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GLOBAL EXISTENCE FOR 3D NAVIER-STOKES EQUATIONS IN A LONG PERIODIC DOMAIN

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Abstract. We consider the global existence of strong solutions of the 3D incompressible Navier-Stokes equations in a long periodic domain. We show by a simple argument that a strong solution exists globally in time when the initial velocity in H^1 and the forcing function in $L^p([0,T);L^2)$, $T>0$, $2\le p\le +\infty$ satisfy a certain condition. This condition commonly appears for the global existence in thin non-periodic domains. Larger and larger initial data and forcing functions satisfy this condition as the thickness of the domain ϵ tends to zero.

1. Introduction

We consider the incompressible Navier-Stokes equations,

$$u_t - \nu\triangle u + (u\cdot\nabla)u + \nabla p = f, \tag{1}$$

$$\nabla\cdot u = 0, \tag{2}$$

in a periodic domain $\Omega = T^3 = [0,l_1]\times[0,l_2]\times[0,l_3]$. Here u denotes the velocity of a homogeneous, viscous incompressible fluid, f is the density of force per unit volume, p denotes the pressure, and ν is the kinematic viscosity. We require that the forcing function f and the initial data u_0 satisfy

$$\nabla\cdot f = \nabla\cdot u_0 = 0.$$

We assume in addition that

$$\int_\Omega f dx = \int_\Omega u dx = 0, \tag{3}$$

which could be achieved by the Galilean transformation with suitable vectors $c(t)$ and e,

$$u(x,t)\to u(x+c(t)+et,t) - \frac{dc}{dt} - e.$$

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Indeed, we can take

$$c(t) = \int_0^t \int_0^r \int f(x,s)dxdsdr, \quad e = \int u_0 dx.$$

By the classical results of Leray and Hopf ([11], [4]), there exists a global weak solution of the Navier-Stokes equations in a three dimensional torus. It is also known that the solution becomes necessarily strong (regular) for all regular data in a two dimensional domain. But in a three dimensional domain, global strong solutions have only been guaranteed for small initial data (See, for example, [2], [3], [14], [15] and the references therein).

In [13], Raugel and Sell treated the problem on thin periodic domain, $\Omega = (0, l_1] \times (0, l_2] \times (0, \epsilon]$ and they obtained a significant existence result on global regular solutions. The main idea is that if the thickness of the domain is small enough, the solution of the Navier-Stokes equations is close to the 2D Navier-Stokes equations. They proved that there are large sets $R(\epsilon) \subset H^1(\Omega)$ and $S(\epsilon) \subset L^\infty((0,\infty), L^2(\Omega))$ such that if $u(0) = u_0 \in R(\epsilon)$ and $f \in S(\epsilon)$, then there exists a strong solution $u(t)$ that remains in $H^1(\Omega)$ for all $t \geq 0$. The sets $R(\epsilon)$ and $S(\epsilon)$ get larger and larger as $\epsilon \to 0$.

Since then, there have been many improvements on the estimates of the size of these sets $R(\epsilon)$ and $S(\epsilon)$ under various boundary conditions (See [1], [5], [12], [6], [7], [8], [9], [16] and the references therein). Roughly, under various boundary conditions except the periodic boundary condition, it has been shown that if

$$\|u_0\|_{H^1} \leq C\epsilon^{-1/2} \quad \text{and} \quad \|f\|_{L^\infty((0,\infty),L^2)} \leq C\epsilon^{-1/2} \tag{4}$$

for some constant $C = C(\nu)$, then the corresponding global strong solution exists (See [1], [16]). We note that the above condition can cover very large initial data and forcing functions if $\epsilon > 0$ is small enough.

However, under the periodic boundary condition, it is not known whether (4) implies the existence of global strong solutions. Until now, it is known that, when $f = 0$, the existence of the global strong solution is guaranteed under the condition ([10])

$$\|u_0\|_{H^1} \leq C\epsilon^{-1/2}|\log \epsilon|^{1/2},$$

or under the following condition ([6])

$$\|(Nu_0)_3\| \leq C\nu\epsilon^{1/2}, \ \|Nf\|_{L^\infty(0,\infty;L^2)} \leq C\nu^2\epsilon^{1/2},$$
$$\|\nabla u_0\| \leq C\nu\epsilon^{-1/2}, \ \|f\|_{L^\infty(0,\infty;L^2)} \leq C\nu^2\epsilon^{-1/2}.$$

Here, N is the average operator with respect to the thin direction. We note that the first two conditions in the above are not so restrictive since Nu_0 and Nf are independent of the third variable and so they are in fact ϵ independent conditions.

