim_k R_{i-d} \\ &= \binom{n+i}{n} - \left[\sum_{j=1}^{r} \binom{n+i-(d-d_j)}{n}\right] + (r-1)\binom{n+i-d}{n}, \end{aligned}$$

as needed. □

3. Some properties of Hilbert functions of star-configurations in $\mathbb{P}^2$

In this section, we introduce a few more interesting results on star-configurations in $\mathbb{P}^2$ when $d_1 = \cdots = d_r = d$.

Proposition 3.1. *Let $\mathbb{X}$ be a star-configuration in $\mathbb{P}^2$ defined by general forms $F_1, \ldots, F_r$ in $R = k[x_0, x_1, x_2]$ of degree d $(d = 1, 2)$ with $r \ge 3$. Then $R/I_{\mathbb{X}}$ has generic Hilbert function, i.e.,*

$$\mathbf{H}_{\mathbb{X}}(-) \;:\; 1 \;\binom{1+2}{2}\;\binom{2+2}{2}\;\cdots\;\binom{2+((r-1)d-1)}{2}\;\binom{2+(r-2)}{2}d^2 \;\to .$$

Proof. By Proposition 1.1, $I_{\mathbb{X}}$ is the ideal of a set of $\binom{r}{2} \times d^2$ points in $\mathbb{P}^2$, and $I_{\mathbb{X}}$ has only r generators in degree $d(r-1)$. Hence it suffices to show that, for $d = 1, 2$,

$$\mathbf{H}_{\mathbb{X}}(d(r-1)) = \deg(\mathbb{X}) = \binom{2+(r-2)}{2}d^2.$$

Case 1. If $d = 1$, then

$$\begin{aligned}\mathbf{H}_{\mathbb{X}}(r-1) &= \dim_k R_{r-1} - r\\ &= \binom{2+(r-1)}{2} - r\\ &= \binom{2+(r-2)}{2}\cdot 1^2\end{aligned}$$

(see also Lemma 7.8, [18]).

Case 2. If $d = 2$, then

$$\begin{aligned}\mathbf{H}_{\mathbb{X}}(2(r-1)) &= \dim_k R_{2(r-1)} - r\\ &= \binom{2+2(r-1)}{2} - r\\ &= \binom{2r}{2} - r\\ &= \binom{2+(r-2)}{2}\cdot 2^2.\end{aligned}$$

Therefore, *Cases 1* and *2* complete the proof. □

The following example, however, shows that Proposition 3.1 does not hold for $d = 3$ and $r = 3$.

Example 3.2. Let F_1, F_2, F_3 be general forms in $R = k[x_0, x_1, x_2]$ of degree 3 and let $I = (F_1F_2, F_1F_3, F_2F_3)$. By Corollary 2.5, the Hilbert function of R/I is

$$1 \quad 3 \quad 6 \quad 10 \quad 15 \quad 21 \quad 25 \quad 27 \quad \rightarrow .$$

In other words,

$$\mathbf{H}(R/I, 3(3-1)) = \mathbf{H}(R/I, 6) = 25 \neq 27 = \binom{3}{2}\cdot 3^2,$$

which does not satisfy Proposition 3.1.

The following proposition shows that the Hilbert function of a star-configuration defined by general forms of degree $d \geq 3$ can never be generic (see Remark 3.4).

Proposition 3.3. *Let $\mathbb{X}$ be a star-configuration in $\mathbb{P}^2$ defined by general forms $F_1, \ldots, F_r$ in $R = k[x_0, x_1, x_2]$ of degree $d \geq 3$ with $r \geq 3$. Then the Hilbert function of $R/I_{\mathbb{X}}$ is*

$$\mathbf{H}_{\mathbb{X}}(i) = \begin{cases} \binom{2+i}{2}, & 0 \leq i \leq d(r-1)-1,\\ \binom{2+i}{2} - \binom{2+(i-d(r-1))}{2}, & d(r-1) \leq i \leq d(r-1)+(d-2),\\ \deg(\mathbb{X}), & i \geq d(r-1)+(d-2).\end{cases}$$

