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# THE ASYMPTOTIC BEHAVIOUR OF THE $m$-TH ORDER CARDINAL $B$-SPLINE WAVELET

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Abstract. It is well-known that the $m$-th order cardinal $B$-spline wavelet, $\psi_m$, decays exponentially. Our aim in this paper is to determine the exact rate of this decay and thereby to describe the asymptotic behaviour of $\psi_m$.

## 1. Introduction

Mallat and Meyer in [6, p. 225] define a wavelet, $\psi$, with the aid of a scaling function, $\varphi$, in $L_2(R)$, which function satisfies, among other things, a relation of the form

$$\varphi(x) = \sum_{j\in\mathbb{Z}} a_j \varphi(2x-j) \tag{1.1}$$

for certain scaling constants $a_j$; indeed,

$$\psi(x) := \sum_{j\in\mathbb{Z}} (-1)^j a_{1-j} \varphi(2x-j). \tag{1.2}$$

The cardinal $B$-spline scaling functions, and hence the wavelets they determine, are given in terms of the cardinal $B$-splines, $N_m$. As in [1, p. 17], the latter are defined inductively by

$$N_1(x) = \chi_{[0,1)}(x) \text{ and } N_m(x) = \int_0^1 N_{m-1}(x-y)dy,$$

$m = 2, 3, \ldots$. One has $N_m \in C^{m-2}(\mathbb{R})$ for $m \geq 2$. Moreover, it is supported in $[0, m]$ and is equal to a polynomial of degree $m-1$ on each interval of the form $[k, k+1), k = 0, 1, \ldots, m-1$.

Though the family $\{N_m(x-j)\}_{j\in\mathbb{Z}}$ is not an orthonormal system, it is possible to construct a scaling function, $\varphi_m$, from it so that $\{\varphi_m(x-j)\}_{j\in\mathbb{Z}}$ is

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such a system. Indeed, one can take

$$\varphi_m(x) := \sum_{j\in\mathbb{Z}} c_j N_m(x-j), \tag{1.3}$$

with

$$c_j = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\cos j\theta}{\sqrt{P_m\left(\cos\frac{\theta}{2}\right)}}\,d\theta, \tag{1.4}$$

in which $P_m$ is a polynomial of degree $m$ and

$$P_m\left(\cos\frac{\theta}{2}\right) = 2\pi\sum_{j\in\mathbb{Z}} |\widehat{N_m}(\theta+2\pi j)|^2$$

(We use the convention $\widehat{f}(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(t)e^{-i\lambda t}dt$, $\lambda\in\mathbb{R}$, for the Fourier transform).

It is well-known that the $m$-th order cardinal $B$-spline wavelet defined by (1.2) in terms of the scaling constants of $\varphi_m$ decays exponentially; see, for example, [6, Corollary 5.4.2, pp. 150–152].

Our aim in this paper is to obtain the exact rate of the decay. Its principal result is:

**Theorem A.** *The $m$-th order cardinal $B$-spline wavelet, $\psi_m, m \geq 2$, has the asymptotic form*

$$\psi_m(x) = \left[\sum_{x-m\leq j\leq x} (-1)^{r_j} E_j \frac{e^{-\alpha_0 r_j}}{\sqrt{r_j}} N_m(x-j)\right][1+o(1)], \tag{1.5}$$

*as $|x|\to\infty$, in which $r_j = \left[\frac{|j|}{2}\right]$ and $E_j$ depends only on the sign and parity of $j\in\mathbb{Z}$. The constant $\alpha_0$ in (1.5) is given by*

$$\alpha_0 = \ln\left[\frac{\sqrt{\mu_{m-1}+1}+\sqrt{\mu_{m-1}}}{\sqrt{\mu_{m-1}+1}-\sqrt{\mu_{m-1}}}\right],$$

*where*

$$\mu_{m-1} = \frac{(\lambda_{m-1}+1)^2}{4|\lambda_{m-1}|}$$

*and $\lambda_{m-1}$ is the $(m-1)$-st smallest negative root of the Euler-Frobenius polynomial of degree $2m-2$.*

The constants $E_j$ are given explicitly for $j\in 2\mathbb{Z}_+$ in the proof of Theorem 6.2. Therefore, they are expressed in terms of constants $D_m = D_m(j)$ and $D_{m+1} = D_{m+1}(j)$, which are themselves specified in the proof of Theorem 5.1.

Estimates similar to (1.5) are given in [2, Theorem 1] for the Franklin functions.

To prove Theorem A we need a representation of $\psi_m$ similar to the one for $\varphi_m$ in (1.3), namely,

$$\psi_m(x) = \sum_{j\in\mathbb{Z}} \gamma_j N_m(x-j). \tag{1.6}$$

As will be shown in Lemma 6.1 below,

$$\gamma_j = (-1)^j \sum_{k\in\mathbb{Z}} (-1)^k a_{k-j+1} c_k, \tag{1.7}$$

$j \in \mathbb{Z}$, where the $a_j$ appearing in (1.7) are the scaling constants for $\varphi_m$. These constants are given by the formula

$$a_j = 2^{-m} \sum_{i=M-m}^{M+1} \binom{m+1}{M+1-i} \sum_{\substack{l\in\mathbb{Z}\\ l\equiv i}} c_{\frac{l-i}{2}} b_{j-l} \tag{1.8}$$

for $j \in \mathbb{Z}$; here,

$$b_j = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sqrt{P_m\left(\cos\frac{\theta}{2}\right)} \cos j\theta d\theta,$$

$j \in \mathbb{Z}$, and $l \equiv i$ means $l = i \bmod 2$. The constants $b_j$ come out of an equation inverse to (1.3), namely,

$$N_m(x) = \sum_{j\in\mathbb{Z}} b_j \varphi_m(x-j), \tag{1.9}$$

$x \in \mathbb{R}$.

The behaviour of $\psi_m(x)$ as $|x| \to \infty$ is determined by that of $\gamma_j$ as $|j| \to \infty$. To describe the latter we need to know the long term behaviour of, successively, the $c_j, b_j$ and $a_j$. This is obtained in Sections 3, 4 and 5, respectively. The proof of Theorem A is then essentially given in Section 6 by the determination of the behaviour of $\gamma_j$ as $|j| \to \infty$.

We begin in the next section with a study of $\frac{1}{\sqrt{P_m\left(\cos\frac{\theta}{2}\right)}}$.