In this paper, we consider the global existence of strong solutions in a long periodic domain, $\Omega = (0, \epsilon] \times (0, \epsilon] \times (0, l]$. We first prove in a simple way that a global strong solution exists whenever the initial and the forcing functions satisfy for any $2 \le p \le \infty$ and $L > 0$,

$$\|\nabla u_0\|_{L^2} \le \frac{C\nu}{L} \quad \text{and} \quad \|f\|_{L^p((0,\infty),L^2)} \le C\nu^{(2p-1)/p}\lambda_1^{(3p-4)/4p} \tag{5}$$

together with a mild condition,

$$\frac{1}{L}\|u_0\|_{L^2} \le 1 \tag{6}$$

for some universal constant C. Here, $\lambda_1 = 4\pi^2/l$ is the first eigenvalue of the Stokes operator. This result is obtained simply by considering a differential inequality for a product of norms, which is comparable to $H^{1/2}$ norm. The most natural choice of L in the condition (6) is $L = \sqrt{|\Omega|}$, which is not practically restrictive since it just means that the spatial average of the square of the velocity is bounded by a suitable constant. Then, when the domain is long rod type $\Omega = (0,\epsilon] \times (0,\epsilon] \times (0,l]$, the choice $L = \sqrt{|\Omega|}$ becomes of order ϵ and the bound on H^1 norm of the velocity in (5) is improved greatly compared to the case of thin domain. We also give a condition independent of the L^2 norm of the velocity. Concretely, we show that the global regularity is guaranteed if

$$\|\nabla u_0\| \le C\nu\epsilon^{-1/2}, \ \|f\|_{p,2} \le C\nu^{(2p-1)/p}\epsilon^{-1/2}$$

for any $2 \le p \le \infty$. The above condition exactly recovers (4) even for more general p and supports that the condition (4) might be enough for the global existence in a thin periodic domain under the periodic boundary condition.

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2. Preliminary estimates

Throughout the paper, $\Omega = (0,\epsilon] \times (0,\epsilon] \times (0,l]$. Here, l is a fixed constant and $\epsilon > 0$ is a small parameter. For convenience's sake, we denote the two dimensional torus $D \equiv D_\epsilon = (0,\epsilon] \times (0,\epsilon]$. The function spaces we work with are

$$H = \{u \in L^2(\Omega) | \nabla \cdot u = 0, \ \int_\Omega u = 0\}$$

and $V = H \cap W^{1,2}(\Omega)$. It is well known that $\|\nabla u\|_{L^2}$ is an equivalent norm for V due to the Poincaré inequality. For convenience's sake, we also denote

$$\|\cdot\|_{L^p} = \|\cdot\|_p, \quad \|\cdot\|_2 = \|\cdot\|, \quad \|\cdot\|_{L^p(0,\infty;L^q(\Omega))} = \|\cdot\|_{p,q},$$

the Leray projection on $L^2(\Omega)$ into H by $\mathbb{P}$, and the Stokes operator by $A = \mathbb{P}(-\Delta)$. We define the bilinear form $B(u,v) = \mathbb{P}(u \cdot \nabla)v$ and the trilinear form $b(u,v,w)$ by

$$b(u,v,w) = \langle B(u,v), w\rangle = \int_\Omega B(u,v) \cdot w dx.$$

We now define an orthogonal projection M on $L^2(\Omega_\epsilon)$ by

$$Mu = \frac{1}{\epsilon^2}\int_0^\epsilon\int_0^\epsilon u(x_1,x_2,x_3)dx_1dx_2 \tag{7}$$

and denote $v \equiv Mu$ and $w \equiv (I-M)u$ for simplicity. Note that the above projection is different from the one in [6]. Here, $v = v(x_3)$ and $\nabla\cdot v = 0$. So, v_3 must be a constant in space. Since we assume (3) from the first, we then get

$$v_3 = \frac{1}{|\Omega|}\int v_3 = \frac{1}{|\Omega|}\int u_3 = 0. \tag{8}$$

It is clear that the following Poincaré inequality holds for $w \in H^1$ since $Mw = 0$:

$$\|w\|^2 \le C\epsilon^2\|\nabla w\|^2. \tag{9}$$

Further, w satisfies the following inequalities, which are basically Gargliardo-Nirenberg inequalities.

