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SOME REMARKS ON VECTOR-VALUED TREE MARTINGALES

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Abstract. Our first aim of this paper is to define maximal operators of a-quadratic variation and of a-conditional quadratic variation for vector-valued tree martingales and to show that these maximal operators and maximal operators of vector-valued tree martingale transforms are all sublinear operators. The second purpose is to prove that maximal operator inequalities of a-quadratic variation and of a-conditional quadratic variation for vector-valued tree martingales hold provided $2 \leq a < \infty$ by means of Marcinkiewicz interpolation theorem. Based on a result of reference [10] and using Marcinkiewicz interpolation theorem, we also propose a simple proof of maximal operator inequalities for vector-valued tree martingale transforms, under which the vector-valued space is a UMD space.

1. Introduction

Tree martingales have been studied by Schipp, Fridli, Weisz, and others but there are still many open problems about them. In 1980s, tree martingale transforms are firstly introduced by Schipp and Fridli [6, 15], Weisz [17], and with the help of predictability and of previsibility and of regularity for tree martingales, maximal inequalities of tree martingale transforms are proved by them, provided tree martingales are previsible or regular, as well as, Burkholder-Davis-Gundy's inequality of predictable tree martingales are proved by them, provided $2 < p < \infty$. Furthermore, the latter has been extended to all $1 < p < \infty$ for a regular tree stochastic basis. As an important application of tree martingales, by formulating an atomic decomposition of the Hardy spaces consisting of tree martingale difference sequences, one of the deepest result in Fourier analysis is proved by Schipp and Weisz [16], it is an expansion of Carleson's convergence theorem which is obtained by Gosselin [8], Young, Wo-Sang [19]. This expansion of Carleson's convergence theorem

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indicates that the one-parameter Walsh-Fourier series and its generalization, the so-called Vilenkin-Fourier series [18, 7] of a function in $L^p(1 < p < \infty)$, converge almost everywhere to the function itself. In addition to this, the foregoing convergence in L^p norm is verified as well. By using the properties of Banach spaces, some vector-valued predictable tree martingale inequalities are investigated by He and Hou [12]. He and Shen [10] proved that if X is a UMD space and tree martingales are regular or previsible, then maximal operators of X-valued predictable tree martingale transforms are norm-bounded in $L^p(X)$. It should be strongly emphasized that the predictability of tree martingales plays an important role in published literature [6, 15, 17, 10, 12]. Lately, with the help of mild conditions, He and Shen [11] proved that tree martingales are isomorphic to a class of special Cairoli-Walsh martingales, and these results show that the family of σ-filtrations of tree martingales can be constructed as σ-filtrations of Cairoli-Walsh martingales [4, 13].

When we tackle tree martingale problems, however, there are two sources of difficulties that need to be overcome and they are both related to the fact that the tree $\mathbf{T}$–cannot be well ordered in a useful way. The first problem is that there is no sensible way to uniquely define tree martingale's stopping times, because it is not clear–and generally not true–that there is a uniquely minimum element $t \in \mathbf{T}$. The second source of difficulty is that tree martingale transforms also cannot be defined as one-parameter martingale transforms.

This paper is concerned with tree martingales with values in uniformly convex spaces and UMD spaces, and their operator inequalities.

2. Preliminaries and definitions

Let $\mathbf{T}$ be a countable, upward-directed index set with respect to the partial ordering $\preceq$ satisfying the following two conditions:

(1) the set $\mathbf{T}^t := \{s \in \mathbf{T} : s \preceq t\}$ is finite for every $t \in \mathbf{T}$;
(2) the set $\mathbf{T}_t := \{s \in \mathbf{T} : t \preceq s\}$ is linearly ordered for every $t \in \mathbf{T}$.

Then $\mathbf{T}$ is a tree and every non-empty subset of $\mathbf{T}$ has at least one minimum. The succeeding element of $t \in \mathbf{T}$, namely, the minimum element of the set $\mathbf{T}_t - \{t\}$ is denoted by t^+.