## 2. The function $\frac{1}{\sqrt{P_m\left(\cos\frac{\theta}{2}\right)}}$

It is shown in [1, p. 90] that

$$P_m\left(\cos\frac{\theta}{2}\right) = \frac{1}{(2m-1)!}\prod_{k=1}^{m-1} \frac{1-2\lambda_k\cos\theta+\lambda_k^2}{|\lambda_k|}, \tag{2.1}$$

in which $0 > \lambda_1 > \cdots > \lambda_{m-1}$ are the first $m-1$ negative simple real roots of the Euler-Frobenius polynomial

$$E_{2m-1}(z) = (2m-1)!z^{m-1} \sum_{k=-m+1}^{m-1} N_{2m}(m+k)z^k,$$

the remaining $m-1$ negative roots, $\lambda_1, \ldots, \lambda_{2m-2}$, being such that $\lambda_1\lambda_{2m-2} = \cdots = \lambda_{m-1}\lambda_m = 1$.

**Lemma 2.1.** *Let* $P_m\left(\cos\frac{\theta}{2}\right)$ *and* $\lambda_k$, $k = 1, \ldots, m-1$, *be as in* (2.1). *Then,*

$$\frac{1}{\sqrt{P_1\left(\cos\frac{\theta}{2}\right)}} = \frac{1}{\sqrt{\frac{1}{3} + \frac{2}{3}\cos^2\frac{\theta}{2}}}, \tag{2.2}$$

$$\frac{1}{\sqrt{P_2\left(\cos\frac{\theta}{2}\right)}} = \frac{1}{\sqrt{\frac{2}{15} + \frac{11}{15}\cos^2\frac{\theta}{2} + \frac{2}{15}\cos^4\frac{\theta}{2}}}, \tag{2.3}$$

*and, for* $m \geq 3$,

$$\frac{1}{\sqrt{P_m\left(\cos\frac{\theta}{2}\right)}} = A\left(1 + \sum_{k=1}^{\infty}(-1)^k\binom{-\frac{1}{2}}{k}(\mu_{m-1}+1)^{-k}\,(1 + B_k^{m-2})\sin^{2k}\frac{\theta}{2}\right). \tag{2.4}$$

*The positive constant* $A$ *is specified in the proof and* $B_k^{m-2}$ *is the final term in the finite recurrence sequence*

$$B_k^1 = \sum_{j=1}^{k} R(j,k)\left(\frac{\mu_2+1}{\mu_1+1}\right)^j,$$

*where*

$$R(j,k) = \frac{\binom{-\frac{1}{2}}{j}\binom{-\frac{1}{2}}{k-j}}{\binom{-\frac{1}{2}}{k}},$$

*and*

$$B_k^{l+1} = \sum_{j=1}^{k} R(j,k)\left(\frac{\mu_{l+2}+1}{\mu_{l+1}+1}\right)^j (1 + B_j^l),$$

$l = 1, \ldots, m-3$, *with*

$$\mu_i = \frac{(\lambda_i+1)^2}{4|\lambda_i|},$$

$i = 1, \ldots, m-1$. *Moreover,* $B = \lim_{k\to\infty} B_k^{m-2}$ *exists.*

*Proof.* The formulas (2.2) and (2.3) can be obtained directly from [1, p. 88, (4.2.10)].

As for (2.4), we observe that, since $1 - 2\lambda\cos\theta + \lambda^2 = -4\lambda\Big(x + \frac{(\lambda+1)^2}{-4\lambda}\Big)$, with $x = \cos^2\left(\frac{\theta}{2}\right)$, one has

$$\begin{aligned}\frac{1}{\sqrt{P_m\left(\cos\frac{\theta}{2}\right)}} &= 2^{-(m-1)}\sqrt{(2m-1)!}\left(\prod_{i=1}^{m-1}(x+\mu_i)\right)^{-\frac{1}{2}} \\ &= A\prod_{i=1}^{m-1}\left(1 + \frac{x-1}{\mu_i+1}\right)^{-\frac{1}{2}}\end{aligned}$$

$$= A \prod_{i=1}^{m-1} \left( 1 + \sum_{k=1}^{\infty} \binom{-\frac{1}{2}}{k} \left( \frac{x-1}{\mu_i + 1} \right)^k \right),$$

where $A = 2^{-(m-1)} \sqrt{(2m-1)!} \prod_{i=1}^{m-1} (\mu_i + 1)^{-\frac{1}{2}}$.

Now,

$$\left[ 1 + \sum_{k=1}^{\infty} \binom{-\frac{1}{2}}{k} \left( \frac{x-1}{\mu_1 + 1} \right)^k \right] \left[ 1 + \sum_{k=1}^{\infty} \binom{-\frac{1}{2}}{k} \left( \frac{x-1}{\mu_2 + 1} \right)^k \right] = 1 + \sum_{k=1}^{\infty} d_k (x-1)^k,$$

in which

$$\begin{aligned} d_k &= \binom{-\frac{1}{2}}{k} (\mu_1 + 1)^{-k} + \binom{-\frac{1}{2}}{k} (\mu_2 + 1)^{-k} \\ &\quad + \sum_{j=1}^{k-1} \binom{-\frac{1}{2}}{j} \binom{-\frac{1}{2}}{k-j} (\mu_1 + 1)^{-j} (\mu_2 + 1)^{-(k-j)} \\ &= \binom{-\frac{1}{2}}{k} (\mu_2 + 1)^{-k} \left[ 1 + B_k^1 \right] \end{aligned}$$

and

$$B_k^1 = \sum_{j=1}^{k} R(j,k) \left( \frac{\mu_2 + 1}{\mu_1 + 1} \right)^j .$$

We claim that $\lim_{k \to \infty} B_k^1$ exists. Indeed, $\mu_1 > \mu_2 > \cdots > \mu_{m-1} > 0$, so the claim will follow from the Lebesgue dominated convergence theorem for sequences once we show $R(j,k)$ is bounded independently of $j$ and $k$, $1 \le j \le k$. But, on expressing the generalized binomial coefficients in terms of gamma functions and using the relation $\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}$, we obtain

$$R(j,k) = \frac{1}{\sqrt{\pi}} \frac{\Gamma(k+1)\Gamma(j+\frac{1}{2})\Gamma(k-j+\frac{1}{2})}{\Gamma(k+\frac{1}{2})\Gamma(j+1)\Gamma(k-j+1)}.$$