Lemma 2.1. *Given $u \in V\cap D(A)$, let $v = Mu$ and $w = (I-M)u$. We have*

$$\|\nabla v\|_\infty \le \frac{C}{\epsilon}\|\nabla v\|^{1/2}\|Av\|^{1/2}, \tag{10}$$

$$\|\nabla w\|_q \le C(\|\nabla_3 w\|\|w\| + \|w\|^2)^{1/2q}\|Aw\|^{\frac{q-1}{q}}, \quad 1 < q \le 3. \tag{11}$$

Here, all C's are independent of ϵ.

Proof. Since $w(\cdot,x_3)$ is average zero on D for any $x_3 \in (0,l]$, w satisfies the following two dimensional Gargliardo-Nirenberg inequality.

$$\|\nabla w\|^q_{L^q(D)} \le C\|\nabla^2 w\|^{q-1}_{L^2(D)}\|w\|_{L^2(D)}.$$

Here, C is independent from ϵ. In fact, the above inequality is scaling invariant. Integrating with respect to x_3, we have

$$\int_0^l dx_3\int_D|\nabla w|^q \le C\int_0^l dx_3\left(\int_D|\nabla^2 w|^2\right)^{\frac{q-1}{2}}\sup_{x_3}\|w\|_{L^2(D)}(x_3). \tag{12}$$

While,

$$\begin{aligned}\|w\|^2_{L^2(D)}(b) &\le \left|\int_a^b dx_3\ \partial_3\|w\|^2_{L^2(D)}(x_3)\right| + \|w\|^2_{L^2(D)}(a)\\ &\le \int_a^b\int_D|\partial_3 w||w|dx + \|w\|^2_{L^2(D)}(a)\\ &\le \|\nabla_3 w\|\|w\| + \|w\|^2_{L^2(D)}(a).\end{aligned}$$

Integrating the above with respect to a over $(0,l]$, we have

$$\sup_{x_3}\|w\|^2_{L^2(D)} \le \|\nabla_3 w\|\|w\| + \frac{1}{l}\|w\|^2.$$

Plugging the above into (12) and using the Hölder inequality, we have

$$\int_\Omega |\nabla w|^q \leq C\|\nabla^2 w\|^{q-1}(\|\nabla_3 w\|\|w\| + \|w\|^2)^{1/2}.$$

Since $\|\nabla^2 w\| \leq C\|Aw\|$, we have the desired inequality (11). Similarly,

$$(\partial_3 v_i)^2(b) = 2\int_a^b dx_3 \partial_3^2 v_i \partial_3 v_i + (\partial_3 v)^2(a).$$

There exists a such that $\partial_3 v_i(a) = 0$ since $\partial_3 v_i$ is average zero. Thus we have

$$(\nabla v_i)^2(b) \leq C\int dx_3 |\nabla^2 v||\nabla v| \leq \frac{C}{\epsilon^2}\|\nabla v\|\|Av\|.$$

Taking supremum with respect to b and adding them up for $i = 1, 2$, we have the desired result (10). □

We now present the following estimates concerning the trilinear form b. We use the above lemma with $q = 3$ to get the estimates.

Lemma 2.2. *Let v and w be as before, we have*

$$|b(w, w, Aw)| \leq C\|w\|^{1/2}\|\nabla w\|^{1/2}\|Aw\|^2, \tag{13}$$

$$\begin{aligned} &|b(v, w, Aw)|,\ |b(w, v, Aw)|,\ |b(w, w, Av)| \\ \leq\ & C\|\nabla v\|^{1/2}\|Av\|^{1/2}\|w\|^{1/2}\|Aw\|^{3/2}. \end{aligned} \tag{14}$$

Here, all C's are independent from ϵ.