Proof. By Corollary 2.3, the minimal free resolution of $R/I_{\mathbb{X}}$ is

$$0 \to R^{r-1}(-dr) \to R^r(-d(r-1)) \to R \to R/I_{\mathbb{X}} \to 0.$$

Hence the Hilbert function of $R/I_{\mathbb{X}}$ is

$$\begin{aligned}\mathbf{H}_{\mathbb{X}}(i) &= \dim_k R_i - r\cdot\dim_k R(-d(r-1))_i + (r-1)\cdot\dim_k R(-dr)_i \\ &= \binom{2+i}{2} - r\cdot\binom{2+i-d(r-1)}{2} + (r-1)\cdot\binom{2+i-dr}{2},\end{aligned}$$

and so, for $d(r-1)\le i\le d(r-1)+(d-2)$, the Hilbert function of $R/I_{\mathbb{X}}$ is now of the form

$$\mathbf{H}_{\mathbb{X}}(i) = \begin{cases}\binom{2+i}{2}, & 0\le i\le d(r-1)-1,\\ \binom{2+i}{2} - r\cdot\binom{2+(i-d(r-1))}{2}, & d(r-1)\le i\le d(r-1)+(d-2).\end{cases}$$

In particular, the Hilbert function of $R/I_{\mathbb{X}}$ in degree $d(r-1)+(d-2)$ is

$$\begin{aligned}\mathbf{H}_{\mathbb{X}}(d(r-1)+(d-2)) &= \binom{2+(d(r-1)+(d-2))}{2} - \binom{2+(d-2)}{2}\cdot r \\ &= \binom{dr}{2} - \binom{d}{2}\cdot r \\ &= \frac{dr(dr-1)}{2} - \frac{d(d-1)}{2}\cdot r \\ &= \binom{r}{2}\cdot d^2 \\ &= \deg(\mathbb{X}),\end{aligned}$$

and therefore the Hilbert function of $R/I_{\mathbb{X}}$ is

$$\mathbf{H}_{\mathbb{X}}(i) = \begin{cases}\binom{2+i}{2}, & 0\le i\le d(r-1)-1,\\ \binom{2+i}{2} - r\cdot\binom{2+(i-d(r-1))}{2}, & d(r-1)\le i\le d(r-1)+(d-2),\\ \deg(\mathbb{X}), & i\ge d(r-1)+(d-2),\end{cases}$$

which completes the proof. □

Here is an alternative proof of Proposition 3.3 counting the number of minimal generators of $I_{\mathbb{X}}$ in each degree. Moreover, the following alternative proof shows what the polynomial basis for $(I_{\mathbb{X}})_d$ is precisely in each degree d.

Alternative Proof of Proposition 3.3. Since the ideal $I_{\mathbb{X}}$ has only r generators in degree $d(r-1)$, we have

$$\mathbf{H}_{\mathbb{X}}(d(r-1)) = \binom{2+d(r-1)}{2} - r.$$

Now consider the set of r forms in $I_{\mathbb{X}}$ of degree $d(r-1)$

$$\bigcup_{i=1}^{r}\{F_1\cdots\hat{F}_i\cdots F_r\}.$$

Then the set $S := \bigcup_{i=1}^{r}\{x_0F_1\cdots\hat{F}_i\cdots F_r, x_1F_1\cdots\hat{F}_i\cdots F_r, x_2F_1\cdots\hat{F}_i\cdots F_r\}$ of $3r$ forms of degree $d(r-1)+1$ in $I_{\mathbb{X}}$ is linearly independent. Assume that $x_0F_1\cdots\hat{F}_i\cdots F_r$ is a linear combination of the rest of $3r-1$ forms in S as follows:

$$\begin{aligned}
&x_0F_1\cdots\hat{F}_i\cdots F_r\\
&=(\alpha_{1,0}x_0+\alpha_{1,1}x_1+\alpha_{1,2}x_2)\hat{F}_1F_2\cdots F_r+\cdots\\
&\quad+(\alpha_{i-1,0}x_0+\alpha_{i-1,1}x_1+\alpha_{i-1,2}x_2)F_1\cdots\hat{F}_{i-1}\cdots F_r\\
&\quad+(\alpha_{i,1}x_1+\alpha_{i,2}x_2)F_1\cdots\hat{F}_i\cdots F_r\\
&\quad+(\alpha_{i+1,0}x_0+\alpha_{i+1,1}x_1+\alpha_{i+1,2}x_2)F_1\cdots\hat{F}_{i+1}\cdots F_r+\cdots\\
&\quad+(\alpha_{r,0}x_0+\alpha_{r,1}x_1+\alpha_{r,2}x_2)F_1\cdots F_{r-1}\hat{F}_r,
\end{aligned}$$

where $\alpha_{i,j}\in k$. Then,

$$F_i \mid (x_0-\alpha_{i,1}x_1-\alpha_{i,2}x_2)F_1\cdots\hat{F}_i\cdots F_r,\quad \text{i.e.,}\quad F_i \mid (x_0-\alpha_{i,1}x_1-\alpha_{i,2}x_2),$$

which is impossible since $\deg F_i = d > 1$. Similarly, we see that $x_1F_1\cdots\hat{F}_i\cdots F_r$ and $x_2F_1\cdots\hat{F}_i\cdots F_r$ cannot be a linear combination of the rest of $3r-1$ forms in S for every $1\le i\le r$. Hence the Hilbert function of $R/I_{\mathbb{X}}$ has the form

$$1\ \binom{2+1}{2}\ \cdots\ \binom{2+(d(r-1)-1)}{2}\ \left[\binom{2+d(r-1)}{2}-r\right]\ \left[\binom{2+(d(r-1)+1)}{2}-3r\right]\ \cdots.$$