Stirling's formula in the form $\Gamma(x) \sim \sqrt{2\pi} e^{-x} x^{x-\frac{1}{2}}$ then yields

$$\begin{aligned} R(j,k) &\sim \sqrt{2e} \left( \frac{k+1}{k+\frac{1}{2}} \right)^k \left( \frac{j+\frac{1}{2}}{j+1} \right)^j \left( \frac{k-j+\frac{1}{2}}{k-j+1} \right)^{k-j} \sqrt{\frac{k+1}{(j+1)(k-j+1)}} \\ &\le \sqrt{2e} \left( \frac{k+1}{k+\frac{1}{2}} \right)^k \\ &= \sqrt{2e} \left( 1 + \frac{1}{2k+1} \right)^k \\ &\le \sqrt{2e^3}. \end{aligned}$$

Again,

$$\left[ 1 + \sum_{k=1}^{\infty} \binom{-\frac{1}{2}}{k} \left( \frac{x-1}{\mu_2 + 1} \right)^k \left[ 1 + B_k^1 \right] \right] \left[ 1 + \sum_{k=1}^{\infty} \binom{-\frac{1}{2}}{k} \left( \frac{x-1}{\mu_3 + 1} \right)^k \right] = 1 + \sum_{k=1}^{\infty} h_k (x-1)^k,$$

with

$$h_k = \binom{-\frac{1}{2}}{k}(1+\mu_3)^{-k}(1+B_k^2)$$

and

$$B_k^2 = \sum_{j=1}^{k} R(j,k)\left(\frac{\mu_3+1}{\mu_2+1}\right)^j (1+B_k^1).$$

An argument similar to the one involving the $B_k^1$ shows $\lim_{k\to\infty} B_k^2$ exists. Continuing like this we finally get

$$\begin{aligned}\frac{1}{\sqrt{P_m\left(\cos\frac{\theta}{2}\right)}} &= A\left[1+\sum_{k=1}^{\infty}\binom{-\frac{1}{2}}{k}\left(\frac{x-1}{\mu_{m-1}+1}\right)^k (1+B_k^{m-2})\right] \\ &= A\left[1+\sum_{k=1}^{\infty}(-1)^k\binom{-\frac{1}{2}}{k}(\mu_{m-1}+1)^{-k}\sin^{2k}\left(\frac{\theta}{2}\right)(1+B_k^{m-2})\right]\end{aligned}$$

and $\lim_{k\to\infty} B_k^{m-2}$ exists, as asserted. □

**Corollary 2.2.** *Let $c_j$ be the Fourier coefficient in* (1.3)*. Then, for $j \gg 1$,*

$$c_j = (-1)^j A(1+B+o(1))\sum_{k=j}^{\infty}(-1)^k\binom{-\frac{1}{2}}{k}\binom{2k}{k-j}(4(\mu_{m-1}+1))^{-k}, \tag{2.5}$$

*where $A$ and $B$ are as in Lemma 2.1.*

*Proof.* According to Lemma 2.1,

$$c_j = \frac{A}{2\pi}\sum_{k=j}^{\infty}(-1)^k\binom{-\frac{1}{2}}{k}(\mu_{m-1}+1)^{-k}(1+B_k^{m-2})\int_{-\pi}^{\pi}\sin^{2k}\frac{\theta}{2}\cos j\theta d\theta. \tag{2.6}$$

Now,

$$\sin^{2k}\theta = \frac{1}{2^{2k}}\binom{2k}{k} + \frac{1}{2^{2k-1}}\sum_{i=0}^{k-1}(-1)^{k-i}\binom{2k}{i}\cos 2(k-i)\theta,$$

so

$$\begin{aligned}\int_{-\pi}^{\pi}\sin^{2k}\frac{\theta}{2}\cos j\theta d\theta &= \frac{1}{2^{2k-1}}\sum_{i=0}^{k-1}(-1)^{k-i}\binom{2k}{i}\int_{-\pi}^{\pi}\cos(k-i)\theta\cos j\theta d\theta \\ &= \begin{cases} 0, & k<j \\ (-1)^j\dfrac{\pi}{2^{2k-1}}\dbinom{2k}{k-j}, & k\geq j.\end{cases}\end{aligned}$$

Substitution in (2.6) and the observation that $B_k^{m-2} = B + o(1)$ for $k \gg 1$ yields (2.5). □

## 3. The constants $c_j$

The purpose of this section is to prove:

**Theorem 3.1.** *Let* $c_j$ *be given by* (1.4). *Then,*

$$c_j \sim (-1)^j K_c \frac{e^{-\alpha_0|j|}}{\sqrt{|j|}} \tag{3.1}$$

*as* $|j| \to \infty$, *where*

$$\alpha_0 = \ln\left[\frac{\sqrt{\mu_{m-1}+1}+\sqrt{\mu_{m-1}}}{\sqrt{\mu_{m-1}+1}-\sqrt{\mu_{m-1}}}\right]$$

*and*

$$K_c = \frac{1}{\sqrt{\pi}} A(1+B)\left(1+\frac{1}{\mu_{m-1}}\right)^{\frac{1}{4}}.$$

*Proof.* According to Corollary 2.2,

$$c_j = c_{|j|} = (-1)^j A(1+B+o(1)) \sum_{k=|j|}^{\infty} (-1)^k \binom{-\frac{1}{2}}{k}\binom{2k}{k-|j|}(4(\mu_{m-1}+1))^{-k}$$

for $|j| \gg 1$. We have, when $j > 0$

$$\sum_{k=j}^{\infty}(-1)^k \binom{-\frac{1}{2}}{k}\binom{2k}{k-j}(4(\mu_{m-1}+1))^{-k}$$
$$= \frac{1}{\sqrt{\pi}}\sum_{k=j}^{\infty}\frac{\Gamma(k+\frac{1}{2})\Gamma(2k+1)}{\Gamma(k+1)\Gamma(k+j+1)\Gamma(k-j+1)}(4(\mu_{m-1}+1))^{-k}.$$

The ratio of the $k+1$-st term to the $k$-th term in the last series is

$$r(n) = \frac{1}{4(\mu_{m-1}+1)}\frac{(2n-1)^2}{(n^2-j^2)},\ n = k+1.$$

As

$$r'(n) = \frac{2(2n-1)(n-(2j)^2)}{4(\mu_{m-1}+1)(n^2-j^2)^2},$$

we conclude

(i) $$\lim_{n\to\infty} r(n) = \frac{1}{\mu_{m-1}+1};$$

(ii) $r(n)$ decreases until $n = 2j^2$, after which it increases;

(iii) $r(n) = 1$ when

$$(4c-1)n^2 - 4cn + c + j^2 = 0,\ c = \frac{1}{4(\mu_{m-1}+1)}$$

or

$$n = \frac{4c-\sqrt{4c+4j^2(1-4c)}}{2(4c-1)} \backsim \frac{j}{\sqrt{1-4c}} = j\sqrt{1+\frac{1}{\mu_{m-1}}}.$$