Proof. First, by integration by parts,

$$\begin{aligned} b(w, w, Aw) &= -\int (w\cdot\nabla)w\cdot\Delta w = \int (\nabla_j w\cdot\nabla)w\cdot\nabla_j w + w\cdot\nabla(\nabla_j w)\nabla_j w \\ &= \int (\nabla_j w\cdot\nabla)w\cdot\nabla_j w. \end{aligned}$$

Thus, using (11) with $q = 3$, (9), and the smallness of ϵ,

$$\begin{aligned} |b(w, w, Aw)| \leq C\|\nabla w\|_3^3 &\leq C\|Aw\|^2(\|\nabla_3 w\|^{1/2}\|w\|^{1/2} + \|w\|) \\ &\leq C\|Aw\|^2\|\nabla w\|^{1/2}\|w\|^{1/2}. \end{aligned}$$

By similar argument,

$$b(v, w, Aw) = \int (\nabla_j v\cdot\nabla)w\cdot\nabla_j w.$$

Then, since v depends only on x_3,

$$\begin{aligned} |b(v, w, Aw)| &\leq \int_0^l dx_3 |\nabla v| \int_D |\nabla w|^2 \\ &\leq C\|\nabla v\|_{L^\infty(0,l)}\|\nabla w\|^2 \leq C\|\nabla v\|^{1/2}\|Av\|^{1/2}\|\nabla w\|\|Aw\| \\ &\leq C\|\nabla v\|^{1/2}\|Av\|^{1/2}\|w\|^{1/2}\|Aw\|^{3/2}. \end{aligned}$$

Here, we used in the last line the interpolation inequality

$$\|\nabla f\|^2 = -\int f\Delta f \le \|f\|\|Af\|. \tag{15}$$

Similarly,

$$\begin{aligned}|b(w,v,Aw)| &\le \int_0^l dx_3 |\nabla v| \|w\|_{L^2(D)} \|Aw\|_{L^2(D)} \\ &\le C\|\nabla v\|_{L^\infty(0,l)} \|w\|\|Aw\| \le C\|\nabla v\|^{1/2}\|Av\|^{1/2}\|w\|\|Aw\|^{3/2}, \\ |b(w,w,Av)| &= \left| \int \nabla_j w \cdot \nabla w \cdot \nabla_j v + w \cdot \nabla\nabla_j w \cdot \nabla_j v \right| \\ &\le C\|\nabla v\|^{1/2}\|Av\|^{1/2}\|w\|^{1/2}\|Aw\|^{3/2}.\end{aligned}$$

□

3. Regularity

In this section, we give our main result. We first reformulate (1)-(2) in the standard nonlinear evolutionary equation on the Hilbert space V,

$$u_t + \nu Au + B(u,u) = \mathbb{P}f. \tag{16}$$

We shall consider solutions of (16) with the initial data u_0 and $f = f(t)$ in the class

$$u_0 \in V, \quad f(t) \in L^p([0,\infty), H), \ p \ge 2. \tag{17}$$

We first present the following theorem, which is simple and shows the underlying idea of our result.

Theorem 3.1. *Given any $p \ge 2$, the Navier-Stokes evolutionary equation* (16) *has a solution*

$$u \in C^0([0,\infty), H) \cap L^\infty((0,\infty), V)$$

if

$$\|u_0\|\|\nabla u_0\| + 2\nu^{-\frac{2p-2}{p}} \lambda_1^{-\frac{3p-4}{2p}} \|f\|_{p,2}^2 \le \frac{\nu^2}{C^2}. \tag{18}$$

Here, λ_1 is the first eigenvalue of the Stokes operator, C is an absolute constant independent of ϵ. Moreover, in this case

$$\|\nabla u\|^2(t) \le \|\nabla u_0\|^2 + 4\nu^{-\frac{2p-2}{p}} \lambda_1^{-\frac{p-2}{p}} \|f\|_{p,2}^2 \tag{19}$$

for all $t > 0$.