A tree $\mathbf{T}$ is a partially ordered set relative to the partially ordering $\preceq$. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and $\mathcal{F} = (\mathcal{F}_t, t \in \mathbf{T})$ be a family of non-decreasing sub-σ-algebras of $\mathcal{F}$ relative to the partial ordering $\preceq$ such that $\mathcal{F} = \sigma(\bigcup_{t\in\mathbf{T}} \mathcal{F}_t)$. Obviously, $(\mathcal{F}_s, s \in \mathbf{T}_t)$ can be ordered linearly for every $t \in \mathbf{T}$. This common σ-algebra is denoted by $\mathcal{F}_t^- = \{\emptyset\}$. Throughout this paper, let X be a Banach space which has the Radon-Nikodym property and $\|\cdot\|_X$ be the norm of a Banach space X. Let E_t be the conditional expectation operators related to σ-filtration $\mathcal{F}_t$. The indicator function of a set $\mathbf{A}$ is denoted by $\chi_{\mathbf{A}}$ and σ-finite measure is denoted by μ. Let $\|\cdot\|_{L^p(X)}$ be the $L^p(X)$-norm, $L^p_*(X)$ be the weak $L^p(X)$-space with norm $\|\cdot\|_{L^p_*(X)}$, and $L^1(X)$ be the space of all Bochner integrable and measurable functions.

In the tree case, since conditional expectation operators are also projections, for a more general result, conditional expectation operators will be replaced by projections. Let $(\phi_t, t \in \mathbf{T})$ be a family of scalar complex-value measurable functions related to a family of sub-σ-filtrations $(\mathcal{F}_t, t \in \mathbf{T})$ with $|\phi_t| = 1$. $(P_t, t \in \mathbf{T})$ is a family of projections for which

$$P_t f = \phi_s E_t(f \bar{\phi_s}), \quad f \in L^1(X) \tag{2.1}$$

for all $s \prec t (s, t \in \mathbf{T})$. Then $(P_t, t \in \mathbf{T})$ is said to be a family of contractive projections with respect to $L^1(X)$ [1, 5]. Operators of this type are said to be *universal contractive projections* [15], that is, if $s \preceq t$, then $P_t P_s = P_s P_t = P_s$ for any comparable $s, t \in \mathbf{T}$.

With the help of universal contractive projections, X-valued *tree martingales*, *predictable* or *previsible tree martingales*, and *regular tree martingales* have been defined in [10], the readers are advised to refer to [17] for further details.

To study some inequalities of X-valued tree martingales, a quasi-norm $\|\cdot\|_{\mathbf{M}^{pq}}$ is going to be introduced. Let $f = (f_t, t \in \mathbf{T})$ be a family of $\mathcal{F}$-measurable functions defined on the complete probability space $(\Omega, \mathcal{F}, P)$. To any $y \geq 0$, put

$$\nu_y^f = \inf\{t \in \mathbf{T} : \|f_t\|_X > y\}.$$

Then it is easy to see that

$$\{t \in \nu_y^f\} = \{\omega \in \Omega : \|f_t(\omega)\|_X > y, \|f_s(\omega)\|_X \leq y, \ \forall s \prec t\},$$

where $s \prec t$ means that $s \preceq t$ but $s \neq t$. Now, the quasi-norm $\|\cdot\|_{M^{pq}}$ can be defined by ν_y^f. Let $0 < p, q < \infty$. Put

$$\|f\|_{\mathbf{M}^{pq}} = \sup_{y>0} y\Big(\int_\Omega \big(\sum_{t \in \mathbf{T}} \chi_{\{t \in \nu_y^f\}}\big)^{\frac{p}{q}}\Big)^{\frac{1}{p}}, \tag{2.2}$$

$$\mathbf{M}^{pq} = \{f = (f_t, t \in \mathbf{T}) : \|f\|_{\mathbf{M}^{pq}} < \infty\},$$

where $\chi_{\{t \in \nu_y^f\}}$ is the indicator function of the set $\{t \in \nu_y^f\}$. Note that the map $\|\cdot\|_{\mathbf{M}^{pq}}$ is a quasi-norm and is non-decreasing in the following sense: if the inequality $\|f_t\|_X \leq \|g_t\|_X$ holds for all $t \in \mathbf{T}$, then

$$\|f\|_{\mathbf{M}^{pq}} \leq \|g\|_{\mathbf{M}^{pq}} \quad (0 < p < \infty, \ 0 < q \leq \infty).$$