Take $k = l + j$, so that

$$c_j = (-1)^j \frac{A}{\sqrt{\pi}}(1 + B + o(1)) \sum_{l=0}^{\infty} \frac{\Gamma(l+j+\frac{1}{2})\Gamma(2l+2j+1)}{\Gamma(l+j+1)\Gamma(l+2j+1)\Gamma(l+1)} (4(\mu_{m-1}+1))^{-(l+j)}.$$

Let

$$l_1 = \left(\sqrt{1 + \frac{1}{\mu_{m-1}}} - 1\right) j = \alpha j.$$

Then, according to [5, p. 274], we have, for $l = l_1 + h$ and $|h| \le j^{\frac{3}{5}}$,

$$\Gamma(l_1+j+\frac{1}{2}+h) = \Gamma(l_1+j+\frac{1}{2})(l_1+j-\frac{1}{2})^h \exp\left(\frac{h^2}{2(l_1+j-\frac{1}{2})}\right)[1+O(j^{-\frac{1}{5}})],$$

$$\Gamma(2l_1+2j+1+2h) = \Gamma(2l_1+2j+1)(2l_1+2j)^{2h} \exp\left(\frac{4h^2}{2(2l_1+2j)}\right)[1+O(j^{-\frac{1}{5}})],$$

$$\Gamma(l_1+j+1+h) = \Gamma(l_1+j+1)(l_1+j)^h \exp\left(\frac{h^2}{2(l_1+j)}\right)[1+O(j^{-\frac{1}{5}})],$$

$$\Gamma(l_1+2j+1+h) = \Gamma(l_1+2j+1)(l_1+2j)^h \exp\left(\frac{h^2}{2(l_1+2j)}\right)[1+O(j^{-\frac{1}{5}})],$$

and

$$\Gamma(l_1+1+h) = \Gamma(l_1+1) l_1^h \exp\left(\frac{h^2}{2l_1}\right)[1+O(j^{-\frac{1}{5}})].$$

Thus, with $l = l_1 + h$, we have, uniformly in $h$, $|h| \le j^{\frac{3}{5}}$,

$$\frac{\Gamma(l+j+\frac{1}{2})\Gamma(2l+2j+1)}{\Gamma(l+j+1)\Gamma(l+2j+1)\Gamma(l+1)}$$

equal to

$$\begin{aligned} &\frac{\Gamma(l_1+j+\frac{1}{2})\Gamma(2l_1+2j+1)}{\Gamma(l_1+j+1)\Gamma(l_1+2j+1)\Gamma(l_1+1)} \\ &\times \exp\left[-\frac{h^2}{2}\left(-\frac{1}{l_1+j-\frac{1}{2}} - \frac{4}{2l_1+2j} + \frac{1}{l_1+j} + \frac{1}{l_1+2j} + \frac{1}{l_1}\right)\right] \\ &\times \left[\frac{(l_1+j-\frac{1}{2})(2l_1+2j)^2}{(l_1+j)(l_1+2j)l_1}\right]^h \\ \sim &\left[\frac{4(l_1+j)^3}{(l_1+j)(l_1+2j)l_1}\right]^h = \left[\frac{4(l_1+j)^2}{l_1(l_1+2j)}\right]^h. \end{aligned}$$

Now,

$$\frac{4(l_1+j)^2}{l_1(l_1+2j)} = 4(\mu_{m-1}+1)$$

and

$$j\left[-\frac{1}{l_1+j-\frac{1}{2}} - \frac{4}{2l_1+2j} + \frac{1}{l_1+j} + \frac{1}{l_1+2j} + \frac{1}{l_1}\right]$$

$$\longrightarrow -\frac{2}{\alpha+1}+\frac{1}{\alpha+2}+\frac{1}{\alpha}=\frac{2}{\alpha(\alpha+1)(\alpha+2)}=\beta.$$

Setting

$$d_l=\left[\frac{\Gamma(l+j+\frac{1}{2})\Gamma(2l+2j+1)}{\Gamma(l+j+1)\Gamma(l+2j+1)\Gamma(l+1)}\right](4(\mu_{m-1}+1))^{-(l+j)},$$

we have shown

$$\begin{aligned}\sum_{|l-l_1|\le j^{\frac{3}{5}}} d_l &\sim \sqrt{j}d_{l_1}\sum_{|l-l_1|\le j^{\frac{3}{5}}}\frac{e^{-\frac{h^2}{2}\beta}}{\sqrt{j}}\\ &\sim \sqrt{j}d_{l_1}\int_{-j^{\frac{1}{10}}}^{j^{\frac{1}{10}}} e^{-\frac{-\beta t^2}{2}}dt, \quad \text{by [5, p. 275]},\\ &\sim \sqrt{j}d_{l_1}\sqrt{\frac{2}{\beta}}\int_{-\infty}^{\infty}e^{-t^2}dt\\ &\sim \sqrt{\frac{2\pi j}{\beta}}d_{l_1}\end{aligned}$$

as $j\to\infty$.

Again, when $|h|>j^{\frac{3}{5}}$ there exists $\rho$, $0<\rho<1$, such that

$$d_l\le \rho^{l_1-j^{\frac{3}{5}}-l}d_{l_1-j^{\frac{3}{5}}}, \quad \text{where} \quad 0\le l\le l_1+j^{\frac{3}{5}},$$

and

$$d_l\le \rho^{l-l_1-j^{\frac{3}{5}}}d_{l_1+j^{\frac{3}{5}}}, \quad \text{where} \quad l_1+j^{\frac{3}{5}}\le l,$$

so

$$\sum_{l=0}^{l_1-j^{\frac{3}{5}}} d_l\le d_{l_1-j^{\frac{3}{5}}}\sum_{l=0}^{l_1-j^{\frac{3}{5}}}\rho^{l_1-l-j^{\frac{3}{5}}}=d_{l_1-j^{\frac{3}{5}}}\sum_{l=0}^{l_1-j^{\frac{3}{5}}}\rho^l\le\frac{d_{l_1-j^{\frac{3}{5}}}}{1-\rho}$$

and, similarly,

$$\sum_{l=l_1+j^{\frac{3}{5}}}^{\infty} d_l\le\frac{d_{l_1+j^{\frac{3}{5}}}}{1-\rho}.$$