By continuing the same process to the degree up to $d(r-1)+(d-2)$, the Hilbert function of $R/I_{\mathbb{X}}$ is now of the form

$$\mathbf{H}_{\mathbb{X}}(i)=\begin{cases}\binom{2+i}{2}, & 0\le i\le d(r-1)-1,\\ \binom{2+i}{2}-r\cdot\binom{2+(i-d(r-1))}{2}, & d(r-1)\le i\le d(r-1)+(d-2).\end{cases}$$

The rest of this proof is the same as the previous proof, so we omit it here. □

Remark 3.4. Let $\mathbb{X}$ be as in Proposition 3.3. Since $d\ge 3$, we see that $d(r-1) < d(r-1)+(d-2)$. Moreover, by Proposition 3.3,

$$\mathbf{H}_{\mathbb{X}}(d(r-1))=\binom{2+d(r-1)}{2}-\binom{2}{2}<\binom{2+d(r-1)}{2}\qquad\text{and}$$

$$\deg(\mathbb{X})=\mathbf{H}_{\mathbb{X}}(d(r-1)+(d-2))$$

$$= \binom{2+d(r-1)+(d-2)}{2} - \binom{2+(d(r-1)+(d-2)-d(r-1)}{2}$$
$$= \binom{2+d(r-1)+(d-2)}{2} - \binom{d}{2}.$$

Hence

$$\begin{aligned}
&\deg(\mathbb{X}) - \mathbf{H}_{\mathbb{X}}(d(r-1)) \\
= &\left[\binom{2+d(r-1)+(d-2)}{2} - \binom{d}{2}\right] - \left[\binom{2+d(r-1)}{2} - \binom{2}{2}\right] \\
= &\left[\binom{2+d(r-1)+(d-2)}{2} - \binom{d}{2}\right] - \left[\binom{2+d(r-1)}{2} - \binom{2}{2}\right] \\
= &\left[\binom{2+d(r-1)+(d-2)}{2} - \binom{2+d(r-1)}{2}\right] - \left[\binom{d}{2} - \binom{2}{2}\right] \\
= &\frac{(d-2)(2d(r-1)+d+1)}{2} - \left[\binom{d}{2} - \binom{2}{2}\right] \\
\geq &\frac{(d-2)(5d+1)}{2} - \frac{d^2-d}{2} + 1 \quad (\text{since } r \geq 3) \\
= &\ 2d(d-2) \\
> &\ 0 \quad (\text{since } d \geq 3).
\end{aligned}$$

This implies that

$$\mathbf{H}_{\mathbb{X}}(d(r-1)) < \min\left\{\binom{2+d(r-1)}{2}, \deg(\mathbb{X})\right\}.$$

Therefore, $\mathbb{X}$ does not have generic Hilbert function, as we wished.

4. The weak Lefschetz property

We start with a proposition on the weak Lefschetz property from [12] and provide an answer to Question 1.3 for $d = 1$.

Let $\mathbb{X}$ be a finite set of points in $\mathbb{P}^n$ and define

$$\sigma(\mathbb{X}) = \min\{\, i \mid \mathbf{H}_{\mathbb{X}}(i-1) = \mathbf{H}_{\mathbb{X}}(i)\}.$$

Proposition 4.1 (Proposition 5.15, [12])**.** *Let $\mathbb{X}$ be a finite set of points in $\mathbb{P}^n$ and let A be an Artinian quotient of the coordinate ring of $\mathbb{X}$. Assume that $\mathbf{H}_A(i) = \mathbf{H}_{\mathbb{X}}(i)$ for all $0 \leq i \leq \sigma(\mathbb{X}) - 1$. Then A has the weak Lefschetz property.*

Theorem 4.2. *Let $\mathbb{X}$ and $\mathbb{Y}$ be linear star configurations in $\mathbb{P}^2$ of types t and s with $s \geq t \geq 3$, respectively. Then $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ is an Artinian ring with the weak Lefschetz property.*