Remark 2.1. The quasi-norm $\|f\|_{\mathbf{M}^{pq}}$ is decreasing with respect to q and is increasing with respect to p for each fixed family of $f = (f_t, t \in \mathbf{T})$. Therefore, the limit does exists as $q \to \infty$ and satisfies

$$\|f\|_{\mathbf{M}^{p\infty}} = \lim_{q \to \infty} \|f\|_{\mathbf{M}^{pq}} = \sup_{y>0} y[\mu(f^* > y)]^{\frac{1}{p}}, \quad 0 < p < \infty. \tag{2.3}$$

The space of all X-valued predictable tree martingales is denoted by $\mathbf{P}^{pq}$ and endows it with the following quasi-norm:

$$\|f\|_{\mathbf{P}^{pq}} = \inf \|\lambda\|_{\mathbf{M}^{pq}} \quad (0 < p < \infty, 0 < q \leq \infty),$$

where the infimum is taken over all predictions $\lambda \in \mathbf{M}^{pq}$ belonging to f. Let $0 < p < \infty$. The space of all X-valued measurable functions f satisfying

$$\|f\|_{L^p_*(X)} = \sup_{y>0} y[\mu(\|f\|_X > y)]^{\frac{1}{p}} < \infty \tag{2.4}$$

is called a weak $L^p(X)$-space, and is denoted by $L^p_*(X)$. It is well-known that $\|\cdot\|_{L^p_*(X)}$ is a quasi-norm on $L^p_*(X)$ and $L^p(X) \subset L^p_*(X)$. For reader's convenience, throughout this paper, we always let

$$\|(S_t^{(a)}(f), t \in \mathbf{T})\|_{\mathbf{M}^{pq}} = \|S^{(a)}(f)\|_{\mathbf{M}^{pq}},$$
$$\|(s_t^{(a)}(f), t \in \mathbf{T})\|_{\mathbf{M}^{pq}} = \|s^{(a)}(f)\|_{\mathbf{M}^{pq}}.$$

It follows from (2.3) and (2.4) that

$$\|f^*\|_{L^p_*(X)} = \|f\|_{\mathbf{M}^{p\infty}} = \|(f_t, t \in \mathbf{T})\|_{\mathbf{M}^{p\infty}}, \tag{2.5}$$

where f^* is the maximal function of the family of functions $f = (f_t, t \in \mathbf{T})$.

In what follows, Marcinkiewicz interpolation theorem will be introduced in Lemma 2.3.

Definition 2.2 ([2])**.** Let X, Y be Banach spaces. An operator T is said to be a sublinear operator from X to Y if

$$\|T(f+g)\|_Y \le \|T(f)\|_Y + \|T(g)\|_Y \tag{2.6}$$

for all X-valued functions f, g.

Lemma 2.3 ([2])**.** *Assume that $1 \le p_0 < p_1 \le \infty$ and that T is a sublinear operator which satisfies*

$$\|T(f)\|_{L^{p_i}_*(X)} = \sup_{y>0} \mu(T(f) > y)^{\frac{1}{p_i}} \le C_i \|f\|_{L^{p_i}(X)} \tag{2.7}$$

for all f and $i = 0, 1$. Let $0 < \theta < 1$ and define p_θ by

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

Then T is a sublinear operator which satisfies

$$\|T(f)\|_{L^{p(\theta)}(X)} \le C_\theta \|f\|_{L^{p_\theta}(X)}, \quad f \in L^{p_0}(X) \cap L^{p_1}(X), \tag{2.8}$$

where $C_\theta \le k C_0^{1-\theta} C_1^\theta$ and $k = 2(\frac{p_\theta}{p_\theta - p_0} + \frac{p_\theta}{p_1 - p_\theta})$.