Altogether, then,

$$c_j\sim(-1)^jA(1+B)\sqrt{\frac{2j}{\beta}}d_{l_1}.$$

Next, Stirling's formula (in the form $\Gamma(x)\sim\sqrt{2\pi}e^{-x}x^{x-\frac{1}{2}}$) yields

$$\begin{aligned}&(4(\mu_{m-1}+1))^{l_1+j}jd_{l_1}\\ &\sim\frac{j}{\sqrt{2\pi}}\exp[-(l_1+j+\frac{1}{2}+2l_1+2j+1-l_1-j-1-l_1-2j-1-l_1-1)]\\ &\times\frac{(l_1+j+\frac{1}{2})^{l_1+j}(2l_1+2j+1)^{2l_1+2j+\frac{1}{2}}}{(l_1+j+1)^{l_1+j+\frac{1}{2}}(l_1+2j+1)^{l_1+2j+\frac{1}{2}}(l_1+1)^{l_1+\frac{1}{2}}}\end{aligned}$$

which equals

$$\frac{e^{\frac{3}{2}}}{\sqrt{2\pi}}\frac{\left(1+\frac{\frac{1}{2}}{l_1+j}\right)^{l_1+j}}{\left(1+\frac{1}{l_1+j}\right)^{l_1+j}}\frac{(2l_1+2j+1)^{2l_1+2j+\frac{1}{2}}}{(l_1+2j+1)^{l_1+2j+\frac{1}{2}}(l_1+1)^{l_1}}\sqrt{\frac{j^2}{(l_1+j+1)(l_1+1)}}$$

$$\sim \frac{e}{\sqrt{2\pi\alpha(\alpha+1)}}4^{l_1+j+\frac{1}{4}}\frac{(l_1+j+\frac{1}{2})^{2(l_1+j)+\frac{1}{2}}}{(l_1+j+j+1)^{l_1+2j+\frac{1}{2}}(l_1+j-j+1)^{l_1}}$$

$$\sim \frac{e}{\sqrt{2\pi\alpha(\alpha+1)}}4^{l_1+j+\frac{1}{4}}\frac{\left(1+\frac{\frac{1}{2}}{l_1+j}\right)^{2(l_1+j)}}{\left(1+\frac{j}{l_1+j}+\frac{1}{l_1+j}\right)^{l_1+2j}\left(1-\frac{j}{l_1+j}+\frac{1}{l_1+j}\right)^{l_1}}\left(\frac{l_1+j+\frac{1}{2}}{l_1+2j+1}\right)^{\frac{1}{2}}$$

$$\sim \frac{4^{l_1+j+\frac{1}{4}}e^2}{\sqrt{2\pi\alpha(\alpha+1)}}\left(\frac{\alpha+1}{\alpha+2}\right)^{\frac{1}{2}}\frac{1}{\left(1+\frac{1}{\alpha+1}+\frac{1}{l_1+j}\right)^{l_1+2j}}\frac{1}{\left(1-\frac{1}{\alpha+1}+\frac{1}{l_1+j}\right)^{l_1}}$$

$$\sim \frac{4^{l_1+j}e^2}{\sqrt{\pi\alpha(\alpha+2)}}(1-\gamma^2)^{-(l_1+j)}\left(\frac{1-\gamma}{1+\gamma}\right)^j\frac{1}{\left(1+\frac{1}{(1+\gamma)(l_1+j)}\right)^{l_1+j}}\frac{1}{\left(1+\frac{1}{(1-\gamma)(l_1+j)}\right)^{l_1+j}}$$

$$\times\left(1+\frac{\gamma}{(1-\gamma)j}\right)^j\left(1+\frac{\gamma}{(1+\gamma)j}\right)^{-j}\quad\left(\gamma=\frac{1}{1+\alpha},\, l_1+j=(\alpha+1)j=\frac{j}{\gamma}\right)$$

$$\sim \frac{(4(\mu_{m-1}+1))^{l_1+j}e^2}{\sqrt{\pi\alpha(\alpha+2)}}e^{-\alpha_0 j}\exp\left[\frac{\gamma}{1-\gamma}-\frac{\gamma}{1+\gamma}-\frac{1}{1-\gamma}-\frac{1}{1+\gamma}\right]$$

$$= \frac{(4(\mu_{m-1}+1))^{l_1+j}}{\sqrt{\pi\alpha(\alpha+2)}}e^{-\alpha_0 j}.$$

Finally, then,

$$c_j \sim (-1)^j K_c\frac{e^{-\alpha_0 j}}{\sqrt{j}}$$

as $j\to\infty$, where

$$K_c = A(1+B)\sqrt{\frac{\alpha+1}{\pi}} = \frac{A(1+B)}{\sqrt{\pi}}\left(1+\frac{1}{\mu_{m-1}}\right)^{\frac{1}{4}}.$$

□

## 4. The constants $b_j$

Using the methods of Section 3 one can prove:

**Theorem 4.1.** *Suppose $A, \alpha_0$ and $\mu_{m-1}$ are as in Section 3 and let*

$$b_j = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sqrt{P_m\left(\cos\frac{\theta}{2}\right)}\cos j\theta d\theta, \tag{4.1}$$

*$j\in\mathbb{Z}$. Then,*

$$b_j \sim (-1)^{j+1}K_b\frac{e^{-\alpha_0|j|}}{|j|^{\frac{3}{2}}},$$

*with*

$$K_b = \frac{1}{\sqrt{\pi}}\frac{1+C}{A}\left(1+\frac{1}{\mu_{m-1}}\right)^{-\frac{1}{4}}$$

*and*

$$C = \lim_{k\to\infty} C_k^{m-2},$$

*where* $C_k^{m-2}$ *is the last term in the finite recurrence sequence*

$$C_k^1 = \sum_{i=1}^{k} S(i,k)\left(\frac{\mu_2+1}{\mu_1+1}\right)^j,$$

*and*

$$C_k^{n+1} = \sum_{i=1}^{k} S(i,k)\left(\frac{\mu_{n+2}+1}{\mu_{n+1}+1}\right)^k (1+C_i^n),\ n = 1,\ldots,m-3;$$

*here,*

$$S(i,k) = \frac{\binom{\frac{1}{2}}{i}\binom{\frac{1}{2}}{k-i}}{\binom{\frac{1}{2}}{k}}.$$

We only remark that the key difference lies in the $d_l$ of Section 3 being replaced by

$$\frac{-\Gamma(l+j-\frac{1}{2})(\Gamma(2l+2j+1))}{\Gamma(l+j+1)\Gamma(l+2j+1)\Gamma(l+1)}(4(\mu_{m-1}+1))^{-(l+j)} = -\frac{d_l}{l+j-\frac{1}{2}}.$$