Proof. By taking the scalar product of (16) with u and using the fact that

$$\int B(u,u)u dx = 0,$$

we find that

$$\frac{d}{dt}\|u\|^2 + 2\nu\|\nabla u\|^2 \le 2\|f\|\,\|u\|. \tag{20}$$

Since v depends only on x_3 and $v_3 = 0$, $\langle B(v,v), Av\rangle = 0$ and $b(v,v,w) = b(w,v,v) = b(v,w,v) = 0$. So,

$$\begin{aligned}\langle B(u,u), Au\rangle &= \langle B(v,w), Aw\rangle + \langle B(w,v), Aw\rangle \\ &\quad + \langle B(w,w), Av\rangle + \langle B(w,w), Aw\rangle \\ &\le C(\|\nabla v\|^{1/2}\|Av\|^{1/2}\|w\|^{1/2}\|Aw\|^{3/2} + \|w\|^{1/2}\|\nabla w\|^{1/2}\|Aw\|^2) \\ &\le C\|w\|^{1/2}\|\nabla u\|^{1/2}\|Au\|^2\end{aligned}$$

by the orthogonality of v and w. Then, taking the scalar product of (16) with Au and using the above estimate, we obtain

$$\begin{aligned}\frac{d}{dt}\|\nabla u\|^2 + 2\nu\|Au\|^2 &\le 2\left|\int fAu\right| + \left|\int B(u,u)Au\right| \\ &\le 2\|f\|\|Au\| + C(\|w\|\|\nabla u\|)^{1/2}\|Au\|^2. \end{aligned} \tag{21}$$

Now, we multiply (20) by $\|\nabla u\|^2$ and (21) by $\|u\|^2$ and adding them to have

$$\begin{aligned}&\frac{d}{dt}(\|u\|^2\|\nabla u\|^2) + 2\nu\|\nabla u\|^4 + 2\nu\|u\|^2\|Au\|^2 \\ &\le 2\|f\|\|u\|(\|\nabla u\|^2 + \|u\|\|Au\|) + C(\|u\|\,\|\nabla u\|)^{1/2}\|u\|^2\|Au\|^2.\end{aligned} \tag{22}$$

By the Young inequality and (15), we have

$$\begin{aligned}2\|f\|\|u\|(\|\nabla u\|^2 + \|u\|\|Au\|) &\le 4\|f\|\|u\|^2\|Au\| \\ &\le 4\|f\|\frac{\|\nabla u\|^{1/2}}{\lambda_1^{1/4}}\|u\|^{1/2}\|u\|\|Au\| \\ &\le \nu\|u\|^2\|Au\|^2 + \frac{4}{\nu\lambda_1^{1/2}}\|f\|^2\|u\|\|\nabla u\|.\end{aligned}$$

Denoting $G^2 = \|u\|^2\|\nabla u\|^2$, we thus arrive at

$$\frac{d}{dt}G^2 + \nu\lambda_1 G^2 \le \left[CG^{1/2} - \nu\right]\|u\|^2\|Au\|^2 + \frac{4}{\nu\lambda_1^{1/2}}\|f\|^2 G.$$

If $G(t) \le \frac{\nu^2}{C^2}$ for all $t > 0$,

$$\frac{d}{dt}G + \nu\lambda_1 G \le \frac{2}{\nu\lambda_1^{1/2}}\|f\|^2.$$

Therefore, by the Grönwall inequality,

$$\begin{aligned}G &\le G(0)e^{-\nu\lambda_1 t} + \frac{2}{\nu\lambda_1^{1/2}}\int_0^t \|f\|^2(s)e^{\nu\lambda_1(s-t)}ds \\ &\le G(0) + \frac{2}{\nu\lambda_1^{1/2}}\|f\|_{p,2}^2\left(\frac{p-2}{p\nu\lambda_1}\right)^{(p-2)/p} \\ &\le G(0) + 2\left(\frac{p-2}{p}\right)^{\frac{p-2}{p}}\nu^{-\frac{2p-2}{p}}\lambda_1^{-\frac{3p-4}{2p}}\|f\|_{p,2}^2\end{aligned}$$

for any $p \geq 2$. Note that the above estimate holds true even for $p = 2$ and ∞. Since $((p-2)/p)^{(p-2)/p} \leq 1$, the typical continuation argument and (18) justifies the above argument and we indeed have $G(t) \leq \frac{\nu^2}{C^2}$ for all $t > 0$. Furthermore, if $G(t) \leq \frac{\nu^2}{C^2}$, we apply the Hölder inequality to (21) to have

$$\frac{d}{dt}\|\nabla u\|^2 + \frac{\nu}{2}\lambda_1\|\nabla u\|^2 \leq \frac{2}{\nu}\|f\|^2.$$

Again, by the Grönwall inequality,

$$\|\nabla u\|^2(t) \leq \|\nabla u_0\|^2 + \int_0^t \frac{2}{\nu}\|f\|^2(s)e^{\nu\lambda_1(s-t)/2}ds,$$

which gives (19) and finishes the proof. □

The condition (18) is in a sense a condition of smallness of the initial data and external force. However, this condition allows for initial data with large H^1 norm provided that the L^2 norm of the initial data and f are small enough. In particular, when $f = 0$, the above theorem tells that there exists a globally regular solution if $\|u_0\|$ is small enough compared with $\nu^2\|\nabla u_0\|^{-1}$. As a corollary of the above theorem, we have the following.