3. Some maximal operators of vector-valued tree martingales

By the definition of tree $\mathbf{T}$, it is clear that $\mathbf{T}_t$ is linearly ordered for any $t \in \mathbf{T}$, in other words, $\mathbf{T}_t$ is a totally ordered set. Then we can define the difference, a-quadratic variation and a-conditional quadratic variation of an one-parameter X-valued martingale $f = (f_t, t \in \mathbf{T}_t)$ for every $t \in \mathbf{T}$ as [14], respectively,

$$d_r f = f_{r^+} - f_r \quad (r \in \mathbf{T}_t);$$

$$S_t^{(a)}(f) = \Big(\sum_{r\in\mathbf{T}_t} \|d_r f\|_X^a\Big)^{\frac{1}{a}}, \quad s_t^{(a)}(f) = \Big(\sum_{r\in\mathbf{T}_t} E_r\|d_r f\|_X^a\Big)^{\frac{1}{a}}.$$

Moreover, their maximal operators can be defined by:

$$S^{(a)}(f) = \sup_{t\in\mathbf{T}} S_t^{(a)}(f), \quad s^{(a)}(f) = \sup_{t\in\mathbf{T}} s_t^{(a)}(f).$$

As mentioned earlier, an X-valued tree martingale transforms can not be defined as a one-parameter martingale for some martingale unless it is already a one-parameter martingale. Similarly, stopping times can not be introduced. Here we define an X-valued tree martingale transform that is more general than the transform of a one-parameter martingale. Firstly, a linear operator shall be introduced.

Definition 3.1. $\pi = (\pi^t, t \in \mathbf{T})$ is a family of linear operators such that for all $f \in L_1(X)$ and $t \in \mathbf{T}$:

(1) $P_{t^+}(\pi^t f) = \pi^t f$;
(2) $P_t(\pi^t f) = 0$;
(3) for every $\mathcal{F}_t$-measurable function ξ one has $\pi^t(\xi f) = \xi\pi^t f$;
(4) $\|\pi^t f\|_X \le RE_t^- \|f\|_X$, where the constant R is independent of t and f.

Next, by Definition 3.1, X-valued tree martingale transforms can be defined.

Definition 3.2. Suppose $f = (f_t, t \in \mathbf{T})$ is an X-valued tree martingale. Then these X-valued tree martingale transforms are defined by

$$\pi_s f = \sum_{t\preceq r\prec s} \pi^r(d_r f) \quad (t \in \mathbf{T}, s \in \mathbf{T}_t), \tag{3.1}$$

and the maximal operators of these X-valued tree martingale transforms are defined by

$$\pi_t^* f = \sup_{s\in\mathbf{T}_t} \|\pi_s f\|_X, \quad \pi f = \sup_{t\in\mathbf{T}} \pi_t^* f. \tag{3.2}$$

Now, it can be identified that tree martingale operators $S^{(a)}(f), s^{(a)}(f)$, πf are sublinear operators [3].

Lemma 3.3. *If $a \ge 1$, then these operators $S^{(a)}(f), s^{(a)}(f)$, πf are all sublinear ones.*

Proof. To every X-valued tree martingale $f = (f_t, t \in \mathbf{T})$, make a decomposition by

$$f = \sum_{i=0}^{1} f^i, \quad f^i = (f_t^i, t \in \mathbf{T}) \ \ (i = 0, 1).$$

Put

$$d^{(t)} f = (d_s f, s \in \mathbf{T}_t), \ \ d^{(t)} f^i = (d_s f^i, s \in \mathbf{T}_t) \ \ (i = 0, 1)$$

for every $t \in \mathbf{T}$. Then $d^{(t)}f = \sum_{i=0}^{1} d^{(t)}f^i$. Since $a \geq 1$,

$$\begin{aligned} S_t^{(a)}(f) &= \Big(\sum_{s\in\mathbf{T}_t} \|d_s f\|_X^2\Big)^{\frac{1}{a}} = \|d^{(t)}f\|_{l^a(X)} \\ &\leq \|d^{(t)}f^0\|_{l^a(X)} + \|d^{(t)}f^1\|_{l^a(X)} = S_t^{(a)}(f^0) + S_t^{(a)}(f^1). \end{aligned} \tag{3.3}$$

It follows from (3.3) that

$$\|S^{(a)}(f)\|_X \leq \|S^{(a)}(f^0)\|_X + \|S^{(a)}(f^1)\|_X.$$

Likewise,

$$\|s^{(a)}(f)\|_X \leq \|s^{(a)}(f^0)\|_X + \|s^{(a)}(f^1)\|_X.$$

Therefore, $S^{(a)}(f), s^{(a)}(f)$ are all sublinear operators. Now, let us identify that πf is also a sublinear operator. Since the cardinality of the set $\{r | t \preceq r \prec s\}$ is finite for every $t \in \mathbf{T}$. Then from Definition 3.1 and Definition 3.2, it is clear that the operator $\pi_s f$ is a linear operator, and

$$\begin{aligned} \pi_s f &= \sum_{t\preceq r\prec s} \pi^r(d_r f^0) + \sum_{t\preceq r\prec s} \pi^r(d_r f^1) \\ &= \pi_s f^0 + \pi_s f^1. \end{aligned} \tag{3.4}$$