Proof. By Proposition 3.1, $R/I_{\mathbb{X}}$ and $R/I_{\mathbb{Y}}$ have generic Hilbert functions. Since $\sigma(\mathbb{X}) \leq \sigma(\mathbb{Y}) \leq \sigma(\mathbb{X} \cup \mathbb{Y})$, by Proposition 3.1 again, we have that

$$\mathbf{H}_{\mathbb{X}}(i) = \mathbf{H}_{\mathbb{Y}}(i) = \mathbf{H}_{\mathbb{X}\cup\mathbb{Y}}(i) = \binom{i+2}{2}$$

for $0 \leq i \leq \sigma(\mathbb{X}) - 1$. Using the following exact sequence

$$0 \quad \to \quad R/I_{\mathbb{X}\cup\mathbb{Y}} \quad \to \quad R/I_{\mathbb{X}} \bigoplus R/I_{\mathbb{Y}} \quad \to \quad R/(I_{\mathbb{X}} + I_{\mathbb{Y}}) \quad \to \quad 0,$$

we obtain that

$$\begin{aligned} \mathbf{H}_{R/(I_{\mathbb{X}}+I_{\mathbb{Y}})}(i) &= \mathbf{H}_{\mathbb{X}}(i) + \mathbf{H}_{\mathbb{Y}}(i) - \mathbf{H}_{\mathbb{X}\cup\mathbb{Y}}(i) \\ &= \mathbf{H}_{\mathbb{X}}(i) \end{aligned}$$

for $0 \leq i \leq \sigma(\mathbb{X}) - 1$. Furthermore, since

$$R/(I_{\mathbb{X}} + I_{\mathbb{Y}}) \simeq (R/I_{\mathbb{X}})/((I_{\mathbb{X}} + I_{\mathbb{Y}})/I_{\mathbb{X}})$$

is an Artinian quotient of the coordinate ring $R/I_{\mathbb{X}}$, by Proposition 4.1 $R/(I_{\mathbb{X}} + I_{\mathbb{Y}})$ has the weak Lefschetz property, as we wished. □

Remark 4.3 (CoCoA, [26]). If $\mathbb{X}$ and $\mathbb{Y}$ are not linear star-configurations in Proposition 3.1, then Theorem 4.2 may not hold in general. For example, assume that $\mathbb{X}$ and $\mathbb{Y}$ are star-configurations defined by general forms of degree 2 of the same type 4. Then, by Proposition 3.1 (see also Corollary 2.5), the Hilbert functions of $R/I_{\mathbb{X}}$ and $R/I_{\mathbb{Y}}$ are

$$1 \quad 3 \quad 6 \quad 10 \quad 15 \quad 21 \quad 24 \quad \to,$$

and thus

$$\sigma(\mathbb{X}) = \sigma(\mathbb{Y}) = 7.$$

Furthermore, the Hilbert function of $R/I_{\mathbb{X}\cup\mathbb{Y}}$, obtained by CoCoA, is

$$1 \quad 3 \quad 6 \quad 10 \quad 15 \quad 21 \quad 28 \quad 36 \quad 45 \quad 48 \quad \to,$$

and thus

$$\begin{aligned} \mathbf{H}(R/I_{\mathbb{X}} + I_{\mathbb{Y}}, 6) &= \mathbf{H}(R/I_{\mathbb{X}}, 6) + \mathbf{H}(R/I_{\mathbb{Y}}, 6) - \mathbf{H}(R/I_{\mathbb{X}\cup\mathbb{Y}}, 6) \\ &= 24 + 24 - 28 \\ &= 20 \\ &\neq \mathbf{H}(R/I_{\mathbb{X}}, 6). \end{aligned}$$

This does not satisfy the conditions in Proposition 4.1, and thus we do not know if Theorem 4.2 still holds for this case when $\mathbb{X}$ and $\mathbb{Y}$ are star-configurations in $\mathbb{P}^2$ defined by general forms of degree d with $d = 2$.

5. Comments

Theorem 2.1 gives the complete answer to Question 1.2 and Theorem 4.2 gives an answer to Question 1.3 for $d = 1$. In other words, Question 1.3 for $d > 1$ is still open. Thus, we restate Question 1.3 as follows.

Question 5.1 (Restated Question 1.3)**.** Let $\mathbb{X}$ and $\mathbb{Y}$ be star-configurations in $\mathbb{P}^2$ defined by general forms of degree $d > 1$. Does the Artinian ring $R/(I_{\mathbb{X}}+I_{\mathbb{Y}})$ have the weak Lefschetz property?

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