## 5. The scaling constants $a_j$

The purpose of this section is to prove:

**Theorem 5.1.** *Let* $a_j$ *be the* $j$*-th scaling constant of the function* $\varphi_m$*, given by* (1.8)*. Then, with* $\alpha_0$ *as in Theorem A and* $r = \left[\frac{|j-m|}{2}\right]$*, one has*

$$a_j \sim (-1)^r D_j \frac{e^{-\alpha_0 r}}{\sqrt{r}} \tag{5.1}$$

*as* $|j| \to \infty$*. Here,* $D_j > 0$ *depends only on the sign and parity of* $j$*.*

To do this we require:

**Lemma 5.2.** *Set* $M = \left[\frac{m}{2}\right]$*. Then, the scaling constant*

$$a_j = 2^{-m}\sum_{i=M-m}^{M+1}\binom{m+1}{M+1-i}\sum_{\substack{k\in\mathbb{Z}\\ k\equiv i}} c_{\frac{k-i}{2}} b_{j-k}, \tag{5.2}$$

$j \in \mathbb{Z}$*; as before,* $k \equiv i$ *means* $k = i \bmod 2$*.*

*Proof.* According to [3, p. 148], the cardinal $B$-spline, $N_m$, satisfies the scaling relation

$$N_m(x) = 2^{-m} \sum_{i=0}^{m+1} \binom{m+1}{i} N_m(2x - M - 1 + i)$$
$$= 2^{-m} \sum_{i=M-m}^{M+1} \binom{m+1}{M+1-i} N_m(2x - i).$$

From formulas (1.3) and (1.9) we then have

$$\begin{aligned}\varphi_m(x) &= \sum_{k\in\mathbb{Z}} c_k N_m(x-k) = \sum_{k\in\mathbb{Z}} c_k 2^{-m} \sum_{i=M-m}^{M+1} \binom{m+1}{M+1-i} N_m(2x-2k-i)\\
&= 2^{-m} \sum_{i=M-m}^{M+1} \binom{m+1}{M+1-i} \sum_{k\in\mathbb{Z}} c_k N_m(2x-2k-i)\\
&= 2^{-m} \sum_{i=M-m}^{M+1} \binom{m+1}{M+1-i} \sum_{\substack{k\in\mathbb{Z}\\ k\equiv i}} c_{\frac{k-i}{2}} N_m(2x-k)\\
&= 2^{-m} \sum_{i=M-m}^{M+1} \binom{m+1}{M+1-i} \sum_{\substack{k\in\mathbb{Z}\\ k\equiv i}} c_{\frac{k-i}{2}} \sum_{l\in\mathbb{Z}} b_l \varphi_m(2x-k-l)\\
&= 2^{-m} \sum_{i=M-m}^{M+1} \binom{m+1}{M+1-i} \sum_{\substack{k\in\mathbb{Z}\\ k\equiv i}} c_{\frac{k-i}{2}} \sum_{j\in\mathbb{Z}} b_{j-k} \varphi_m(2x-j)\\
&= \sum_{j\in\mathbb{Z}} \left[ 2^{-m} \sum_{i=M-m}^{M+1} \binom{m+1}{M+1-i} \sum_{\substack{k\in\mathbb{Z}\\ k\equiv i}} c_{\frac{k-i}{2}} b_{j-k} \right] \varphi_m(2x-j).\end{aligned}$$

Thus, (5.2) holds, in view of the scaling relation (1.1). □

*Proof of Theorem 5.1.* For $j >> 1$. Fix $i$ in (5.2), say $i = m+2n_i$, and consider

$$\sum_{\substack{k\in\mathbb{Z}\\ k\equiv i}} c_{\frac{k-i}{2}} b_{j-k} = \sum_{\substack{k\in\mathbb{Z}\\ k\equiv m}} c_{\frac{k-i}{2}} b_{j-k}$$
$$= \sum_{n\in\mathbb{Z}} c_{n-n_i} b_{j-m-2n}.$$

To begin,

$$\left| \sum_{n=-\infty}^{0} c_{n-n_i} b_{j-m-2n} \right| = \left| \sum_{n=0}^{\infty} c_{n+n_i} b_{j-m+2n} \right|$$

$$\begin{aligned}
&\le K_b \sum_{n=0}^{\infty} |c_{n+n_i}| \frac{e^{-\alpha_0(j-m+2n)}}{(j-m+2n)^{\frac{3}{2}}} \\
&\le K \frac{e^{-\alpha_0(j-m)}}{(j-m)^{\frac{3}{2}}} \sum_{n=0}^{\infty} e^{-2\alpha_0 n} \\
&= o\left(\frac{e^{-\alpha_0 r}}{\sqrt{r}}\right).
\end{aligned}$$

Next,

$$\begin{aligned}
\left|\sum_{n=1}^{r-[\sqrt{r}]} c_{n-n_i} b_{j-m-2n}\right| &\le K \sum_{n=1}^{r-[\sqrt{r}]} \frac{e^{-\alpha_0 n}}{\sqrt{n}} \frac{e^{-\alpha_0(j-m-2n)}}{(j-m-2n)^{\frac{3}{2}}} \\
&\le K e^{-\alpha_0(j-m)} \sum_{n=1}^{r-[\sqrt{r}]} \frac{1}{\sqrt{n}} \frac{e^{\alpha_0 n}}{(j-m-2n)^{\frac{1}{2}}} \\
&\le K e^{-\alpha_0} e^{-\alpha_0[\sqrt{r}]} \sum_{n=1}^{r-[\sqrt{r}]} \frac{1}{\sqrt{n}(r-n)^{\frac{3}{2}}} \\
&= o\left(\frac{e^{-\alpha_0 r}}{\sqrt{r}}\right).
\end{aligned}$$

Again,

$$\begin{aligned}
&\sum_{n=r-[\sqrt{r}]+1}^{\infty} c_{n-n_i} b_{j-m-2n} \\
&\sim \sum_{n=r-[\sqrt{r}]+1}^{\infty} (-1)^{n-n_i} K_c \frac{e^{-\alpha_0(n-n_i)}}{\sqrt{n-n_i}} b_{j-m-2n} \\
&= K_c \frac{e^{-\alpha_0 r}}{\sqrt{r}} \sum_{n=r-[\sqrt{r}]+1}^{\infty} (-1)^{n-n_i} e^{-\alpha_0(n-n_i-r)} \sqrt{\frac{r}{n-n_i}} b_{j-m-2n} \\
&= K_c \frac{e^{-\alpha_0 r}}{\sqrt{r}} \sum_{k=-n_i-[\sqrt{r}]-1}^{\infty} (-1)^{k+r} e^{-\alpha_0 k} \sqrt{\frac{r}{r+k}} b_{j-m-2k-2n_i-2r} \\
&\sim (-1)^r K_i^m \frac{e^{-\alpha_0 r}}{\sqrt{r}},
\end{aligned}$$

where

$$K_i^m = K_c \sum_{k\in\mathbb{Z}} (-1)^k e^{-\alpha_0 k} b_{2k+2r-(j-m)+2n_i}.$$