Furthermore, from (3.4) we can derive that

$$\pi f \leq \pi f^0 + \pi f^1, \tag{3.5}$$

which implies that

$$\|\pi f\|_X \leq \|\pi f^0\|_X + \|\pi f^1\|_X.$$

This shows that the maximal operator π is a sublinear one. □

4. Tree martingales with valued in uniformly convex spaces and UMD spaces

Based on Pisier's [14] work, He and Hou [12] extended F. Schipp [15], F. Weisz's [17] work to vector-valued predictable tree martingales.

Lemma 4.1 ([12])**.** *Assume that* $1 \leq p$, $\max(a, p) \leq q < \infty$ *and that* X *is isomorphic to an* a*-uniformly convex space* $(2 \leq a < \infty)$, *and that* X*-valued predictable tree martingales* $f = (f_t, t \in \mathbf{T})$ *are predictable. Then there exists a constant* C_{pq} *depending only on* p *and* q *such that*

$$\Big\|\Big(S_t^{(a)}(f), t \in \mathbf{T}\Big)\Big\|_{\mathbf{M}^{p\infty}} \leq C_{pq}\Big\|f\Big\|_{\mathbf{P}^{pq}}, \tag{4.1}$$

$$\Big\|\Big(s_t^{(a)}(f), t \in \mathbf{T}\Big)\Big\|_{\mathbf{M}^{p\infty}} \leq C_{pq}\Big\|f\Big\|_{\mathbf{P}^{pq}}. \tag{4.2}$$

Lemma 4.2. *Let* $0 \leq \lambda \leq 1,\ 1 < s \leq \infty$ *and define* p, q *by*

$$\frac{1}{p} = \frac{1-\lambda}{2} + \frac{\lambda}{s}, \quad \frac{1}{q} = \frac{1-\lambda}{2}.$$

Suppose $f = (f_t, t \in \mathbf{T}) \in L^p(X)$ *are* X*-valued tree martingales. Then*

(i) *if* $f = (f_t, t \in \mathbf{T})$ *is previsible, then there exists a constant* C_{pq} *depending only on* p *and* q *such that*

$$\Big\| f \Big\|_{\mathbf{P}^{pq}} \le C_{pq} \Big\| f \Big\|_{L^p(X)}. \tag{4.3}$$

(ii) *if* $f = (f_t, t \in \mathbf{T})$ *is regular, then the inequality* (4.3) *also holds.*

Proof. The proof of Lemma 4.2 is the same as Corollary 4.10 in F. Weisz's book [17]. □

The next result comes immediately from Lemma 4.1 and Lemma 4.2.

Theorem 4.3. *Assume that the assumptions of Lemma 4.1 and Lemma 4.2 are valid, and that* X*-valued tree martingales* $f = (f_t, t \in \mathbf{T}) \in L^p(X)$*. Then*

(i) *if* $f = (f_t, t \in \mathbf{T})$ *is previsible, then there exists a constant* C_p *depending only on* p *such that*

$$\Big\| \Big(S_t^{(a)}(f), t \in \mathbf{T} \Big) \Big\|_{M^{p\infty}} \le C_p \Big\| f \Big\|_p \tag{4.4}$$

and

$$\Big\| \Big(s_t^{(a)}(f), t \in \mathbf{T} \Big) \Big\|_{M^{p\infty}} \le C_p \Big\| f \Big\|_p. \tag{4.5}$$

(ii) *if* $f = (f_t, t \in \mathbf{T})$ *is regular, then the inequalities* (4.4) *and* (4.5) *also hold.*

Combining Marcinkiewicz interpolation theorem with Theorem 4.3, it can be obtained that the maximal operator inequalities of the a-quadratic variation and of a-conditional quadratic variation for X-valued tree martingales hold, provided $a \ge 2$.