Corollary 3.2. *There exists a globally regular solution if initial data satisfies* (5) *and* (6) *with* $L = \epsilon$ *when* $\Omega = [0, l] \times [0, \epsilon] \times [0, \epsilon]$.

Applying the projections M and $(I - M)$ to the equation (16) and using $MB(v, v) = B(v, v) = 0$ and $MB(v, w) = MB(w, v) = 0$, we get the equation for v,

$$\frac{dv}{dt} + \nu Av = Mf - MB(w, w), \tag{23}$$

and the equation for w,

$$\frac{dw}{dt} + \nu Aw = (I - M)f - B(w, v) - B(v, w) - (I - M)B(w, w). \tag{24}$$

Theorem 3.3. *There exists a globally regular solution* u *of* (16) *in* Ω *if, for some constant* C,

$$\|\nabla u_0\|^2 + 2\nu^{-\frac{2p-2}{p}}\lambda_1^{-\frac{p-2}{p}}\|f\|_{p,2}^2 < \frac{\nu^2}{4C^2}\epsilon^{-1}. \tag{25}$$

Proof. We start from (21). By (9), (21) becomes

$$\frac{d}{dt}\|\nabla u\|^2 + 2\nu\|Au\|^2 \leq 2\|f\|\|Au\| + C(\epsilon\|\nabla w\|\|\nabla u\|)^{1/2}\|Au\|^2.$$

Then,

$$\frac{d}{dt}\|\nabla u\|^2 + (\nu - C\epsilon^{1/2}\|\nabla u\|)\|Au\|^2 \leq \frac{1}{\nu}\|f\|^2$$

since $\|\nabla u\| \geq \|\nabla w\|$. Now, we apply the Grönwall lemma to the above inequality with typical smallness argument. That is, since $\|\nabla u_0\| < \frac{\nu}{2C}\epsilon^{-1/2}$

from (25), if $\|\nabla u\|(t) > \frac{\nu}{2C}\epsilon^{-1/2}$ for some $t > 0$, there would be the first time $t = T$ such that $\|\nabla u\|(T) = \frac{\nu}{2C}\epsilon^{-1/2}$. However, for $0 < t < T$,

$$\frac{d}{dt}\|\nabla u\|^2 + \frac{\nu}{2}\lambda_1\|\nabla u\|^2 < \frac{1}{\nu}\|f\|^2.$$

Applying the Grönwall lemma to the above inequality, we would have

$$\begin{aligned}\|\nabla u\|^2(T) &< \|\nabla u_0\|^2 + \frac{1}{\nu}\int_0^T \|f\|^2 e^{\nu\lambda_1(s-T)/2}ds\\ &\le \|\nabla u_0\|^2 + \frac{1}{\nu}\left(\frac{2(p-2)}{\nu\lambda_1 p}\right)^{(p-2)/p}\|f\|_{p,2}^2\\ &\le \|\nabla u_0\|^2 + 2\nu^{-\frac{2p-2}{p}}\lambda_1^{-\frac{p-2}{p}}\|f\|_{p,2}^2.\end{aligned}$$

If (25) holds true, this leads a contraction. Therefore, $\|\nabla u\|^2 < \frac{\nu^2}{4C^2}\epsilon^{-1}$ for all $t > 0$ and we finish the proof. □

Clearly, the condition (25) in particular implies that there exists a globally regular solution if, for suitable C,

$$\|\nabla u_0\| \le C\nu\epsilon^{-1/2},\ \|f\|_{p,2} \le C\nu^{(2p-1)/p}\epsilon^{-1/2}$$

since $\lambda_1 = \frac{4\pi^2}{l^2}$ is fixed.

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