When $i = m+1+2n_i$ we arrive at

$$\sum_{\substack{k\in\mathbb{Z} \\ k\equiv m+1}} c_{\frac{k-i}{2}} b_{j-k} \sim (-1)^r K_i^{m+1} \frac{e^{-\alpha_0 r}}{\sqrt{r}},$$

in which
$$K_i^{m+1} = K_c \sum_{k\in\mathbb{Z}} (-1)^k e^{-\alpha_0 k} b_{2k+2r-(j-m-1)+2n_i}.$$

Altogether, then, (5.1) holds, with
$$\begin{aligned} D_j &= D_m + D_{m+1} \\ &= 2^{-m} \sum_{\substack{i=M-m \\ i\equiv m}}^{M+1} \binom{m+1}{M+1-i} K_i^m + 2^{-m} \sum_{\substack{i=M-m \\ i\equiv m+1}}^{M+1} \binom{m+1}{M+1-i} K_i^{m+1}. \end{aligned}$$
□

*Remark* 5.3. (1) $D_j$ is a constant over all $j > 0$ having the same parity. Thus, if $j \equiv m$, one has $2r = j - m$, while if $j \equiv m+1, 2r = j - m - 1$.

(2) When $j << -1, r = \left[\frac{-j+m}{2}\right]$,
$$K_i^m = K_c \sum_{k\in\mathbb{Z}} (-1)^k e^{-\alpha_0 k} b_{2k+2r+j-m+2n_i} \tag{5.3}$$
and
$$K_i^{m+1} = K_c \sum_{k\in\mathbb{Z}} (-1)^k e^{-\alpha_0 k} b_{2k+2r+j-m-1+2n_i}. \tag{5.4}$$

## 6. The constants $\gamma_j$

We here study the asymptotic behaviour of the constants $\gamma_j$ in (1.6). A formula for them is given in:

**Lemma 6.1.** *The constants $\gamma_j$ are given in terms of the constants $a_j$ and $c_j$ by*
$$\gamma_j = (-1)^j \sum_{k\in\mathbb{Z}} (-1)^k a_{k-j+1} c_k, \tag{6.1}$$
$j \in \mathbb{Z}$.

*Proof.* Using the formulas (1.2) and (1.3) for $\psi_m$, and $\varphi_m$ respectively, one obtains
$$\begin{aligned} \psi_m(x) &= \sum_{l\in\mathbb{Z}} (-1)^l a_{1-l} \varphi_m(2x-l) \\ &= \sum_{l\in\mathbb{Z}} (-1)^l a_{1-l} \sum_{n\in\mathbb{Z}} c_n N_m(2x-l-n) \\ &= \sum_{l\in\mathbb{Z}} (-1)^l a_{1-l} \sum_{j\in\mathbb{Z}} c_{j-l} N_m(2x-j) \\ &= \sum_{j\in\mathbb{Z}} \left[ \sum_{l\in\mathbb{Z}} (-1)^l a_{1-l} c_{j-l} \right] N_m(2x-j) \\ &= \sum_{j\in\mathbb{Z}} \left[ \sum_{k\in\mathbb{Z}} (-1)^{j-k} a_{k-j+1} c_k \right] N_m(2x-j), \end{aligned}$$

which proves (6.1). □

**Theorem 6.2.** *Let* $\gamma_j$ *be given by* (6.1) *and suppose* $\alpha_0$ *is as in Theorem A. Then, with* $r = \left[\frac{|j|+1}{2}\right]$,

$$\gamma_j \sim (-1)^r E_j \frac{e^{-\alpha_0 r}}{\sqrt{r}}$$

*as* $|j| \to \infty$, *in which* $E_j$ *depends only on the sign and parity of* $j$.

*Proof.* For $j >> 1$ and $j \equiv 0$. We write

$$\begin{aligned} \gamma_j &= \left( \sum_{k=-\infty}^{-\frac{j}{2}-1} + \sum_{k=-\frac{j}{2}}^{\frac{j}{2}} + \sum_{k=\frac{j}{2}+1}^{j-1} + \sum_{k=j}^{\infty} \right) (-1)^k a_{k-j+1} c_k \\ &= S_1 + S_2 + S_3 + S_4. \end{aligned}$$

Consider $S_2$ first. One has

$$\begin{aligned} S_2 &= \sum_{k=-\frac{j}{2}}^{\frac{j}{2}} (-1)^k a_{k-j+1} c_k \\ &\sim \sum_{\substack{k=-\frac{j}{2} \\ k-j\equiv m+1}}^{\frac{j}{2}} (-1)^k (-1)^{\frac{j-k+m-1}{2}} D_m \frac{e^{-\alpha_0(\frac{j-k+m-1}{2})}}{\sqrt{\frac{j-k+m-1}{2}}} c_k \\ &\quad + \sum_{\substack{k=-\frac{j}{2} \\ k-j\equiv m}}^{\frac{j}{2}} (-1)^k (-1)^{\frac{j-k+m}{2}} D_{m+1} \frac{e^{-\alpha_0(\frac{j-k+m}{2})}}{\sqrt{\frac{j-k+m}{2}}} c_k \\ &= (-1)^{\frac{j}{2}} \frac{e^{-\alpha_0 \frac{j}{2}}}{\sqrt{\frac{j}{2}}} [ \sum_{\substack{k=-\frac{j}{2} \\ k-j\equiv m+1}}^{\frac{j}{2}} (-1)^{\frac{k+m+1}{2}} D_m \sqrt{\frac{\frac{j}{2}}{\frac{j}{2} - \frac{k-m+1}{2}}} e^{-\alpha_0(\frac{m-k-1}{2})} c_k \\ &\quad + \sum_{\substack{k=-\frac{j}{2} \\ k-j\equiv m}}^{\frac{j}{2}} (-1)^{\frac{k+m}{2}} D_{m+1} \sqrt{\frac{\frac{j}{2}}{\frac{j}{2} - \frac{k-m}{2}}} e^{-\alpha_0(\frac{m-k}{2})} c_k ] \\ &\sim (-1)^{\frac{j}{2}} E_j \frac{e^{-\alpha_0 \frac{j}{2}}}{\sqrt{\frac{j}{2}}}, \end{aligned}$$