Theorem 4.4. *Assume that the assumptions of Lemma 4.1 and Lemma 4.2 are valid, and that* X*-valued tree martingales* $f = (f_t, t \in \mathbf{T}) \in L^p(X)$*. Then*

(i) *if* $f = (f_t, t \in \mathbf{T})$ *is previsible, then there exists a constants* C_θ *such that*

$$\Big\| \Big(S_t^{(a)}(f), t \in \mathbf{T} \Big) \Big\|_{L^{p_\theta}(X)} \le C_\theta \Big\| f \Big\|_{L^{p_\theta}(X)}, \tag{4.6}$$

$$\Big\| \Big(s_t^{(a)}(f), t \in \mathbf{T} \Big) \Big\|_{L^{p_\theta}(X)} \le C_\theta \Big\| f \Big\|_{L^{p_\theta}(X)}, \tag{4.7}$$

where $\frac{1}{p_\theta} = \frac{1-\theta}{p_{\lambda_0}} + \frac{\theta}{p_{\lambda_1}}$, $C_\theta \le k C_0^{1-\theta} C_1^\theta$, $\theta \in (0,1)$, *and* k, λ_0, λ_1 *are satisfied the following conditions*

$$k = 2\left(\frac{p_\theta}{p_\theta - p_{\lambda_0}} + \frac{p_\theta}{p_{\lambda_1} - p_\theta} \right), \quad \lambda_0, \lambda_1 \in [0,1] \quad \text{and} \quad \lambda_0 < \lambda_1. \tag{4.8}$$

(ii) *if* $f = (f_t, t \in \mathbf{T})$ *is regular, then the inequalities* (4.6) *and* (4.7) *also hold.*

Proof. If $2 > s \geq 1$, choose $\lambda_0, \lambda_1 \in [0,1]$ such that $\lambda_0 < \lambda_1$,

$$\frac{1}{p_{\lambda_0}} = \frac{\lambda_0}{2} + \frac{1-\lambda_0}{s}, \quad \text{and} \quad \frac{1}{p_{\lambda_1}} = \frac{\lambda_1}{2} + \frac{1-\lambda_1}{s}. \tag{4.9}$$

Then from (4.9) we can derive that $p_{\lambda_0} < p_{\lambda_1}$ (Note that if $s \geq 2$, then choose $\lambda_0, \lambda_1 \in [0,1]$ such that $\lambda_1 < \lambda_0$, $p_{\lambda_0} < p_{\lambda_1}$). Using Theorem 4.3, one obtains that

$$\left\| S^{(a)}(f) \right\|_{L_*^{p_{\lambda_0}}(X)} \leq C_{p_{\lambda_0}} \left\| f \right\|_{L^{p_{\lambda_0}}(X)}, \tag{4.10}$$

$$\left\| S^{(a)}(f) \right\|_{L_*^{p_{\lambda_1}}(X)} \leq C_{p_{\lambda_1}} \left\| f \right\|_{L^{p_{\lambda_1}}(X)}. \tag{4.11}$$

From Lemma 3.3, we know that $S^{(a)}(f)$ is a sublinear operator. Applying Lemma 2.3 (Marcinkiewicz interpolation theorem) to (4.10), (4.11), one can derive that

$$\|S^{(a)}(f)\|_{L^{p_\theta}(X)} \leq C_\theta \|f\|_{L^{p_\theta}(X)}, \tag{4.12}$$

where $\frac{1}{p_\theta} = \frac{1-\theta}{p_{\lambda_0}} + \frac{\theta}{p_{\lambda_1}}$, $C_\theta \leq k C_0^{1-\theta} C_1^\theta$, $\theta \in (0,1)$, and

$$k = 2\left(\frac{p_\theta}{p_\theta - p_{\lambda_0}} + \frac{p_\theta}{p_{\lambda_1} - p_\theta} \right). \tag{4.13}$$

This completes the inequality (4.6). Similarly, the inequality (4.7) also holds. □

In Theorem 4.4, noticing that $\theta \in (0,1)$, it is easy to see that $1 < p_\theta < \infty$. The maximal operator inequalities of the a-quadratic variation and of a-conditional quadratic variation for X-valued tree martingales comes from Theorem 4.4 immediately, provided $a \geq 2$.

Theorem 4.5. *Assume that the assumptions of Lemma 4.1 and Lemma 4.2 are valid, and that X-valued tree martingales $f = (f_t, t \in \mathbf{T}) \in L^p(X)$. Then*

(i) *if $f = (f_t, t \in \mathbf{T})$ is previsible, then there exists a constant C_p depending only on p such that*

$$\left\| \left(S_t^{(a)}(f), t \in \mathbf{T} \right) \right\|_{L^p(X)} \leq C_p \left\| f \right\|_{L^p(X)}, \tag{4.14}$$

$$\left\| \left(s_t^{(a)}(f), t \in \mathbf{T} \right) \right\|_{L^p(X)} \leq C_p \left\| f \right\|_{L^p(X)}. \tag{4.15}$$

(ii) *if $f = (f_t, t \in \mathbf{T})$ is regular, then the inequalities* (4.14) *and* (4.15) *also hold.*

He and Shen [10] showed that if X is an UMD space and X-valued tree martingales are regular or previsible then the maximal operators of X-valued tree martingale transforms are norm-bounded in $L^p(X)$. Here, based on the following lemma, a simpler proof of the maximal operator inequalities for X-valued tree martingale transforms shall be given by using Marcinkiewicz interpolation theorem.

Theorem 4.6. *Assume that the assumptions of Lemma 4.2 are valid, and that X-valued tree martingales $f = (f_t, t \in \mathbf{T}) \in L^p(X)$. Let X be a* UMD *space. Then*

(i) *if $f = (f_t, t \in \mathbf{T})$ is previsible, then there exists a constant C_θ such that*

$$\left\|\pi f\right\|_{L^{p_\theta}(X)} \leq C_\theta \left\|f\right\|_{L^{p_\theta}(X)}, \tag{4.16}$$

where $\frac{1}{p_\theta} = \frac{1-\theta}{p_{\lambda_0}} + \frac{\theta}{p_{\lambda_1}}$, $C_\theta \leq k C_0^{1-\theta} C_1^\theta$, $\theta \in (0,1)$, *and*

$$k = 2\left(\frac{p_\theta}{p_\theta - p_{\lambda_0}} + \frac{p_\theta}{p_{\lambda_1} - p_\theta}\right). \tag{4.17}$$

(ii) *if $f = (f_t, t \in \mathbf{T})$ is regular, then the inequality* (4.16) *also holds.*

Proof. If $2 > s \geq 1$, choose $\lambda_0, \lambda_1 \in [0,1]$ such that $\lambda_1 < \lambda_0$,

$$\frac{1}{p_{\lambda_0}} = \frac{\lambda_0}{2} + \frac{1-\lambda_0}{s}, \quad \text{and} \quad \frac{1}{p_{\lambda_1}} = \frac{\lambda_1}{2} + \frac{1-\lambda_1}{s}. \tag{4.18}$$

Then from (4.18) we can derive that $p_{\lambda_0} < p_{\lambda_1}$. Using Theorem 4.9 in [10, p. 6604], one obtains that

$$\left\|\pi f\right\|_{L_*^{p_{\lambda_0}}(X)} \leq C_{p_{\lambda_0}} \left\|f\right\|_{L^{p_{\lambda_0}}(X)}, \tag{4.19}$$

$$\left\|\pi f\right\|_{L_*^{p_{\lambda_1}}(X)} \leq C_{p_{\lambda_1}} \left\|f\right\|_{L^{p_{\lambda_1}}(X)}. \tag{4.20}$$

From Lemma 3.3, we know that $\pi(f)$ is a sublinear operator, then applying Lemma 2.3 (Marcinkiewicz interpolation theorem) to (4.19), (4.20), one can derive that

$$\|\pi f\|_{L^{p_\theta}(X)} \leq C_\theta \|f\|_{L^{p_\theta}(X)}, \tag{4.21}$$

where $\frac{1}{p_\theta} = \frac{1-\theta}{p_{\lambda_0}} + \frac{\theta}{p_{\lambda_1}}$, $C_\theta \leq k C_0^{1-\theta} C_1^\theta$, $\theta \in (0,1)$, and

$$k = 2\left(\frac{p_\theta}{p_\theta - p_{\lambda_0}} + \frac{p_\theta}{p_{\lambda_1} - p_\theta}\right). \tag{4.22}$$

This completes the inequality (4.16). □

It is noted that $1 < p_\theta < \infty$ in Theorem 4.6, the maximal operator inequality of X-valued tree martingale transforms comes from Theorem 4.6 immediately.