with

$$E_j = \sum_{\substack{k\in\mathbb{Z} \\ k-j\equiv m+1}} (-1)^{\frac{k+m+1}{2}} D_m e^{-\alpha_0(\frac{m-k-1}{2})} c_k$$

$$+ \sum_{\substack{k\in\mathbb{Z}\\ k-j\equiv m}} (-1)^{\frac{k+m}{2}} D_{m+1} e^{-\alpha_0(\frac{m-k}{2})} c_k$$
$$\sim K_c e^{-\alpha_0 \frac{(m-1)}{2}} \sum_{\substack{k\in\mathbb{Z}\\ k-j\equiv m}} (-1)^{\frac{k+m+1}{2}-|k|} D_m e^{-\alpha_0(|k|-\frac{k}{2})}$$
$$+ e^{-\alpha_0 \frac{m}{2}} \sum_{\substack{k\in\mathbb{Z}\\ k-j\equiv m}} (-1)^{\frac{k+m}{2}-|k|} D_{m+1} e^{-\alpha_0(|k|-\frac{k}{2})}.$$

Next,

$$|S_1| = \left| \sum_{k=\frac{j}{2}+1}^{\infty} (-1)^k a_{j+k-1} c_k \right|$$
$$\leq K \left[ \sum_{\substack{k=\frac{j}{2}+1\\ j+k\equiv m+1}}^{\infty} \frac{e^{-\alpha_0(\frac{j+k-1-m}{2})}}{\sqrt{\frac{j+k-1-m}{2}}} \frac{e^{-\alpha_0 k}}{\sqrt{k}} + \sum_{\substack{k=\frac{j}{2}+1\\ j+k\equiv m}}^{\infty} \frac{e^{-\alpha_0(\frac{j+k-m}{2})}}{\sqrt{\frac{j+k-m}{2}}} \frac{e^{-\alpha_0 k}}{\sqrt{k}} \right]$$
$$\leq K \frac{e^{-\alpha_0 \frac{j}{2}}}{\sqrt{\frac{j}{2}}} \left[ \sum_{\substack{k=\frac{j}{2}+1\\ j+k\equiv m+1}}^{\infty} \sqrt{\frac{\frac{j}{2}}{\frac{j}{2}+\frac{k-m-1}{2}}} e^{-\alpha_0(\frac{3k}{2}-\frac{(m+1)}{2})} + \sum_{\substack{k=\frac{j}{2}+1\\ j+k\equiv m}}^{\infty} \sqrt{\frac{\frac{j}{2}}{\frac{j}{2}+\frac{k-m}{2}}} e^{-\alpha_0(\frac{3k}{2}-\frac{m}{2})} \right]$$
$$= o\left( \frac{e^{-\alpha_0 \frac{j}{2}}}{\sqrt{\frac{j}{2}}} \right).$$

Again,

$$|S_3| = \left| \sum_{k=\frac{j}{2}+1}^{j-1} (-1)^k a_{k-j+1} c_k \right|$$
$$\leq K \left[ \sum_{\substack{k=\frac{j}{2}+1\\ k-j\equiv m+1}}^{j-1} \frac{e^{-\alpha_0(\frac{j-k-1-m}{2})}}{\sqrt{\frac{j-k-1-m}{2}}} \frac{e^{-\alpha_0 k}}{\sqrt{k}} + \sum_{\substack{k=\frac{j}{2}+1\\ k-j\equiv m}}^{j-1} \frac{e^{-\alpha_0(\frac{j-k-m}{2})}}{\sqrt{\frac{j-k-m}{2}}} \frac{e^{-\alpha_0 k}}{\sqrt{k}} \right]$$
$$\leq K \frac{e^{-\alpha_0 \frac{j}{2}}}{\sqrt{\frac{j}{2}}} \left[ \sum_{\substack{k=\frac{j}{2}+1\\ k-j\equiv m+1}}^{j-1} e^{-\alpha_0(\frac{k}{2}-m-1)} + \sum_{\substack{k=\frac{j}{2}+1\\ k-j\equiv m}}^{j-1} e^{\alpha_0(\frac{k}{2}-m)} \right]$$
$$= o\left( \frac{e^{-\alpha_0 \frac{j}{2}}}{\sqrt{\frac{j}{2}}} \right).$$

Finally,

$$\begin{aligned}
|S_4| &= \left|\sum_{k=j}^{\infty}(-1)^k a_{k-j+1}c_k\right| \\
&\leq K\left[\sum_{\substack{k=j\\k\equiv m+1}}^{\infty}\frac{e^{-\alpha_0(\frac{k-j+1-m}{2})}}{\sqrt{\frac{k-j+1-m}{2}}}\frac{e^{-\alpha_0 k}}{\sqrt{k}}+\sum_{\substack{k=j\\k\equiv m}}^{\infty}\frac{e^{-\alpha_0(\frac{k-j-m}{2})}}{\sqrt{\frac{k-j-m}{2}}}\frac{e^{-\alpha_0 k}}{\sqrt{k}}\right] \\
&\leq K\frac{e^{-\alpha_0\frac{j}{2}}}{\sqrt{\frac{j}{2}}}\left[\sum_{\substack{k=j\\k\equiv m+1}}^{\infty} e^{-\alpha_0(\frac{3k}{2}-j-m+1)}+\sum_{\substack{k=j\\k\equiv m+1}}^{\infty} e^{-\alpha_0(\frac{3k}{2}-j-m)}\right] \\
&= o\left(\frac{e^{-\alpha_0\frac{j}{2}}}{\sqrt{\frac{j}{2}}}\right).
\end{aligned}$$

□

*Remark* 6.3. As mentioned in Introduction, the result of Theorem 6.2 essentially gives us Theorem A. Similarly, Theorem 3.1 yields, for $m \geq 2$,

$$\varphi_m(x) = K_c\left[\sum_{x-m\leq j\leq x}\frac{e^{-\alpha_0|j|}}{\sqrt{|j|}}N_m(x-j)\right][1+o(1)]$$

as $|x| \to\to \infty$.

Finally, as $m$ gets large, both $\psi_m$ and $\phi_m$ decay less and less rapidly. Indeed, $\psi_m$ and $\phi_m$ converge uniformly to the Shannon wavelet and scaling function, respectively, each of which decays only as $\frac{1}{|x|}$ when $|x| \longrightarrow \infty$; see [4]. We are grateful to the referee for bringing this paper to our attention.

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