Theorem 4.7. *Assume that $1 < p < \infty$, and that X-valued tree martingales $f = (f_t, t \in \mathbf{T}) \in L^p(X)$. Let X be a* UMD *space. Then*

(i) *if $f = (f_t, t \in \mathbf{T})$ is previsible, then there exists a constant C_p such that*

$$\left\|\pi f\right\|_{L^p(X)} \leq C_p \left\|f\right\|_{L^p(X)}. \tag{4.23}$$

(ii) *if $f = (f_t, t \in \mathbf{T})$ is regular, then the inequality* (4.23) *also holds.*

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References

[1] T. Ando, *Contractive projections in L^p spaces*, Pacific J. Math. **17** (1966), 391–405.

[2] J. Bergh and J. Löfström, *Interpolation Spaces: An Introduction*, Vol. 223, Springer-Verlag, Berlin, 1976.

[3] D. L. Burkholder and R. F. Gundy, *Extrapolation and interpolation of Quasi-linear operators on martingales*, Acta Math. **124** (1970), no. 1, 249–304.

[4] R. Cairoli and J. B. Walsh, *Stochastic integrals in the plane*, Acta Math. **134** (1975), 111–183.

[5] L. E. Dor and E. Odell, *Monotone bases in L^p*, Pacific J. Math. **60** (1975), no. 2, 51–61.

[6] S. Fridli and F. Schipp, *Tree-martingales*, The 5th Pannonian Symposium on Mathematical Statistics, 1985, Grossmann, W. and Pflug, G. Ch. and Vincze I. and And Wertz W., 53–63, Visegrád, Hungary, Akademiai Kiado.

[7] G. Gát, *On $(C, 1)$ summability for Vilenkin-like systems*, Studia Math. **144** (2001), no. 2, 101–120.

[8] J. Gosselin, *Almost everywhere convergence of Vilenkin-Fourier series*, Trans. Amer. Math. Soc. **185** (1973), 345–370.

[9] T.-J. He and Y. L Hou, *Some inequalities for tree martingales*, Acta Math. Appl. Sin. Engl. Ser. **21** (2005), no. 4, 671–682.

[10] T.-J. He and Y. Shen, *Maximal operators of tree martingale transforms and their maximal operator inequalities*, Trans. Amer. Math. Soc. **12** (2008), no. 12, 6595–6609.

[11] ———, *Decomposition and convergence for tree martingales*, Stochastic Process. Appl. **119** (2009), no. 8, 2625–2644.

[12] T.-J. He, Y. X. Xiao, and Y. L. Hou, *Inequalities for vector-valued tree martingales*, International Conference on Functional Space Theory and Its Applications, 2004, Liu, P. D., 67-75, Wuhan, Research Information Ltd UK.

[13] D. Khoshnevisan, *Multiparameter Processes*, Springer-Verlag, New York, 2002.

[14] G. Pisier, *Martingales with values in uniformly convex spaces*, Israel J. Math. **20** (1975), no. 3-4, 326–350.

[15] F. Schipp, *Universal contractive projections and a.e. convergence*, Probability theory and applications, 221–233, Math. Appl., 80, Kluwer Acad. Publ., Dordrecht, 1992.

[16] F. Schipp and F. Weisz, *Tree martingales and a.e. convergence of Vilenkin-Fourier series*, Math. Pannon. **8** (1997), no. 1, 17–35.

[17] F. Weisz, *Martingale Hardy spaces and their applications in Fourier analysis*, Lecture Notes in Mathematics, 1568. Springer-Verlag, Berlin, 1994.

[18] ———, *Almost everywhere convergence of Banach space-valued Vilenkin-Fourier series*, Acta Math. Hungar. **116** (2007), no. 1-2, 47–59.

[19] W.-S. Young, *Mean convergence of generalized Walsh-Fourier series*, Trans. Amer. Math. Soc. **218** (1976), 311–320.

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